

Surface Integrated Field Equations Method to Solve 3D Electromagnetic Problems

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Abstract This paper describes how the Surface Integrated Field Equations method (SIFE) is implemented to solve 3D Electromagnetic (EM) problems on substrates in which high contrast materials occur. It gives an account of the promising results that are obtained with it when compared to traditional approaches. Advantages of the method are the highly improved flexibility and accuracy for a given discretization level, at the cost of higher computational complexity.

1 Introduction

In our previous work, we have used the Surface Integrated Equations(SIFE) method for solving 2D electromagnetic problems[1], in which domains are present that exhibit highly contrasting material properties (electric and/or magnetic) with each other. In this paper, we develop the method further to solve 3D electromagnetic problems. Limitation of space prohibits us from giving a detailed description of the SIFE method and its spatial and temporal discretization schemes. For more details, we refer the readers to our previous papers [1, 2], and a full paper documenting the underlying theory will be published soon.

In strongly heterogeneous media such as modern chips, the constitutive parameters can jump by large amounts upon crossing the material interfaces. On a global scale, the EM field components are not differentiable and Maxwell's equations in differential form cannot be used, one has to resort to the original integral form of the EM field relations as the basis for the computational method.

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Let \mathcal{D} be the domain of interest with boundary $\partial\mathcal{D}$, S be any (sufficiently smooth and small) surface ($S \in \mathcal{D}$) with boundary ∂S . For any S , Maxwell's equations in integrated form are:

$$-\oint_{\partial S} \mathbf{H} \cdot d\mathbf{l} + \partial_t \int_S \mathbf{D} \cdot d\mathbf{A} = -\int_S \mathbf{J}^{\text{tot}} \cdot d\mathbf{A}, \quad (1)$$

$$\oint_{\partial S} \mathbf{E} \cdot d\mathbf{l} + \partial_t \int_S \mathbf{B} \cdot d\mathbf{A} = 0, \quad (2)$$

where \mathbf{E} is the electric field strength, \mathbf{H} the magnetic field strength, \mathbf{D} the electric flux density, and \mathbf{B} the magnetic flux density. Moreover, $\mathbf{J}^{\text{tot}} = \mathbf{J} + \mathbf{J}^{\text{ext}}$, where \mathbf{J} is the induced (field dependent) electric-current density, and \mathbf{J}^{ext} is the external electric-current densities. In addition to Maxwell's equations, the compatibility equations have to be satisfied as well. In integrated form, these equations are

$$\oint_{S'} (\partial_t \mathbf{D} + \mathbf{J}^{\text{tot}}) \cdot d\mathbf{A} = 0 \quad \text{and} \quad \partial_t \oint_{S'} \mathbf{B} \cdot d\mathbf{A} = 0,$$

where this time S' is a smooth and closed surface. We also have to describe the type of matter that we are dealing with. The constitutive relations provide us with such a description and for the materials that we consider these relations are:

$$\mathbf{J} = \sigma \mathbf{E}, \quad \mathbf{D} = \varepsilon \mathbf{E}, \quad \text{and} \quad \mathbf{B} = \mu \mathbf{H}, \quad (3)$$

where σ is the conductivity, ε the permittivity, and μ the permeability. These three material parameters are all position dependent and are piecewise continuous in general. We are mostly concerned with media for which the medium parameters are piecewise constant, however, and at source-free interfaces where the parameters exhibit a jump, the tangential components of the electric and magnetic field strength have to be continuous, while the normal components of the electric and magnetic field strength are discontinuous because of the contrast.

2 Discretization Scheme

To satisfy the partial continuity conditions on material interfaces, we construct a so called "consistent linear discretization scheme" [3–5] that meets the continuity requirements across interfaces exactly, using a tetrahedron mesh combined with a consistent linear interpolation of electric and magnetic field strengths.

In this section we briefly present our discretization scheme. In all the experiments that we shall present, we use a nonuniform tetrahedron mesh generated by netgen [6] or msh [7]. For good results it is necessary that the tetrahedrons are "well formed", i.e. that they are not too skewed or too flat in one or more directions so that a vector decomposition along the edges yields a numerically accurate representation. We assume that the material parameters in each tetrahedron are constant (actually taking average values). This is consistent with the fact that a piecewise

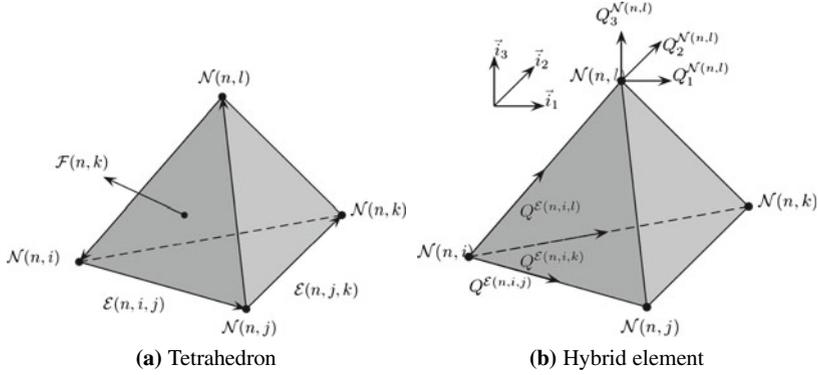


Fig. 1: **a** Tetrahedron \mathcal{T}_n and some geometrical quantities defined on it. (i, j, k, l) is an even permutation of $(0, 1, 2, 3)$. **b** The coefficients of the linear, hybrid expansion functions on \mathcal{T}_n

constant approximation for material parameters is sufficient to ensure continuity of the solutions. A more refined approximation such as continuous linear splines is certainly possible.

2.1 Geometrical Quantities

Before introducing the linear expansion functions, we define a few geometrical quantities as shown in Fig. 1a.

- The coordinate vector is $\mathbf{x} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3$.
- A node with global node index n is denoted as \mathcal{N}_n , and \mathbf{x}_n is its coordinate vector.
- A tetrahedron with global tetrahedron index n is denoted as \mathcal{T}_n .
- The four nodes delimiting \mathcal{T}_n are locally denoted as $\mathcal{N}(n, i), i = \{0, 1, 2, 3\}$.
- For every node with local label $\mathcal{N}(n, i)$, a unique global node index m can be found, and: $\mathcal{N}(n, i) = \mathcal{N}_m$, $\mathbf{x}(n, i)$ denotes its coordinate vector.
- $\mathbf{x}_n^b = \frac{1}{4} \sum_{i=\{0,1,2,3\}} \mathbf{x}(n, i)$ is the coordinate vector of the barycentre of \mathcal{T}_n .
- Let ε be an arbitrary small, positive real number, $\bar{\mathbf{x}}(n, i) = \mathbf{x}(n, i) + \varepsilon [\mathbf{x}_n^b - \mathbf{x}(n, i)]$.
- $\mathcal{E}(n, i, j); j \neq i$ denotes the edge pointing from $\mathcal{N}(n, i)$ to $\mathcal{N}(n, j)$.
- Let $\mathbf{e}(n, i, j)$ be the vectorial length of $\mathcal{E}(n, i, j)$: $\mathbf{e}(n, i, j) = \mathbf{x}(n, j) - \mathbf{x}(n, i)$.
- Let $\mathcal{F}(n, k)$ be the facet of \mathcal{T}_n , which is not delimited by $\mathcal{N}(n, k)$.
- Let $\mathbf{A}(n, k)$ be the vectorial area of $\mathcal{F}(n, k)$, e.g $\mathbf{A}(n, 0) = \frac{1}{2} [\mathbf{e}(n, 1, 2) \times \mathbf{e}(n, 2, 3)]$.
- Let $V(n)$ be the volume of \mathcal{T}_n : $V(n) = \frac{1}{3} [\mathbf{x}(n, 1) - \mathbf{x}(n, 0)] \cdot \mathbf{A}(n, 0)$.
- Let $\phi_i(\mathcal{T}_n, \mathbf{x})$ be the local linear scalar interpolation function,

$$\phi_i(\mathcal{T}_n, \mathbf{x}) = 1/4 - (\mathbf{x} - \mathbf{x}_n^b) \cdot \frac{\mathbf{A}(n, i)}{3V(n)}, \forall \mathbf{x} \in \mathcal{T}_n,$$

2.2 Spatial Discretization Scheme

Let $\mathbf{Q}(\mathbf{x})$ be a vectorial function of space representing electric field strength or magnetic field strength at a time instance t ; that is: $\mathbf{Q}(\mathbf{x})$ represents $\mathbf{E}(\mathbf{x}, t)$ or $\mathbf{H}(\mathbf{x}, t)$. Its tangential component is continuous across the interface while its normal component is discontinuous. We can represent the value of $\mathbf{Q}(\mathbf{x})$ on the nodes delimiting a tetrahedron \mathcal{T}_n with the well defined components. The value of $\mathbf{Q}(\mathbf{x})$ inside \mathcal{T}_n can be interpolated with linear, hybrid expansion functions which are built upon *continuity nodes* for nodes inside domains with uniform material parameters and *discontinuity nodes* on the interfaces.

2.2.1 Continuity Node

Let $\mathcal{N}(n, i)$ be a node where all components of $\mathbf{Q}(\mathbf{x})$ on this node are continuous, $\mathbb{N}_{\mathbf{Q}}^C$ be the set of nodes in the mesh where $\mathbf{Q}(\mathbf{x})$ is totally continuous. The value of $\mathbf{Q}(\mathbf{x})$ on node $\mathcal{N}(n, i)$ can be represented as:

$$\mathbf{Q}^{\mathcal{N}(n, i)} = \sum_{j=1,2,3} Q_j^{\mathcal{N}(n, i)} \mathbf{i}_j, \forall \mathcal{N}(n, i) \in \mathbb{N}_{\mathbf{Q}}^C \quad (4)$$

where $\mathbf{Q}^{\mathcal{N}(n, i)}$ denotes the value of $\mathbf{Q}(\mathbf{x})$ on the node $\mathcal{N}(n, i)$.

2.2.2 Discontinuity Node

Let $\mathcal{N}(n, i)$ be a node on the interface of material discontinuity, $\mathbb{N}_{\mathbf{Q}}^D$ be the set of nodes in the mesh where $\mathbf{Q}(\mathbf{x})$ is discontinuous in its normal component. The normal component of $\mathbf{Q}(\mathbf{x})$ is not well defined for the nodes on the interfaces. However, if we offset the node into the tetrahedron \mathcal{T}_n , then the value of $\mathbf{Q}(\mathbf{x})$ on that node can be represented as:

$$\mathbf{Q}^{\mathcal{N}(n, i)} = \sum_{j=\{0,1,2,3\}, j \neq i} -Q^{\mathcal{E}(n, i, j)} \frac{|\mathbf{e}(n, i, j)|}{3V(n)} \mathbf{A}(n, j), \forall \mathcal{N}(n, i) \in \mathbb{N}_{\mathbf{Q}}^D. \quad (5)$$

$Q^{\mathcal{E}(n, i, j)}$ is the projection of $\mathbf{Q}(\mathbf{x})$ on the node $\mathcal{N}(n, i)$ to the direction $\mathbf{e}(n, i, j)$.

2.2.3 Linear, Hybrid Expansion

Let $[\mathbf{Q}](\mathcal{T}_n, \mathbf{x})$ be the local linear approximation of $\mathbf{Q}(\mathbf{x})$ in the tetrahedron \mathcal{T}_n :

$$[\mathbf{Q}](\mathcal{T}_n, \mathbf{x}) = \sum_{i \in \{0,1,2,3\}} \phi_i(\mathcal{T}_n, \mathbf{x}) \mathbf{Q}^{\mathcal{N}(n, i)}, \forall \mathbf{x} \in \mathcal{T}_n \quad (6)$$

in which $\mathbf{Q}^{\mathcal{N}(n,i)}$ is defined by Eq.4 or Eq.5, $Q_j^{\mathcal{N}(n,i)}$ and $Q^{\mathcal{E}(n,i,j)}$ are the linear, hybrid expansion coefficients as shown in Fig. 1b. A list of properties of the linear, hybrid interpolation functions follows:

- The linear, hybrid expansion functions are consistently linear functions [1], they permit a completely linear expansion of the partially continuous vectorial function $\mathbf{Q}(\mathbf{x})$ inside each tetrahedron. the approximation errors of the linear, hybrid expansion functions are of order $O(h^2)$.
- Assuming the *discontinuity nodes* are assigned in the right place, the linear, hybrid expansions functions ensure continuity in tangential components across material interfaces, and ensure total continuity in homogeneous sub-domains .
- With the linear, hybrid expansion functions, it is easy to apply the boundary conditions that prescribe tangential components.

We refer to the elements with linear, hybrid expansion functions as “hybrid elements”, and refer to the elements with only *continuity nodes* as “nodal elements”.

2.2.4 Field Strength Discretization

With all these properties above, the linear, hybrid expansion functions are a very good choice for interpolating electric field strength and magnetic field strength. Note that an interface can be with electric and/or magnetic contrast. The set of *discontinuity nodes* for magnetic field strength $\mathbb{N}_{\mathbf{H}}^D$ does not have to be the same as that for the electric field strength $\mathbb{N}_{\mathbf{E}}^D$. With the graphic user interface we implemented, it is very easy to assign the *discontinuity nodes*.

2.3 Temporal Discretization Scheme

To implement a time stepping scheme for the spatially discretized Maxwell’s equations, let t_0 be the initial time and $\Delta t > 0$ be the time step size, we introduce the time instance $t_n = n\Delta t + t_0$ and integrate Maxwell’s equations from $t = t_{n-1}$ to $t = t_n$. All integrals that can not be computed analytically are approximated using the trapezoidal rule, which is known to be unconditionally stable in time. The approximation of the trapezoidal rule is of order $O(\Delta t^2)$, which is verified in Section 4.

3 The Surface Integrated Field Equations Method

In this section, we briefly present the Surface Integrated Field Equations method for solving electromagnetic problems in the time domain. We apply the Ampere’s equation (Eq.1) and constitutive relations (Eq. 3) on every facet of every element product a time interval, i.e. $\mathcal{F}(n,i) \times [t_{m-1}, t_m]; \Delta t = t_m - t_{m-1}$, and use the linear

spatial and temporal approximation presented in Sec. 2.2 and Sec. 2.3 to discretize the electromagnetic field strengths, we obtain discretized Ampère equations:

$$\begin{aligned} & \frac{\Delta t}{4} \left[\mathbf{e}(n, l, k) \cdot \mathbf{H}^{\mathcal{N}(n, j)}(t_m) + \mathbf{e}(n, j, l) \cdot \mathbf{H}^{\mathcal{N}(n, k)}(t_m) + \mathbf{e}(n, k, j) \cdot \mathbf{H}^{\mathcal{N}(n, l)}(t_m) \right] \\ & - \sum_{h=j, k, l} \left[\frac{\Delta t}{6} \sigma(\bar{\mathbf{x}}(n, h)) + \frac{1}{3} \varepsilon(\bar{\mathbf{x}}(n, h)) \right] \mathbf{A}(n, i) \cdot \mathbf{E}^{\mathcal{N}(n, h)}(t_m) = - \\ & \frac{\Delta t}{4} \left[\mathbf{e}(n, l, k) \cdot \mathbf{H}^{\mathcal{N}(n, j)}(t_{m-1}) + \mathbf{e}(n, j, l) \cdot \mathbf{H}^{\mathcal{N}(n, k)}(t_{m-1}) + \mathbf{e}(n, k, j) \cdot \mathbf{H}^{\mathcal{N}(n, l)}(t_{m-1}) \right] \\ & + \sum_{h=j, k, l} \left[\frac{\Delta t}{6} \sigma(\bar{\mathbf{x}}(n, h)) - \frac{1}{3} \varepsilon(\bar{\mathbf{x}}(n, h)) \right] \mathbf{A}(n, i) \cdot \mathbf{E}^{\mathcal{N}(n, h)}(t_{m-1}) \\ & + \sum_{h=j, k, l} \frac{1}{3} \int_{t=t_{m-1}}^{t_m} \left[\mathbf{A}(n, i) \cdot \mathbf{J}^{\text{ext}}(\bar{\mathbf{x}}(n, h), t) \right] dt \end{aligned}$$

for any \mathcal{T}_n , and $\{i, j, k, l\}$ an even permutation of $\{0, 1, 2, 3\}$. With similar procedure applied on the surface-time integrated Faraday's equation, we obtain the discretized Faraday's equation. Note that discontinuous field quantities are not well defined for the nodes on the interfaces. Therefore, we take values pertaining to a vanishing offset towards the barycentre of \mathcal{T}_n , where all field quantities are well defined.

With appropriate boundary conditions, we shall have an over-determined system of linear discrete surface-time integrated field equations. Such a system may have no solution at all. The best thing we can do is to find an approximate solution which minimizes a quadratic functional. With the weighted least-squares method[8], we can easily construct normal equations, which we then solve iteratively. After solving the system for the coefficients, we get the approximated electromagnetic field strength in the domain of computation (we use a CG iterative solver with preconditioner).

4 Numerical Experiment

We verify the accuracy of the temporal discretization scheme and spatial discretization scheme using a 3D Electromagnetic time domain problem with high contrast in permeability for which an analytic solution is known. The theoretical solution is a 'steady state' solution at a single frequency, containing a source term that continuously injects current. We use the steady solution at $t = 0$ as initial state, and start integrating from there in the time domain. The computational domain $\mathcal{D} = \{0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, 0 \leq x_3 \leq 1\}$ is bounded by PEC boundaries. Let

$$h(\mathbf{x}, t) = \frac{\sin(\omega t)}{\mu(\mathbf{x})\omega} \quad \text{and} \quad g(\mathbf{x}, t) = \sigma(\mathbf{x}) \cos(\omega t) - \varepsilon(\mathbf{x})\omega \sin(\omega t)$$

The source density distributions be:

Table 1: Configuration of the four sub-domains

\mathcal{D}_i	Definition of sub-domains	ϵ_r	σ	μ_r
\mathcal{D}_0	$0 \leq x_1 < 0.5, 0 \leq x_2 < 0.5, 0 \leq x_3 \leq 1$	1	0	1000
\mathcal{D}_1	$0.5 \leq x_1 < 1, 0 \leq x_2 < 0.5, 0 \leq x_3 \leq 1$	1	0	1
\mathcal{D}_2	$0 \leq x_1 < 0.5, 0.5 \leq x_2 < 1, 0 \leq x_3 \leq 1$	1	0	1
\mathcal{D}_3	$0.5 \leq x_1 < 1, 0.5 \leq x_2 < 1, 0 \leq x_3 \leq 1$	1	0	10

$$\mathbf{J}^{\text{ext}}(\mathbf{x}, t) = [-2\pi^2 h(\mathbf{x}, t) - g(\mathbf{x}, t)] \sin(\pi x_1) \sin(\pi x_2) \mathbf{i}_3, .$$

The exact field strengths are:

$$\mathbf{E}(\mathbf{x}, t) = \sin(\pi x_1) \sin(\pi x_2) \cos(\omega t) \mathbf{i}_3,$$

$$\mathbf{H}(\mathbf{x}, t) = -\pi h(\mathbf{x}, t) \sin(\pi x_1) \cos(\pi x_2) \mathbf{i}_1 + \pi h(\mathbf{x}, t) \cos(\pi x_1) \sin(\pi x_2) \mathbf{i}_2.$$

The angular frequency ω is chosen to be $2\pi 10^9$ rad/s corresponding to a source frequency of 1GHZ. The configuration will be computed for 10 wave periods ($0 \leq t \leq 10^{-8}$ s). The whole domain is divided into four homogeneous sub-domains defined in Tab. 1. We compute this configuration with the SIFE method based on hybrid elements and the weighted ($w = 2 \times 10^{-3}$) Galerkin method based on nodal elements (see [9]). The computational domain is discretized with an interface conforming tetrahedron mesh (5853 nodes and 30208 tetrahedrons). Series of experiments have been done with different time step sizes. Note that, the contrast only exists for the magnetic field strength. Therefore, *discontinuity nodes* are used only when interpolating magnetic field strength on the material interfaces; since the electric field strength is totally continuous. *Discontinuity nodes* are not used for interpolating the electric field strength. As shown in Fig.2, the SIFE method based on hybrid elements has second order accuracy in time even in the presence of high contrast. The accuracy plots show that the SIFE method produces the correct result while the standard packages do not.

5 Conclusions

The SIFE method holds considerable promise to solve three dimensional time domain electromagnetic problems, in which high contrasts between different types of materials exist and irregular structures are present. Its accuracy and stability has now been demonstrated and verified with numerical experiments. With these basic properties being established, we are now working on improvements of the computational properties of the method in terms of numerical complexity and versatility.

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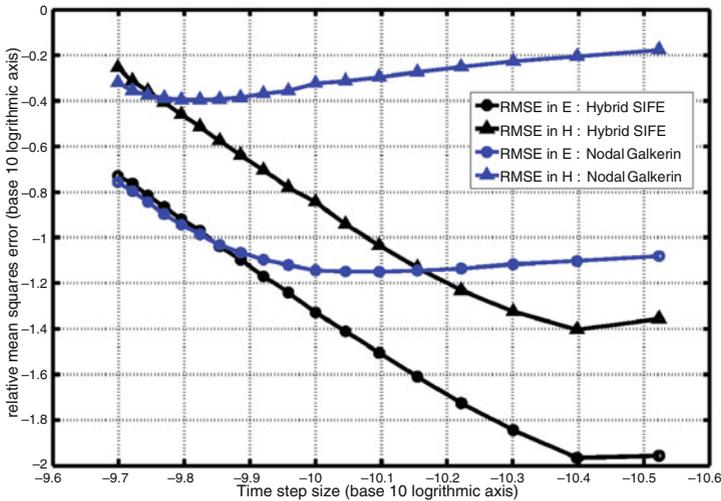


Fig. 2: BICG-stable + nest-dissection reordering + ICC(4) is used for the SIFE method, BICG-stable + nest-dissection reordering + ILU(4) is used for the weighted Galerkin's method. The accuracy of these iterative solvers is set to be 10^{-20}

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References

1. Sheng, Z., Remis, R.F., Dewilde, P.: A least-squares implementation of the field integrated method to solve time domain electromagnetic problems. *Computational Electromagnetics in Time-Domain, 2007. CEM-TD 2007. Workshop* on pp. 1–4 (15-17 Oct. 2007). DOI 10.1109/CEMTD.2007.4373514
2. Sheng, Z., Dewilde, P.M., Remis, R.F.: The field integrated method to solve time domain electromagnetic problems. In: *IEEE/ProRISC workshop on Circuits, Systems and Signal Processing*, Veldhoven (NL), IEEE, (November 2007)
3. Lager, I.E.: Finite element modelling of static and stationary electric and magnetic fields. PhD thesis, Delft University of Technology (1996)
4. Mur, G.: The finite-element modeling of three-dimensional electromagnetic fields using edge and nodal elements. *IEEE transactions on antennas and propagation* **41**(7) (1993)
5. Jorna, P.: Integrated field equations methods for the computation of electromagnetic fields in strongly inhomogeneous media. PHD thesis, TUDelft (2005)
6. Schoberl, J.: Netgen - an advancing front 2d/3d-mesh generator based on abstract rules. *Comput. Visual. Sci* **1**, 41–52 (1997)
7. van der Kolk, K., van der Meijs, N.: On the implementation of a 3-dimensional delaunay-based mesh generator. In: G. Ciuprina; D. Ioan (Ed.), *SCEE 2006 Book of Abstracts*, Sinaia, RO, pp. 171-172, 2006. ISBN: 978-973-718-520-4. (2006)
8. Jiang, B.N.: *The Least-Squares Finite Element Method: Theory and Applications in Computational Fluid Dynamics and Electromagnetics* (Scientific Computation). Springer (2006)
9. Sitapati, K.: Mixed-field finite element computations. PhD thesis, Virginia Polytechnic Institute and State University (2004)