Direction of Arrival Estimation Using Sparse Ruler Array Design

Siavash Shakeri *, Dyonisius Dony Ariananda†, Geert Leus †
Faculty of EEMCS, Delft University of Technology,
2628CD, Delft, The Netherlands
E-mail : *s.shakeri@student.tudelft.nl, †{d.a.dyonisius, g.j.t.leus} @tudelft.nl

***Abstract*—In this paper, a new direction of arrival (DOA) estimation approach is addressed for the case of more sources than physical receiving antennas by considering a novel non-uniform array design. The new design utilizes the concept of minimum sparse rulers which are rulers having incomplete marks. The differences between marks in a sparse ruler cover all lags of the autocorrelation. In array processing, this set of differences can be used as a basis to construct a virtual uniform linear array having a higher number of antennas than the actual linear array. In order to attain the required rank condition of the observation matrix, the most recent spatial smoothing method is used. The MUSIC algorithm can then be applied leading to the desired high resolution result. It is also possible to compromise the resolution for a lower complexity level by exploiting the least-squares approach to generate the angular spectrum.**

***Index Terms*—direction of arrival (DOA) estimation, non-uniform array, sparse rulers, MUSIC algorithm, spatial smoothing**

I. INTRODUCTION

In recent years, high resolution direction of arrival estimation is one of the main topics in the research field of antenna arrays. Many methods have been proposed for the case of uniform linear arrays (ULAs). It is generally impossible to locate the DOAs of all sources if the total number of sources is larger than the number of receiving antennas. For instance, a ULA having N receiving antennas can only detect up to $N-1$ sources.

This issue has spurred researchers to construct non-uniform arrays for DOA estimation when we have more sources than actual physical antennas. One of these approaches is [1] which uses the Khatri-Rao subspace based algorithm for the case of quasi-stationary signals. In [2], the authors use an array structure called nested array and also propose a spatial smoothing method to generate a positive semi-definite spatial covariance matrix. In contrast to [1], this method does not require quasi-stationarity of the received signals. In [3], a new array design is proposed based on the coprime sampling scheme, which employs two ULAs with different antenna spacings. By using the correlation sequence between the resulting samples, a steering vector matrix for a longer ULA can be generated. As in [2], the spatial smoothing is also employed in [3] to generate a spatial covariance matrix having a suitable rank and this covariance matrix is later used for DOA estimation based on the MUSIC algorithm.

The idea of using numerical theories for designing non-uniform linear arrays was first utilized in minimum redundancy arrays which was introduced by Moffet in [4]. Consider an $N-$element uniform linear array in which the antennas are positioned at $\{d_i\}_{i=1}^{N}$. The co-array set of this linear array can then be defined by all possible pairwise antenna separations. Therefore, it can be defined by $D = \{d_i - d_j\}_{i,j=1}^{N}$. Each number in the co-array set corresponds to the spatial correlation lag between an antenna pair with the same antenna separation value. Therefore, the members of the set D can be interpreted as the lags of the spatial correlation matrix. For an $N-$element ULA, the set D has $N(N-1)/2$ possible pairs of separation. However, there are some repeated values (lags) in this set. Therefore, it is said that the co-array has some redundancies and the array is called a redundancy array. By reducing these redundant values from the co-array, it is possible to increase the array aperture and also get a higher spatial resolution as shown in [4]. This goal can be achieved by designing a non-uniform linear array with specific inter-element separation. Zero redundancy arrays can only be achieved for arrays with less than or equal to 4 elements. For more than 4 elements, the goal is to reduce the redundancies to the lowest possible value. If there is any missing lag in the co-array set, then that missing lag is considered as a hole or gap [5][6]. Minimum redundancy arrays (MRAs) are arrays with the lowest possible redundancy and without any holes in their co-array set and therefore they have the highest possible aperture. Minimum hole arrays (MHAs) are arrays with a minimum number of holes and zero redundancy. By using difference bases as mentioned in number theory, MRAs and MHAs can consequently be designed by restricted difference bases and Golomb rulers, which are related to the ruler framework [5]. Non-uniform linear arrays which are designed according to the concept of Golomb rulers are called Golomb arrays [6][7]. In these arrays, the antennas are positioned according to the marks of the Golomb ruler. In [8], a non-uniform array design is proposed for active sensing based on Singer's theorem. Singer's theorem basically allows us to form large aperture arrays for the case of MIMO radars with few receiving and transmitting antennas.

In this paper, we introduce a special antenna array design based on the sparse ruler concept for the case of passive sensing. A sparse ruler is a ruler that is able to measure all integer distances from 0 to $N-1$ with only $M < N$ marks. This ruler with M marks is considered as a minimal sparse ruler if there is no ruler with $M-1$ marks that is also able to measure all integer distances from 0 to $N-1$. With the minimal sparse ruler, it is possible to generate the same set of differences mentioned in [3] but with fewer number of antennas. The structure of the paper is given as follows. First, some preliminary information related to the formulation of uniform linear arrays and the sparse ruler concept is presented in Section II. In Section III, the antenna array design is proposed and the spatial autocorrelation of this array is constructed to demonstrate that a virtual uniform linear array can be achieved with this configuration. In Section IV, the spatial smoothing method proposed in [2] is used to construct a spatial covariance matrix for a ULA with a higher number of antennas which is produced by the difference set derived in Section III. The derived covariance matrix is then used as an input for the MUSIC algorithm in order to estimate the DOAs of the impinging sources. In Section V, we use the same minimal sparse ruler based antenna array design to generate the so-called angular spectrum by using least-squares. This least-squares approach offers less complexity though the resulting angular spectrum might not have sufficient resolution to separate two signals having extremely small differences in their angle of arrival.

II. PRELIMINARIES

A. Antenna Array Model

Consider a uniform linear array (ULA) consisting of N antennas/sensors for receiving the impinging signal of K narrowband planar wavefront sources. The delay along the array can be defined as a phase shift due to the narrowband assumption. In addition, the separation between the antennas should be less than half of the wavelength of the impinging signal to avoid spatial aliasing. Therefore, the array output can be written as:

$$\mathbf{x}(t) = \sum_{k=1}^{K} \mathbf{a}(\theta_k) s_k(t) + \mathbf{n}(t) \tag{1}$$

where $t = 1, ..., T$ is the time index, $\mathbf{x}(t)$ is the $N \times 1$ received vector, $s_k(t)$ is the signal from direction θ_k, $\mathbf{n}(t)$ is the $N \times 1$ additive noise vector, and $\mathbf{a}(\theta_k)$ is the $N \times 1$ steering vector whose elements are the phase shifts experienced by the received signal at the elements of the array. If the first element of the array is considered as a reference point, the steering vector can be written as:

$$\mathbf{a}(\theta_i) = [1, \phi_i, \dots, \phi_i^{N-1}]^T \tag{2}$$

where

$$\phi_i = \exp(j 2\pi \frac{d}{\lambda} \sin(\theta_i)) \tag{3}$$

with d the distance between the antennas. The output equation (1) can also be written as

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \tag{4}$$

where $\mathbf{A} = [\mathbf{a}(\theta_1)\ , \mathbf{a}(\theta_2)\ , \dots\ , \mathbf{a}(\theta_K)]$ is the $N \times K$ matrix containing the steering vectors and $\mathbf{s}(t) = [s_1(t), s_2(t), ..., s_K(t)]^T$ is the $K \times 1$ source signal. The spatial correlation can be defined as $\mathbf{R}_x = E[\mathbf{x}(t)\mathbf{x}^H(t)]$, where $E[.]$ is the expectation operator, which can be estimated by taking an average over T time instances (total number of snapshots).

B. Minimal Length Sparse Rulers

In general, a length-$(N-1)$ ruler contains all the N integer marks starting from 0 to $N-1$. However, it is actually possible to measure all integer distances from 0 to $N-1$ without having the complete ruler. A length-$(N-1)$ sparse ruler is a ruler that is able to measure all integer distances between 0 and $N-1$ with only M marks $\{d_n\}_{n=0}^{M-1}$ where $M < N$ and

$$0 = d_0 < d_1 < ... < d_{M-1} = N-1.$$

In other words, all the differences measured by the marks in a length-$(N-1)$ sparse ruler can be used to generate all integers between 0 and $N-1$. The length-$(N-1)$ sparse ruler with M marks is considered a minimum sparse ruler if there is no length-$(N-1)$ sparse ruler having $M-1$ marks. For further information about the minimal sparse ruler problem and how to solve it, see [9].

III. SPARSE RULER ARRAY APPROACH

In this section, we define a non-uniform linear array by considering a length-$(N-1)$ sparse ruler with marks $S = \{d_n\}_{n=0}^{M-1}$. The cardinality of the set S defines the number of antennas and the value of the marks illustrates the distance between every antenna element and the reference antenna element as shown in Fig. 1.

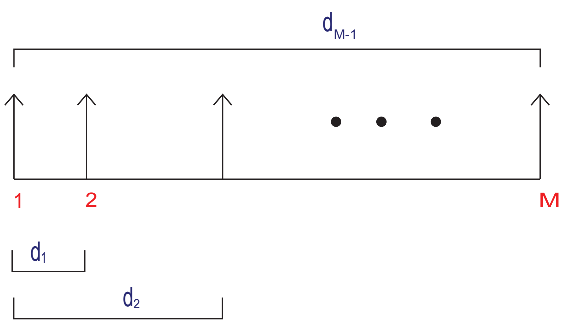

Fig. 1. Non-uniform linear array with the antenna position arranged according to the marks of the sparse ruler.

Let us assume that the narrowband source impinging on the antenna array from direction θ_i has power σ_i^2 where $i = 1, 2, \dots, K$. The output of the array will be similar to (4), namely:

$$\mathbf{x}_s(t) = \mathbf{A}_s \mathbf{s}(t) + \mathbf{n}_s(t) \tag{5}$$

where the steering vector matrix is given by $\mathbf{A}_s = [\mathbf{a}_s(\theta_1), \mathbf{a}_s(\theta_2), \dots, \mathbf{a}_s(\theta_K)]$. The steering vector for the i-th source can be written as

$$\mathbf{a}_s(\theta_i) = [1, {\psi_i}^{d_1}, {\psi_i}^{d_2}, \dots, {\psi_i}^{d_{M-1}}]^T \tag{6}$$

where

$$\psi_i = \exp(\, j\frac{2\pi}{\lambda}\sin(\theta_i)\,).$$

The spatial correlation matrix for this sparse ruler can be written as

$$\mathbf{R}_{x_s} = E\{\mathbf{x}_s(t)\mathbf{x}_s^H(t)\}$$

$$= E\{\sum_{i=1}^{K}\sum_{j=1}^{K} \mathbf{a}_s(\theta_i)s_i(t)s_j^*(t)\mathbf{a}_s^H(\theta_j) + \tag{7}$$

$$\sum_{i=1}^{K}\mathbf{n}_s(t)s_i^*(t)\mathbf{a}_s^H(\theta_i) + \sum_{i=1}^{K}\mathbf{a}_s(\theta_i)s_i(t)\mathbf{n}_s^H(t) +$$

$$+\mathbf{n}_s(t)\mathbf{n}_s^H(t)\}.$$

In this equation, the expectation operator can be estimated by taking an average over the total number of time snapshots. Let us assume that the sources are uncorrelated from each other and that the i-th source has variance σ_i^2. Let us also assume that the noises on the different antennas are mutually uncorrelated with variance σ_n^2. In this case, the spatial correlation matrix can be written as:

$$\mathbf{R}_{x_s} = \sum_{i=1}^{K} \mathbf{a}_s(\theta_i)\mathbf{a}_s^H(\theta_i)\sigma_i^2 + \sigma_n^2\mathbf{I} \tag{8}$$

where $\mathbf{I}$ is an $M \times M$ identity matrix. Similar to [1] and [2], we vectorize $\mathbf{R}_{x_s}$ to get the following equation

$$\mathbf{y} = \mathrm{vec}(\mathbf{R}_{x_s}) = \mathbf{B}\mathbf{p} + \sigma_n^2\hat{\mathbf{e}} \tag{9}$$

where $\mathbf{B}$ has size $M^2 \times K$ and each column of $\mathbf{B}$ is the Kronecker product between the steering vector in a certain direction with its conjugate. Therefore, $\mathbf{B}$ can be written as

$$\mathbf{B} = [\mathbf{a}_s^*(\theta_1)\otimes\mathbf{a}_s(\theta_1), \mathbf{a}_s^*(\theta_2)\otimes\mathbf{a}_s(\theta_2), ..., \mathbf{a}_s^*(\theta_K)\otimes\mathbf{a}_s(\theta_K)].$$

Further, $\mathbf{p}$ is given by

$$\mathbf{p} = [\sigma_1^2, \sigma_2^2, ...\sigma_K^2]^T$$

and $\hat{\mathbf{e}}$ is given by $\hat{\mathbf{e}} = [\mathbf{e}_1^T, \mathbf{e}_2^T, ..., \mathbf{e}_M^T]^T$ with $\mathbf{e}_i$ a vector of all zeros having a one in the i-th position.

Each column of $\mathbf{B}$ contains all the differences between the ruler markers given by $\{d_i - d_j\}_{i,j=0}^{M-1}$. As a result of the properties of the sparse ruler, these differences can be used to develop a new ruler having a complete set of integer marks from 0 to $N-1$. Therefore, each column of $\mathbf{B}$ has $2N-1$ distinct values. By comparing (9) with (4), we can find that the distinct rows of $\mathbf{B}$ provide an array manifold matrix for a longer uniform linear array with $2N-1$ sensors. In order to exploit the obtained differences, spatial smoothing is proposed in Section IV, which constructs a covariance matrix having rank $N-1$. As a consequence, it is possible to identify the DOAs of at most $N-1$ sources by applying the MUSIC algorithm on this covariance matrix. This also means that it is actually possible to identify the DOAs of all sources when the number of physical antennas (M) in an array is less than the number of sources (K).

IV. SPATIAL SMOOTHING METHOD

In this section, the spatial smoothing method is used for the rank enhancement of the observation matrix. This rank enhancement using spatial smoothing is actually proposed in [2]. In Section III, a longer uniform linear array is constructed based on the differences between the ruler markers. Note that $\mathbf{p}$ can be considered as a source vector and spatial smoothing can be applied on the signals to construct a proper covariance matrix. Because the spatial smoothing method can only be applied to uniform linear arrays, there should not be any missing distance in the resulting difference set. This is one of the essential conditions for applying the spatial smoothing method and a sparse ruler can be used to achieve this condition.

This method starts by extracting all the distinct differences from the rows of $\mathbf{B}$. Then we construct a new $(2N-1)\times K$ matrix $\mathbf{B}_1$ whose columns consist of $2N-1$ distinct differences sorted from $-N+1$ to $N-1$. This is equal to considering an array with $2N-1$ antennas where the N-th antenna is considered as a reference point. Therefore, the received signals by the other antennas can be modelled using a phase shift related to this element. Hence, we can write it as

$$\mathbf{y}_1 = \mathbf{B}_1\mathbf{p} + \sigma_n^2\grave{\mathbf{e}} \tag{10}$$

where $\grave{\mathbf{e}}$ is a vector of all zeros except for the N-th element where the zero difference occurs and the entry is one. Let us assume a subarray with N antennas, then it is possible to consider N overlapping subarrays. The positions of the antennas in the i-th subarray is given by :

$$\{(-i+1+n)d\}_{n=0}^{N-1}.$$

The i-th subarray extracts the $(N+1-i)$-th to $(2N-i)$-th element of $\mathbf{y}_1$. We can define it as :

$$\mathbf{y}_{1i} = \mathbf{B}_{1i}\mathbf{p} + \sigma_n^2\grave{\mathbf{e}}_i \tag{11}$$

where $\mathbf{B}_{1i}$ is a $N \times K$ matrix for the i-th subarray. $\grave{\mathbf{e}}_i$ is again a vector of all zeros except for the i-th entry which is one. For each subarray, we can define

$$\mathbf{R}_i = \mathbf{y}_{1i}\mathbf{y}_{1i}^H\ . \tag{12}$$

If we average over the total number of subarrays, we get

$$\mathbf{R}_{sx} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{R}_i \tag{13}$$

where $\mathbf{R}_{sx}$ is a spatially smoothed matrix. By using Theorem 1, which is mentioned in [3], it can be verified that $\mathbf{R}_{sx}$ has the same form as the covariance matrix of a ULA with N antennas. By applying the MUSIC algorithm to it, it is possible to identify up to $N-1$ sources. For further information about the proof of the theorem, see [2].

V. LEAST-SQUARES SOLUTION

Observe that the $(2N-1)\times 1$ vector $\mathbf{y}_1$ in (10) contains all the spatial correlation values between lag $-N+1$ and $N-1$. For clarity, we can write the elements of $\mathbf{B}_1$ in (10) as $\mathbf{B}_1 = [\mathbf{b}(\theta_1), \mathbf{b}(\theta_2), \ldots, \mathbf{b}(\theta_K)]$ where $\mathbf{b}(\theta_i)$ is given by

$$\mathbf{b}(\theta_i) = [\phi_i^{-N+1}, \ldots, \phi_i^{-1}, 1, \phi_i, \ldots, \phi_i^{N-1}]^T \tag{14}$$

with ϕ_i defined in (3). Based on (10) and (14), we can express our DOA estimation problem as a least-squares problem by writing the known observation $\mathbf{y}_1$ as

$$\mathbf{y}_1 = \hat{\mathbf{B}}\hat{\mathbf{p}} + \sigma_n^2\hat{\mathbf{e}} \tag{15}$$

where $\hat{\mathbf{B}} = [\mathbf{b}(\hat{\theta_1}), \mathbf{b}(\hat{\theta_2}), \ldots, \mathbf{b}(\hat{\theta_L})]$ and $\left\{\hat{\theta_i}\right\}_{i=1}^{L}$ is an arbitrary set of angles to be investigated. We want to emphasize here that $\left\{\hat{\theta_i}\right\}_{i=1}^{L}$ is not necessarily the same as the set of actual DOAs $\{\theta_i\}_{i=1}^{K}$ in (14), which are basically unknown from the receiver point of view. The $\hat{\mathbf{p}}$ in (15) can be written as:

$$\hat{\mathbf{p}} = [\hat{\sigma_1}^2, \hat{\sigma_2}^2, \ldots.\hat{\sigma_L}^2]^T$$

where $\hat{\sigma_i}^2$ is the unknown received power at the investigated direction $\hat{\theta_i}$. In order to be able to solve (15) using least-squares, the number of investigated directions L should satisfy $L \leq 2N-1$ and the system matrix $\hat{\mathbf{B}}$ should have full column rank. In other words, although we can compute the received power at up to $2N-1$ investigated angles $\hat{\theta_i}$, the selection of the investigated angles $\hat{\theta_i}$ is not arbitrary due to the need to guarantee the full rank condition of $\hat{\mathbf{B}}$. When $\hat{\mathbf{B}}$ has full column rank, we can solve (15) as:

$$\hat{\mathbf{p}} = (\hat{\mathbf{B}}^H\hat{\mathbf{B}})^{-1}\hat{\mathbf{B}}^H\mathbf{y}_1. \tag{16}$$

Assuming d in (3) equals 0.5λ, the easiest way is to divide a range from -1 to 1 uniformly into $2N-1$ grid points, that is $\left\{\frac{-2N+2}{2N-1}, \ldots, \frac{-2}{2N-1}, 0, \frac{2}{2N-1}, \ldots, \frac{2N-2}{2N-1}\right\}$. Computing the $\sin^{-1}$ at those $2N-1$ grid points will result in $2N-1$ investigated angles. When this is the case, it can be easily found that the $\hat{\mathbf{B}}$ is a permuted version of the Inverse Fast Fourier Transform (IFFT) matrix and thus the resulting $\hat{\mathbf{p}}$ can be regarded as the angular spectrum. In this case, the estimates of the actual DOAs can be found by locating the peaks of this angular spectrum.

Observe how the LS approach can simplify the DOA estimation process. Since spatial smoothing and MUSIC are not employed here, the complexity can thus be reduced at the price of a reduced estimation accuracy.

VI. SIMULATION RESULTS

In this section, some numerical results are shown and the results are compared to the coprime sampling method in [3]. In addition, the two proposed methods are compared and their results are evaluated. Let us assume that we have a length-35 sparse ruler with 10 marks $S = \{0, 1, 4, 10, 16, 22, 28, 30, 33, 35\}$. Therefore the inter-element separation is $\{1, 3, 6, 6, 6, 6, 2, 3, 2\}$.

First, we compare the performance of the coprime sampling and that of the sparse ruler. In this first simulation, both of them employ MUSIC algorithm and spatial smoothing. We generate 17 sources with a 10 degrees separation from -80 to 80 degrees while temporally and spatially white noise are considered. 1600 time snapshots are collected and a 0 dB signal to noise ratio is chosen. The result is illustrated in Fig. 2. Observe how the 17 sources can be detected by the coprime sampling method employing two uniform linear arrays with 7 and 9 antennas. However, the same performance can be achieved by a sparse ruler based non-uniform linear array having only 10 antennas.

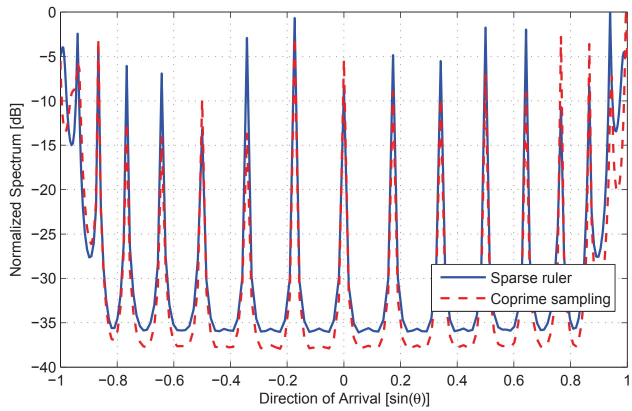

Fig. 2. MUSIC spectrum versus direction of arrival with 17 sources, 10 ruler marks and SNR = 0 dB for the coprime sampling and sparse ruler methods

Next, we evaluate the performance of the least-squares (LS) method by considering the same sparse ruler configuration and number of sources. Recall that the main requirement for the LS method is the full column rank condition of the $\hat{\mathbf{B}}$ matrix in (15). One possible way to satisfy this requirement is to divide a range from -1 to 1 into 71 grid points and investigate the 71 angles $\hat{\theta_i}$s, which are given by the $\sin^{-1}$ of the 71 grid values. The received power at those 71 $\hat{\theta_i}$s can be obtained by solving (16). Fig. 3 illustrates the DOA estimates produced by the LS method and MUSIC. Observe how the LS method performs worse due to the higher sidelobes at the direction where there is no active source. On the other hand, the MUSIC algorithm offers better resolution leading to better DOA separation. Note however that this performance of LS is achieved with lower complexity since in LS, we do not need to perform spatial smoothing during the estimation process.

To further compare the performance of the LS method with that of the MUSIC method, we consider a continuous source from 30 to 40 degrees. This is done by generating finite number of sources between 30 and 40 degrees with very small degree of separation. The result is illustrated by Fig. 4. Observe how the MUSIC method generally performs better. The MUSIC algorithm generally introduces lower power level at the direction where no source exists. However, we are also able to locate the range of DOAs with LS method although its resolution is worse than that of MUSIC. Again, this LS performance is achieved with lower complexity. Therefore, the trade off in these two methods is between resolution and complexity.

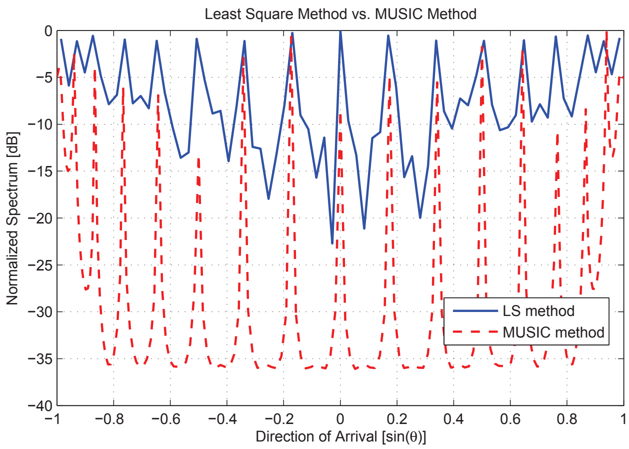

Fig. 3. Spectrum in dB versus direction of arrival with 17 sources, 10 ruler marks and SNR = 0 dB for the MUSIC method and LS method

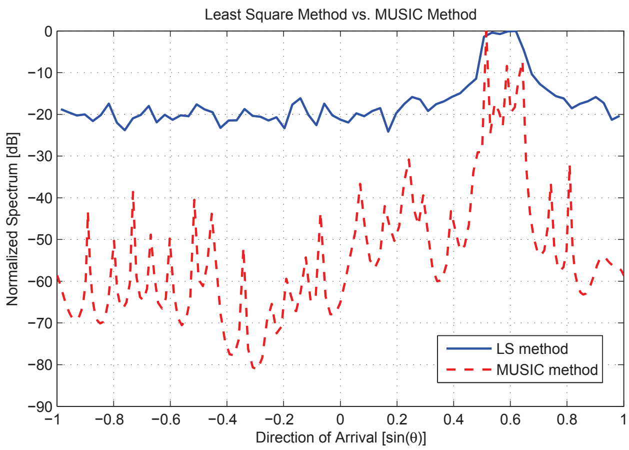

Fig. 4. Comparison between the two proposed methods (LS and MUSIC) with a continuous source from 30 to 40 degrees

VII. Conclusion

In this paper, a new non-uniform linear array design is proposed based on the concept of minimum sparse rulers. With a length-$(N-1)$ sparse ruler, whose distance marks define the inter-element separation, it is possible to create a virtual uniform linear array with $2N-1$ antennas. With the spatial smoothing method, a spatial covariance matrix of suitable rank is constructed. By applying the MUSIC algorithm to this covariance matrix, it is possible to detect $N-1$ sources even when the actual number of physical receiving antennas is less than the number of sources. In the second method, we generate the same virtual ULA by solving a least-squares problem. By finding the angular power spectrum, it is possible to find the DOAs with lower complexity. Simulation results have proven the performance of the proposed methods. The comparison between the two proposed methods has shown the trade off between resolution and complexity. Extending this method to the case of colored source signals is currently under investigation.

References

[1] W.-K. Ma, T.-H. Hsieh, and C.-Y. Chi, "Doa estimation of quasi-stationary signals via khatri-rao subspace," in *IEEE International Conference onAcoustics, Speech and Signal Processing, 2009. ICASSP 2009.*, april 2009, pp. 2165 –2168.

[2] P. Pal and P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *IEEE Transactions on Signal Processing*, vol. 58, no. 8, pp. 4167 –4181, aug. 2010.

[3] ——, "Coprime sampling and the music algorithm," in *Digital Signal Processing Workshop and IEEE Signal Processing Education Workshop (DSP/SPE), 2011 IEEE*, jan. 2011, pp. 289 –294.

[4] A. Moffet, "Minimum-redundancy linear arrays," *IEEE Transactions on Antennas and Propagation*, vol. 16, no. 2, pp. 172 – 175, mar 1968.

[5] D. Linebarger, I. Sudborough, and I. Tollis, "Difference bases and sparse sensor arrays," *IEEE Transactions on Information Theory*, vol. 39, no. 2, pp. 716 –721, mar 1993.

[6] Y. Abramovich, N. Spencer, and A. Gorokhov, "Positive-definite toeplitz completion in doa estimation for nonuniform linear antenna arrays. ii. partially augmentable arrays," *IEEE Transactions on Signal Processing*, vol. 47, no. 6, pp. 1502 –1521, jun 1999.

[7] C. Chambers, T. Tozer, K. Sharman, and T. Durrani, "Temporal and spatial sampling influence on the estimates of superimposed narrowband signals: when less can mean more," *IEEE Transactions on Signal Processing*, vol. 44, no. 12, pp. 3085 –3098, dec 1996.

[8] C. Weng and P. Vaidyanathan, "Nonuniform sparse array design for active sensing," accepted in Asilomar 2011.

[9] J. Leech, "On representation of 1, 2,, n by differences," *Journal of the London Mathematical Society*, vol. 31, p. 160169, April 1956.