

ORTHOGONAL EMBEDDING THEORY FOR CONTRACTIVE TIME-VARYING SYSTEMS

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ABSTRACT

The paper discusses a constructive solution of the problem of the realization of a given (strictly) contractive time-varying system as the partial transfer operator of a lossless system. The construction is done in a state space context and gives rise to a time-varying Riccati-type equation. It is the generalization to the time-varying case of the time-invariant Darlington synthesis.

1. INTRODUCTION

In this paper, we will solve the lossless embedding problem (Darlington problem [1, 2]) for strictly contractive time-varying systems in a state space context. The problem setting is the following.

Problem 1. (The embedding problem) Let be given the transfer operator T of a contractive causal linear time-varying system with n_1 inputs and n_0 outputs and finite dimensional state space, and let \mathbf{T} be a given state space realization of T (as in [3]):

$$\mathbf{T} = \begin{bmatrix} A & C \\ B & D \end{bmatrix}, \quad Y = UT \Leftrightarrow \begin{cases} XZ^{-1} & = & XA + UB \\ Y & = & XC + UD \end{cases}$$

Then determine a unitary and causal multi-port Σ (corresponding to a lossless system)

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix},$$

with state space realization Σ , such that $T = \Sigma_{11}$. See figure 1.

Without loss of generality we can in addition require Σ to be a unitary realization: ($\Sigma\Sigma^* = I$, $\Sigma^*\Sigma = D$). Since $T^*T + T_c^*T_c = I$, (where $T_c = \Sigma_{21}$), this will be possible only if T is contractive (not in the strict sense). While it is clear that contractivity is a necessary condition, it will be shown in the sequel that strict contractivity of T is sufficient to construct a solution to the embedding problem. (The extension to the boundary case is non-trivial.) Note that the condition [Σ is lossless] implies that the number of inputs of Σ is equal to its number of outputs, and that the condition [Σ is unitary] implies that Σ is a “diagonal of square matrices” and hence that the system order of the embedding is constant (even if the system order of T is not).

⁰Proc. Int. Symp. on MTNS-91, ed. H. Kimura. MITA press, Japan (1992), volume II, pp. 513-518.

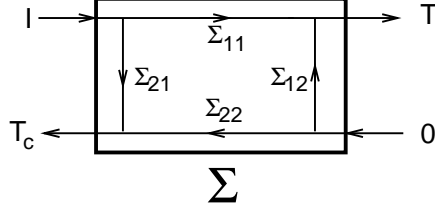


Fig. 1. Embedding of T .

2. NOTATION

The notation in this paper is an inclusive extension of the notation in [4, 3]. We consider a generalization of ℓ_2 sequences $X = [\cdots X_{-1} X_0 X_1 \cdots]$, in which each of the entries X_i is an element of a (row) vector space \mathbf{C}^{N_i} , with varying dimensions $N_i \in \mathbf{N}$, and such that the total energy $\|X\|_2^2 = \sum_{-\infty}^{\infty} \|X_i\|_2^2$ is bounded. We denote the set $(\mathbf{Z} \rightarrow \mathbf{N})$ of index sequences by \mathcal{I} , and with $N \in \mathcal{I}$ say that the above X is an element of $\ell_2(\mathbf{C}^1, \mathbf{C}^N)$, or \mathbf{C}_2^N for brevity. We adopt the shorthand “ $\cdot n$ ” for the index sequence N with all N_i equal to the same integer n .

Let $N, P \in \mathcal{I}$. Following [4], we denote by $\mathcal{X}(\mathbf{C}^N, \mathbf{C}^P)$ the class of bounded operators $(\mathbf{C}_2^N \rightarrow \mathbf{C}_2^P)$. E.g., a system transfer operator T with n_1 input ports and n_0 output ports is an operator in $\mathcal{X}(\mathbf{C}^{n_1}, \mathbf{C}^{n_0})$. Standard subsets of \mathcal{X} are the space of upper (causal), lower and diagonal operators:

$$\begin{aligned} \mathcal{U} &= \{A \in \mathcal{X} : A_{ij} = 0, i > j\} \\ \mathcal{L} &= \{A \in \mathcal{X} : A_{ij} = 0, i < j\} \\ \mathcal{D} &= \mathcal{U} \cap \mathcal{L}. \end{aligned}$$

For every index sequence $N \in \mathcal{I}$, we define the k -th shift $N^{(k)}$ by $(N^{(k)})_i = N_{i-k}$. We will use the shorthand N^+ for $N^{(1)}$, and likewise $N^- = N^{(-1)}$. The shift operator $Z : \mathbf{C}^N \rightarrow \mathbf{C}^{N^+}$ is defined by $(XZ)_i = X_{i-1}$, and the k -th diagonal shift on $X \in \mathcal{X}$ is $X^{(k)} = Z^{*k} X Z^k$.

Let $A \in \mathcal{X}$. We define the j -th diagonal $A_{[j]} \in \mathcal{D}$ of A by $(A_{[j]})_i = A_{i-j, i}$. Hence $A_{[0]}$ is the main diagonal of the operator A , and for positive j , $A_{[j]}$ is the j -th subdiagonal above $A_{[0]}$. With this notation, A can formally be written in terms of its diagonals as $A = \sum_{-\infty}^{\infty} Z^j A_{[j]}$, although this expression need not converge at all. A class of operators that do allow this representation is the set of Hilbert-Schmidt operators [4]:

$$\mathcal{X}_2 = \{A \in \mathcal{X} : \|A\|_{HS}^2 = \sum_{i,j} \|A_{ij}\|_2^2 < \infty\}$$

along with inner product $\langle A, B \rangle = \text{trace}(AB^*)$, and norm $\|A\|_{HS}^2 = \langle A, A \rangle = \text{trace}(AA^*)$. Standard subspaces in \mathcal{X}_2 are $\mathcal{U}_2 = \mathcal{U} \cap \mathcal{X}_2$, $\mathcal{L}_2 = \mathcal{L} \cap \mathcal{X}_2$, $\mathcal{D}_2 = \mathcal{L}_2 \cap \mathcal{U}_2$, and standard projectors onto these spaces are $\mathbf{P}_0 = \mathbf{P}_{\mathcal{D}_2}$ and $\mathbf{P} = \mathbf{P}_{\mathcal{U}_2}$.

For $X \in \mathcal{U}_2$, the diagonal expansion of X is \tilde{X} , defined by

$$\begin{aligned} X &= X_{[0]} + ZX_{[1]} + Z^2 X_{[2]} + \cdots = X_{[0]} + X_{[1]}^{(-1)} Z + X_{[2]}^{(-2)} Z^2 + \cdots \\ \tilde{X} &= \begin{bmatrix} X_{[0]} & X_{[1]}^{(-1)} & X_{[2]}^{(-2)} & \cdots \end{bmatrix}. \end{aligned}$$

For $X \in \mathcal{L}_2 Z^{-1}$, the diagonal expansion of X is also designated by \tilde{X} , now defined by

$$\begin{aligned} X &= Z^{-1} X_{[-1]} + Z^{-2} X_{[-2]} + \cdots = X_{[-1]}^{(+1)} Z^{-1} + X_{[-2]}^{(+2)} Z^{-2} + \cdots \\ \tilde{X} &= \begin{bmatrix} X_{[-1]}^{(+1)} & X_{[-2]}^{(+2)} & \cdots \end{bmatrix} \end{aligned}$$

These definitions keep entries of X that are on the same row in X also on the same row in \tilde{X} .

3. PRELIMINARY

In the sequel, we will need the notion of a “top left” part of an operator T in \mathcal{U} in the sense of that part of T that maps inputs in the “past”, \mathcal{L}_2Z^{-1} , to outputs in the past, \mathcal{L}_2Z^{-1} , which will be shown to correspond to the top left part of the matrix representation of T . To this end, define the operators K_T and V_T in the following way.

Definition 1. Given a system transfer operator $T \in \mathcal{U}$. Then the operators K_T and V_T are defined as

$$\begin{aligned} K_T: \mathcal{L}_2Z^{-1} &\rightarrow \mathcal{L}_2Z^{-1}, & UK_T &= \mathbf{P}_{\mathcal{L}_2Z^{-1}}(UT) \\ V_T: \mathcal{L}_2Z^{-1} &\rightarrow \mathcal{D}_2, & UV_T &= \mathbf{P}_0(UT). \end{aligned}$$

We can define operators \tilde{K}_T and \tilde{V}_T that act on diagonal expansions \tilde{U} and \tilde{Y} of U and Y . Unlike K_T and V_T , these operators have a matrix representation.

Theorem 1. If $Y = UK_T \in \mathcal{L}_2Z^{-1}$ and $D = UV_T \in \mathcal{D}_2$, with $U \in \mathcal{L}_2Z^{-1}$, then the matrix representations of the operators \tilde{K}_T and \tilde{V}_T such that $\tilde{Y} = \tilde{U}\tilde{K}_T$ and $D = \tilde{U}\tilde{V}_T$ is given by

$$\tilde{K}_T = \begin{bmatrix} T_{[0]}^{(+1)} & 0 & \cdots & & \\ T_{[1]}^{(+1)} & T_{[0]}^{(+2)} & 0 & \cdots & \\ T_{[2]}^{(+1)} & T_{[1]}^{(+2)} & T_{[0]}^{(+3)} & 0 & \cdots \\ \vdots & & & \ddots & \ddots \end{bmatrix} \quad \tilde{V}_T = \begin{bmatrix} T_{[1]} \\ T_{[2]} \\ T_{[3]} \\ \vdots \end{bmatrix}.$$

It is clear from the above that \tilde{K}_T satisfies the relation

$$\tilde{K}_T^{(-1)} = \begin{bmatrix} T_{[0]} & 00 \cdots \\ \tilde{V}_T & \tilde{K}_T \end{bmatrix}. \quad (1)$$

Connection of T with K_T and V_T . Let $T \in \mathcal{U}$. Then \tilde{K}_T is represented by an infinite-size matrix with diagonal entries in \mathcal{D} . Construct (infinite size) submatrices K_i ($-\infty < i < \infty$) of \tilde{K}_T by selecting the i -th entry of each diagonal in \tilde{K}_T . The K_i can be viewed as *time-varying* matrices that would be Toeplitz in the time-invariant case. The K_i are double-mirrored “top-left” submatrices of T (see figure 2). In the same way, let V_i be the vector representation of the operator V_T , obtained by selecting the i -th entry of the diagonal representation of \tilde{V}_T .

4. CONTRACTIVITY

Definition 2. A hermitian operator A in \mathcal{X} is strictly positive definite if there exists an $\varepsilon > 0$ such that, for all U in \mathcal{X}_2 , $\mathbf{P}_0(UAU^*) \geq \varepsilon \mathbf{P}_0(UU^*)$. Notation: $A \gg 0$.

It is a known result that an operator A is strictly positive definite if and only if $A = A^*$ and A^{-1} exists in \mathcal{X} .

Definition 3. Let T be a system transfer operator in \mathcal{U} . T is strictly contractive if $I - TT^* \gg 0$.

Because of the identity $I + T^*(I - TT^*)^{-1}T = (I - T^*T)^{-1}$ it is clear that $I - TT^* \gg 0$ implies that $I - T^*T \gg 0$ also.

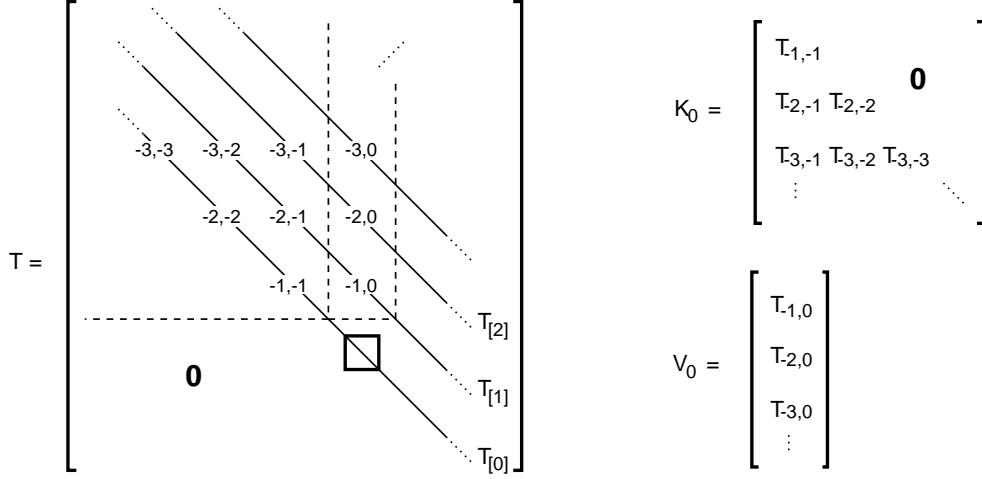


Fig. 2. K_i and V_i matrices are submatrices of T .

Lemma 1. Let T be a system transfer operator in \mathcal{U} . If T is strictly contractive, then K_T is strictly contractive on its domain: $I - K_T K_T^* \gg 0$, $I - K_T^* K_T \gg 0$. We also have that \tilde{K}_T is strictly contractive: $I - \tilde{K}_T \tilde{K}_T^* \gg 0$, $I - \tilde{K}_T^* \tilde{K}_T \gg 0$.

At this point, we remark that if \tilde{K}_T is strictly contractive, then all K_i are strictly contractive, and letting $i \rightarrow \infty$ it follows that T is strictly contractive. Hence contractivity of T , K_T and \tilde{K}_T are equivalent.

Theorem 2. Let $T \in \mathcal{U}$ be a system transfer operator. If T is strictly contractive, then

$$I - T_{[0]}^* T_{[0]} - \tilde{V}_T^* (I - \tilde{K}_T \tilde{K}_T^*)^{-1} \tilde{V}_T \gg 0.$$

PROOF Since T is strictly contractive, \tilde{K}_T and $\tilde{K}_T^{(-1)}$ are also strictly contractive. Using equation (1), we have that

$$I - \tilde{K}_T^{(-1)*} \tilde{K}_T^{(-1)} = \begin{bmatrix} I - T_{[0]}^* T_{[0]} - \tilde{V}_T^* \tilde{V}_T & -\tilde{V}_T^* \tilde{K}_T \\ -\tilde{K}_T^* \tilde{V}_T & I - \tilde{K}_T^* \tilde{K}_T \end{bmatrix}$$

From an application of Schur's inversion formula (see *e.g.*, [5]), it is seen that this expression is positive definite iff

$$\begin{cases} (1) & I - \tilde{K}_T^* \tilde{K}_T \gg 0 \\ (2) & I - T_{[0]}^* T_{[0]} - \tilde{V}_T^* \tilde{V}_T - \tilde{V}_T^* \tilde{K}_T (I - \tilde{K}_T^* \tilde{K}_T)^{-1} \tilde{K}_T^* \tilde{V}_T \gg 0. \end{cases}$$

The first condition is satisfied because T is strictly contractive. The second condition is equal to the result. \square

5. CONTRACTIVITY OF A STATE SPACE REALIZATION

Let $T \in \mathcal{U}$ have a state space realization $\{A, B, C, D\}$, with $A \in \mathcal{D}(\mathbf{C}^N, \mathbf{C}^N)$. We denote by \mathcal{C} the controllability operator:

$$\mathcal{C} = \begin{bmatrix} B^{(+1)} \\ B^{(+2)} A^{(+1)} \\ B^{(+3)} A^{(+2)} A^{(+1)} \\ \vdots \end{bmatrix}$$

and we shall say that a realization is uniformly controllable if $\mathcal{C}^* \mathcal{C} \gg 0$. (\mathcal{C} is an extension of the usual controllability operator to the present context. Its derivation and

many related issues are discussed elsewhere [6].) It can be derived that $\tilde{V}_T = \mathcal{C} \cdot C$, and by using the above structure of \mathcal{C} we also have

$$\tilde{V}_T^{(-1)} = \begin{bmatrix} B \\ CA \end{bmatrix} \cdot C^{(-1)}. \quad (2)$$

Let the hermitian operator M in $\mathcal{D}(\mathbf{C}^N, \mathbf{C}^N)$ be defined by

$$M = \mathcal{C}^* (I - \tilde{K}_T \tilde{K}_T^*)^{-1} \mathcal{C}. \quad (3)$$

M is well-defined if T is strictly contractive. It will play an important role in the embedding theory to follow in the next section. In that respect, the following observation is important. The contractivity condition implies that $M \geq 0$. If in addition the state space realization is uniformly controllable, $\mathcal{C}^* \mathcal{C} \gg 0$, then also $M \gg 0$ and invertible.

Theorem 3. Let $T \in \mathcal{U}$ be a system transfer operator with state space realization $\{A, B, C, D\}$. If T is strictly contractive, then the above defined M satisfies the relations $I - D^* D - C^* M C \gg 0$, and

$$M^{(-1)} = A^* M A + B^* B + [A^* M C + B^* D] (I - D^* D - C^* M C)^{-1} [D^* B + C^* M A].$$

If in addition the state space realization is uniformly controllable, then $M \gg 0$.

PROOF The proof uses the definition of M , equations (1,2), and Theorem 2. \square

6. ORTHOGONAL EMBEDDING

We will construct a solution to the embedding problem as stated in the Introduction under the following conditions.

Theorem 4. Let T be a bounded causal LTV system in $\mathcal{U}(\mathbf{C}^{n_1}, \mathbf{C}^{n_0})$, with state space realization \mathbf{T} . Suppose $A \in \mathcal{D}(\mathbf{C}^N, \mathbf{C}^{N'})$. A solution to the embedding problem can be constructed if T is strictly contractive and the given realization \mathbf{T} is uniformly controllable. This construction will yield a lossless realization Σ for the embedding system Σ with the following properties.

(1) Σ is in $\mathcal{U}(\mathbf{C}^n, \mathbf{C}^n)$, with $n = n_1 + n_0$, i.e., the embedding adds n_0 more inputs and n_1 more outputs to those of T . This n cannot be smaller.

(2) $\Sigma = \{A_\Sigma, B_\Sigma, C_\Sigma, D_\Sigma\}$ has $A_\Sigma \in \mathcal{D}(\mathbf{C}^m, \mathbf{C}^m)$, where $m = \max_i(N_i)$. This m cannot be smaller.

Step 1. of the construction is to find a state transformation R and matrices B_2 and D_{21} such that the columns of Σ_2 ,

$$\Sigma_2 = \begin{bmatrix} R & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} A & C \\ B & D \\ B_2 & D_{21} \end{bmatrix} \begin{bmatrix} R^{(-1)} & \\ & I \end{bmatrix}$$

are unitary, i.e., $(\Sigma_2)^* \Sigma_2 = I$.

Lemma 2. A solution to step 1. is obtained by putting $M = R^* R$ and solving for M in

$$M^{(-1)} = A^* M A + B^* B + [A^* M C + B^* D] (I - D^* D - C^* M C)^{-1} [D^* B + C^* M A].$$

The solution M exists under the condition [T is strictly contractive] and is strictly positive definite if [\mathbf{T} is uniformly controllable]. Because of Theorem 3, it is given in closed

form by equation (3). $B_2 \in \mathcal{D}(\mathbf{C}^{n_0}, \mathbf{C}^{N^-})$ and $D_{21} \in \mathcal{D}(\mathbf{C}^{n_0}, \mathbf{C}^{n_0})$ are determined as

$$\begin{cases} D_{21} &= (I - D^*D - C^*MC)^{1/2} \\ B_2 &= -(I - D^*D - C^*MC)^{-1/2} [D^*B + C^*MA] \end{cases}$$

PROOF To solve step 1, compute $(\Sigma_2)^*\Sigma_2$, and put $M = R^*R$. From the orthogonality conditions the equations mentioned in the theorem follow directly. At this point, recall Theorem 3, and observe that the solution to the last equation is precisely given by

$$M = C^*(I - \tilde{K}_T \tilde{K}_T^*)^{-1}C.$$

Since the realization is uniformly controllable, Theorem 3 asserts that this $M \gg 0$, so that it can be factored as $M = R^*R$ with R invertible. It also follows that $D_{21}^*D_{21} \gg 0$, so that D_{21} cannot have less than n_0 rows, and hence no less than n_0 inputs must be added to T to yield a lossless embedding. \square

Step 2. Define Σ'_2 as Σ_2 extended by zero rows to $\Sigma'_2 \in \mathcal{D}(\mathbf{C}^{(m+n_1+n_0)}, \mathbf{C}^{N^-})$:

$$\Sigma'_2 = \begin{bmatrix} 0_{(m-N) \times N^-} \\ \Sigma_2 \end{bmatrix} = \begin{bmatrix} I & & \\ & R & \\ & & I \end{bmatrix} \begin{bmatrix} 0_{(m-N) \times N^-} \\ A & C \\ B & D \\ B_2 & D_{21} \end{bmatrix} \begin{bmatrix} R^{(-1)} & \\ & I \end{bmatrix}$$

Find matrices $\Sigma_1 \in \mathcal{D}(\mathbf{C}^{(m+n_1+n_0)}, \mathbf{C}^{m-N^-})$ and $\Sigma_3 \in \mathcal{D}(\mathbf{C}^{(m+n_1+n_0)}, \mathbf{C}^{n_1})$ such that

$$\Sigma = [\Sigma_1 \quad \Sigma'_2 \quad \Sigma_3]$$

is in $\mathcal{D}(\mathbf{C}^{(m+n_1+n_0)}, \mathbf{C}^{(m+n_0+n_1)})$: a diagonal of square unitary matrices of constant size $(m + n_1 + n_0)$. Put into this form, step 2. is always possible and reduces to a standard exercise in linear algebra.

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