Noise Analysis of Non-Linear Dynamic Integrated Circuits

Amir Zjajo, Qin Tang, Michel Berkelaar, Nick van der Meijs
Delft University of Technology
Mekelweg 4, 2628 CD Delft, The Netherlands

***Abstract*— A time-domain methodology for noise analysis of non-linear dynamic integrated circuits with arbitrary excitations is presented. A non-stationary stochastic noise process is described as an Ito system of stochastic differential equations and a numerical solution for such a set of equations is found. Statistical simulation of specific circuits fabricated in 65 nm CMOS process shows that the proposed numerical methods offer accurate and efficient solution of stochastic differentials for noise analysis of integrated circuits.**

I. INTRODUCTION

Noise limitations pose a rudimentary issue for the robust circuit design and its evaluation has been subject of numerous studies [1]-[3]. The most important types of electrical noise sources (thermal, shot, and flicker noise) in passive elements and integrated circuit devices have been investigated extensively, and appropriate models have been derived [1] as stationary and in [4] as non-stationary noise sources. In this paper we adapt model description as defined in [4], where thermal and shot noise are expressed as delta-correlated noise processes having independent values at every time point, modeled as modulated white noise processes. These noise processes correspond to the current noise sources which are included in the models of the integrated-circuit devices. The noise performance of a circuit can be analyzed in terms of the small-signal equivalent circuits by considering each of the uncorrelated noise sources in turn and separately computing its contribution at the output. Unfortunately, this method is only applicable to circuits with fixed operating points and is not appropriate for noise simulation of circuits with changing bias conditions. A widespread approach for noise simulation in time-domain is Monte Carlo analysis. However, accurately determining the noise content requires a large number of simulations, so consequently, Monte Carlo method becomes very *cpu*-time consuming if the chip becomes large.

In this paper, we treat the noise as a non-stationary stochastic process, and introduce an Ito system of stochastic differential equations as a convenient way to represent such a process. Recognizing that the variance-covariance matrix when backward Euler is applied to such a matrix can be written in the continuous-time Lyapunov matrix form, we then provide numerical solution to such a set of linear time-varying equations.

II. STOCHASTIC MNA FOR NOISE ANALYSIS

In general, for time-domain analysis, modified nodal analysis (MNA) leads to a nonlinear ordinary differential equation (ODE) or differential algebraic equation (DAE) system that, in most cases, is transformed into a nonlinear algebraic system by means of linear multistep integration methods [5]-[6] and, at each integration step, a Newton-like method is used to solve this nonlinear algebraic system. Therefore, from a numerical point of view, the equations modeling a dynamic circuit are transformed to equivalent linear equations at each iteration of the Newton method and at each time instant of the time-domain analysis. Thus, we can say that the time-domain analysis of a nonlinear dynamic circuit consists of the successive solutions of many linear circuits approximating the original (nonlinear and dynamic) circuit at specific operating points.

Consider a linear circuit with $N+1$ nodes and B voltage-controlled branches (two-terminal resistors, independent current sources, and voltage-controlled n-ports), the latter grouped in set B. We then introduce the source current vector $\hat{i} \in R^B$ and the branch conductance matrix $\mathbf{G} \in R^{B\times B}$. By assuming that the branches (one for each port) are ordered element by element, the matrix is block diagonal: each 1×1 block corresponds to the conductance of a one-port and in any case is nonzero, while $n\times n$ blocks correspond to the conductance matrices of voltage-controlled n-ports. More in detail, the diagonal entries of the $n\times n$ blocks can be zero and, in this case, the nonzero off-diagonal entries, on the same row or column, correspond to voltage-controlled current sources (VCCSs). Now, consider MNA and circuits embedding, besides voltage-controlled elements, independent voltage sources, the remaining types of controlled sources and noise sources. We split the set of branches B in two complementary subsets: B_V of voltage-controlled branches (v-branches) and B_C of current-controlled branches (c-branches). Conventional nodal analysis (NA) is extended to MNA [6] as follows: currents of c-branches are added as further unknowns and the corresponding branch equations are appended to the NA system. The $N\times B$ incidence matrix $\mathbf{A}$ can be partitioned as $\mathbf{A}=[\mathbf{A}_v\ \mathbf{A}_c]$, with $\mathbf{A}_v \in R^{N\times Bv}$ and $\mathbf{A}_c \in R^{N\times Bc}$. As in conventional NA, constitutive relations of v-branches are written, using the conductance submatrix $\mathbf{G}^{\cdot} \in R^{Bc\times Bv}$ in the form

$$\mathbf{i}_v = \mathbf{G}^{\cdot}\mathbf{v}_v \tag{1}$$

978-1-4244-5760-1/10/$26.00 ©2010 IEEE

while the characteristics of the c-branches, including independent voltage sources and controlled sources except VCCSs, are represented by the implicit equation

$$\mathbf{B}_c\mathbf{v}_c + \mathbf{R}_c\mathbf{i}_c + \hat{\mathbf{v}}_c + \mathbf{F}_c\boldsymbol{\lambda} = 0 \quad (2)$$

where $\mathbf{B}_c$, $\mathbf{R}_c$, $\mathbf{F}_c$ $\in R^{Bc\times Bc}$, $\hat{\mathbf{v}}_c$=$(\mathbf{A}^T\mathbf{v}_c)\in R^{Bc}$ [5] and $\boldsymbol{\chi}$ $\in R^{Bc}$ is a random vector accounting for noise. By using the above notations, we obtain the system, written in the compact form

$$F(\mathbf{r}',\mathbf{r},t) + B(\mathbf{r},t)\cdot\boldsymbol{\chi} = 0 \quad (3)$$

where $\mathbf{r}=[\mathbf{v}_c\ \mathbf{i}_v]^T$ is the vector of stochastic processes that represents the state variables (e.g. node voltages) of the circuit, $\boldsymbol{\chi}$ is a vector of white Gaussian processes and $B(\mathbf{r},t)$ is state and time dependent modulation for the vector of noise sources. Every column of $B(\mathbf{r},t)$ corresponds to $\boldsymbol{\chi}$, and has normally either one or two nonzero entries. The rows correspond to either a node equation or a branch equation of an inductor or a voltage source. Equation (3) represents a system of nonlinear stochastic differential equations, which formulate a system of stochastic algebraic and differential equations that describe the dynamics of the nonlinear circuit that lead to the MNA equations when the random sources $\boldsymbol{\chi}$ are set to zero. Solving (3) means to determine the probability density function P of the random vector $\mathbf{r}(t)$ at each time instant t. However, generally it is not possible to handle this distribution directly. Hence, it may be convenient to look for an approximation that can be found after partitioning the space of the stochastic source variables $\boldsymbol{\chi}$ in a given number of subdomains, and then solving the equation in each subdomain by means of a piecewise-linear truncated Taylor approximation. Since the magnitude of the noise content in a signal is much smaller in comparison to the magnitude of the signal itself in any functional circuit, a system of nonlinear stochastic differential equations described in (3) can be piecewise-linearized; it is then possible to combine the partial results and obtain the desired approximated solution to the original problem. Now, including the noise content description, (3) can be expressed in general form as

$$\lambda'(t) = E(t)\lambda + F(t)\chi \quad (4)$$

where $\boldsymbol{\lambda}$= $[(\mathbf{r}-r_0)^T,(\boldsymbol{\chi}-\chi_0)^T]^T$. We will interpret (4) as an Ito system of stochastic differential equations. Now rewriting (4) in the more natural differential form

$$d\lambda(t) = E(t)\lambda dt + F(t)dw \quad (5)$$

where we substituted $dw(t)=\chi(t)dt$ with a vector of Wiener process w. If the functions $E(t)$ and $F(t)$ are measurable and bounded on the time interval of interest, there exists a unique solution for every initial value $\lambda(t_0)$ [7].

If λ is a Gaussian stochastic process, then it is completely characterized by its mean and correlation function. From Ito's theorem on stochastic differentials

$$\begin{aligned} d(\lambda(t)\lambda^T(t))/dt &= \lambda(t)\cdot d(\lambda^T(t))/dt \\ &+ d(\lambda(t))/dt\cdot\lambda^T(t) + F(t)\cdot F^T(t)dt \end{aligned} \quad (6)$$

and expanding (6) with (5), noting that λ and dw are uncorrelated, variance-covariance matrix $\mathbf{K}(t)$ of $\boldsymbol{\lambda}(t)$ with the initial value $K(0)=E[\lambda\ \lambda^T]$ can be expressed in differential Lyapunov matrix equation form as [7]

$$d\mathbf{K}(t)/dt = \mathbf{E}(t)\mathbf{K}(t) + \mathbf{K}(t)\mathbf{E}^T(t) + \mathbf{F}(t)\mathbf{F}^T(t) \quad (7)$$

Note that the mean of the noise variables is always zero for most integrated circuits. In view of the symmetry of $\mathbf{K}(t)$, (7) represents a system of linear ordinary differential equations with time-varying coefficients. To obtain a numerical solution, (7) has to be discretized in time using a suitable scheme, such as any linear multi-step method, or a Runge-Kutta method. For circuit simulation, implicit linear multi-step methods, and especially the trapezoidal method and the backward differentiation formula were found to be most suitable [8]. If backward Euler is applied to (7), the differential Lyapunov matrix equation can be written in a special form referred to as the continuous-time algebraic Lyapunov matrix equation

$$\mathbf{P}_r\mathbf{K}(t_r) + \mathbf{K}(t_r)\mathbf{P}_r^T + \mathbf{Q}_r = 0 \quad (8)$$

$\mathbf{K}(t)$ at time point t_r is calculated by solving the system of linear equations in (8). Such continuous time Lyapunov equations have a unique solution $\mathbf{K}(t)$, which is symmetric and positive semidefinite.

Several iterative techniques have been proposed for the solution of the algebraic Lyapunov matrix equation (11) arising in some specific problems where the matrix $\mathbf{P}_r$ is large and sparse [9]-[10], such as the Bartels-Stewart method [11], and Hammarling's method [7], which remains the one and only reference for directly computing the Cholesky factor of the solution $\mathbf{K}(t_r)$ of (8). Large dense Lyapunov equations can be solved by sign function based techniques [12]. Krylov subspace methods, which are related to matrix polynomials have been proposed [13] as well. Relatively large sparse Lyapunov equations can be solved by iterative approaches, e.g., [14]. In this paper, we apply a low rank version of the iterative method, which is related to rational matrix functions. The postulated iteration for the Lyapunov equation (8) is given by $\mathbf{K}(0) = 0$ and

$$\begin{aligned} (P_r + \gamma_i I_n)K_{i-1/2} &= -Q_r - K_{i-1}(P_r^T - \gamma_i I_n) \\ (P_r + \overline{\gamma}_i I_n)K_i^T &= -Q_r - K_{i-1/2}^T(P_r^T - \overline{\gamma}_i I_n) \end{aligned} \quad (9)$$

for i = $1,2,\ldots$ This method generates a sequence of matrices $\mathbf{K}_i$ which often converges very fast towards the solution, provided that the iteration shift parameters γ_i are chosen (sub)optimally. For a more efficient implementation of the method, we replace iterates by their Chlesky factors, i.e., $K_i=L_iL_i^H$ and reformulate in terms of the factors L_i. The low rank Cholesky factors L_i are not uniquely determined. Different ways to generate them exist [15].

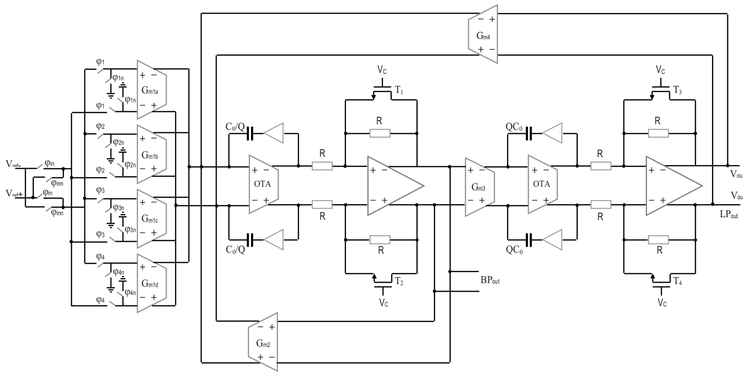

Figure 1: Gm-C-OTA biquad filter

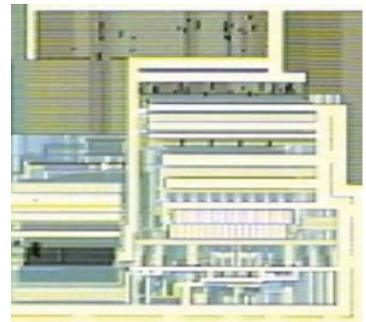

Figure 2: Prototype micrograph

Note that the number of iteration steps i_{max} needs not be fixed a priori. However, if the Lyapunov equation should be solved as accurate as possible, correct results are usually achieved for low values of stopping criteria that are slightly larger than the machine precision.

III. Experimental Results

The proposed method and all sparse techniques have been implemented in Matlab. All the experimental results are carried out on a single processor Linux system with Intel Core 2 Duo CPUs with 2.66 Ghz and 3 GB of memory. The proposed method solves the set of linear time-varying equations (8) including the noise content description to get the steady state value of the time-varying covariance matrix. This gives the variance at the output node and its cross-correlation with other nodes in the circuit. The covariance matrix is periodic with the same period as either the input signal (e.g., translinear circuits) or the clock (in circuits such as switched capacitor circuits). Moreover, with the proposed method, it is also possible to compute all the cross-correlations between any electrical quantity and any stochastic source. This makes it possible for the designer to evaluate the devices that most affect a particular performance, so that design efforts can be addressed to the most critical section of the circuit.

The effectiveness of the proposed approaches was evaluated on several circuits exhibiting different distinctive features in a variety of applications. As a representative example of the results that can be obtained, we show an application of noise analysis to the characterization of the continuous-time bandpass G_m-*C-OTA* biquad filter fabricated in standard 65 nm CMOS technology (Figure 1 and Figure 2) with the frequency response illustrated in Figure 3. The implemented double feedback structure yields an overall improvement on the filter linearity performance. With the opposite phase of the distortion amount introduced by the transconductors in the feedback path, the smaller loop (with G_{m2}) partially attenuates the nonlinearity deriving from transconductor G_{m3}, whereas the larger loop (with G_{m4}) attenuates the nonlinearity deriving from the input G_{m1}. The transconductor G_{m2} introduces some partial positive feedback (acts as a negative resistor) so that the quality factor can be made as high as desired, only limited by parasitics and stability issues. The filter cut-off frequency is controlled through G_{m3} and G_{m4}, the Q-factor is controlled through a G_{m2}, and the gain can be set with G_{m1}. In the simulation we have included only the shot and thermal noise sources as including the flicker noise sources increases the simulation time due to the large time constants introduced by the networks for flicker noise source synthesis. We assumed that the time series **r** are composed of a smoothly varying function, plus additive Gaussian white noise χ (Figure 4), and that at any point **r** can be represented by a low order polynomial (a truncated local Taylor series approximation). This is achieved by trimming off the tails of the distributions and then using percentiles to reverse the desired variance. However, this process increases simulation time and introduces bias in the results. Inadvertently, this bias is a function of the series length and as such predictable, so the last steps in noise estimation are to filter out that predicted bias from the estimated variance. Noise estimation is robust to a few arbitrary spikes or discontinuities in the function or its derivatives (Figure 5). The results of the estimation of the noise variance are illustrated in Figure 6. In comparison with 1500 Monte Carlo iterations, the difference is less then 1% and 4% for mean and variance, respectively, with considerable *cpu*-time reduction (1241.7 sec versus 18.6 sec).

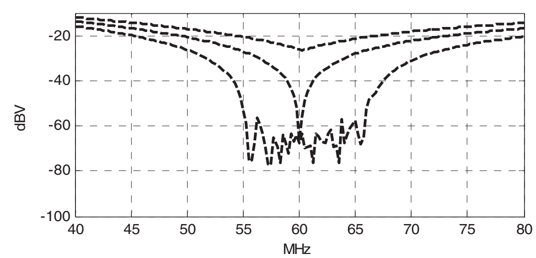

Figure 3: G_m-C-OTA biquad filter frequency response. Middle line designates the nominal behavior.

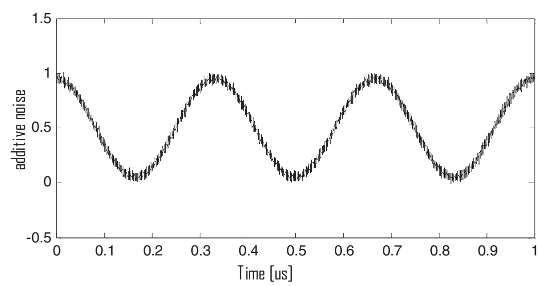

Figure 4: Time series with additive Gaussian noise.

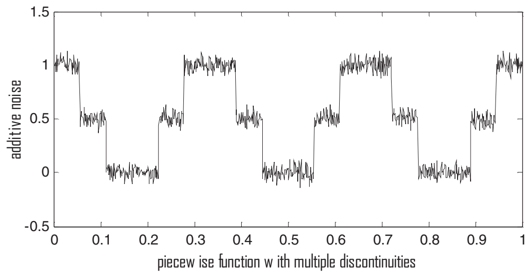

Figure 5: Noise estimation for functions with multiple discontinuities.

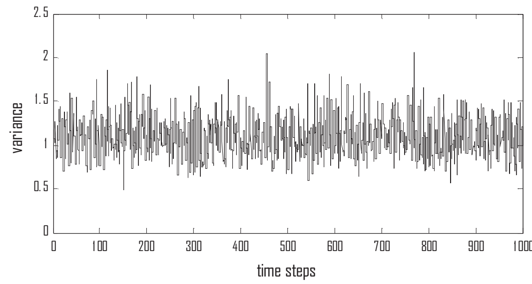

Figure 6: Estimation of noise variance.

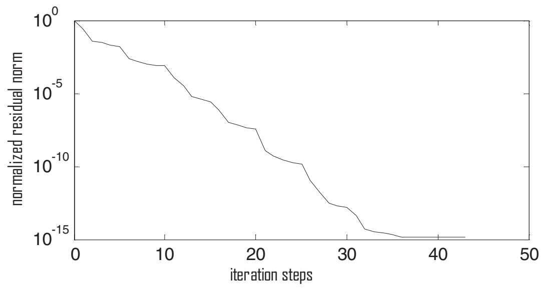

Figure 7: Stopping criterion: maximal number of iteration steps.

Similarly, the measured noise figure at the filter output node (measured across 25 prototype samples) is within 5% of the simulated noise figure obtained as average noise power calculated over the periodic noise variance waveform. The Bartels-Stewart algorithm and Hammarling's method carried out explicitly (as done in Matlab) can exploit the advantages provided by modern high performance computer hardware, which contains several levels of cache memories. For the recursive algorithms presented here it is observed that a faster lowest level kernel solver (with suitable block size) leads to an efficient solver of triangular matrix equations. For models with large dimension N_c and N_v, usually the matrix $\mathbf{P}_r$ has a banded or a sparse structure and applying the Bartels-Stewart type algorithm becomes impractical due to the Schur decompositions (or Hessenberg-Schur), which cost expensive $O(N^3)$ flops. In comparison with the standard Matlab function *lyap.m*, the *cpu*-time shows that computing the Cholesky factor directly is faster by approximately N flops. Similarly, when the original matrix equation is real, using real arithmetic is faster than using complex arithmetic. Hence, we resort to iterative projection methods when N_c and N_v are large. The approximate solution of the Lyapunov equation is given by the low rank Cholesky factor L, for which $LL^H \sim K$. L has typically fewer columns than rows. In general, L can be a complex matrix, but the product LL^H is real. More precisely, the complex low rank Cholesky factor delivered by the iteration is transformed into a real low rank Cholesky factor of the same size, such that both low rank Cholesky factor products are identical. However, doing this requires additional computation. The iteration is stopped after a priori defined iteration steps (Figure 7).

IV. Conclusions

Statistical simulation is one of the foremost steps in the evaluation of successful high-performance IC designs due to circuit noise that strongly affect devices behavior in today's deep submicron technologies. As circuit noise is non-stationary process, Ito stochastic differentials are introduced and effective solution for Lyapunov equation found. The effectiveness of the proposed approaches was evaluated on several circuits with the continuous-time biquad filter as a representative example. As the results indicate, the suggested numerical method provides accurate and efficient solutions of stochastic differentials for noise analysis.

V. References

[1] P.R. Gray, R.G. Meyer, *Analysis and Design of Analog Integrated Circuits,* Wiley, 1984

[2] J. Roychowdhury, "Reduced-Order Modeling of Time-Varying Systems", *IEEE Transactions on Circuits and Systems-II: Analog and Digital Signal Processing*, vol. 46, no. 10, pp. 1273-1288, 1999

[3] V. Vasudevan, "A Time-Domain Technique for Computation of Noise-Spectral Density in Linear and Nonlinear Time-Varying Circuits", *IEEE Transactions on Circuits and Systems-I: Regular Papers*, vol. 51, no. 2, pp. 422-433, 2004

[4] A. Demir, E. Liu, A. Sangiovanni-Vincentelli, "Time-Domain Non-Monte Carlo Noise Simulation for Nonlinear Dynamic Circuits with Arbitrary Excitations", *Proceedings of IEEE/ACM International Conference on Computer Aided Design,* pp. 598-603, 1994

[5] J. Vlach, K. Singhal, *Computer Methods for Circuit Analysis and Design*, New York: Van Nostrand Reinhold, 1983

[6] L.O. Chua, C.A. Desoer, E.S. Kuh, *Linear and Nonlinear Circuits*, New York: Mc Graw-Hill, 1987

[7] L. Arnold, *Stochastic Differential Equations: Theory and Application*, New York: Wiley, 1974

[8] A. Sangiovanni-Vincentelli, "Circuit Simulation" in *Computer Design Aids for VLSI Circuits*, Sijthoff and Noordhoff, The Netherlands, 1980

[9] P. Heydari, M. Pedram, "Model-Order Reduction Using Variational Balanced Truncation With Spectral Shaping", *IEEE Transaction on Circuits and Systems-I: Regular Papers*, vol. 53, no.4, pp. 879-891, 2006

[10] K.H. Lim, K.P. Seng, L.-M. Ang, S.W. Chin, "Lyapunov Theory-Based Multilayered Neural Network", *IEEE Transactions on Circuits and Systems-II: Express Briefs*, vol. 56, no. 4, pp. 305-309, 2009

[11] R.H. Bartels, G.W. Stewart, "Solution of the Matrix Equation AX+XB=C", *Communication of the Association of Computer Machinery*, vol. 15, pp. 820-826, 1972

[12] P. Benner, E. Quintana-Orti, "Solving Stable Generalized Lyapunov Equations with the Matrix Sign Function", *Numerical Algebra*, vol. 20, pp. 75-100, 1999

[13] I. Jaimoukha, E. Kasenally, "Krylov Subspace Methods for Solving Large Lyapunov Equations", *SIAM Journal of Numerical Analysis*, vol. 31, pp. 227-251, 1994

[14] E. Wachspress, "Iterative Solution of the Lyapunov Matrix Equation", *Application Mathematical Letters*, vol. 1, pp. 87-90, 1998

[15] J. Li, F. Wang, J. White, "An Efficient Lyapunov Equation-Based Approach for Generating Reduced-Order Models of Interconnect", *Proceedings. of IEEE Design Automation Conference*, pp. 1-6, 1999