Blind Source Separation: The Location of Local Minima in the Case of Finitely Many Samples

Amir Leshem, *Senior Member, IEEE*, and Alle-Jan van der Veen, *Fellow, IEEE*

***Abstract*—Cost functions used in blind source separation are often defined in terms of expectations, i.e., an infinite number of samples is assumed. An open question is whether the local minima of finite sample approximations to such cost functions are close to the minima in the infinite sample case. To answer this question, we develop a new methodology of analyzing the finite sample behavior of general blind source separation cost functions. In particular, we derive a new probabilistic analysis of the rate of convergence as a function of the number of samples and the conditioning of the mixing matrix. The method gives a connection between the number of available samples and the probability of obtaining a local minimum of the finite sample approximation within a given sphere around the local minimum of the infinite sample cost function. This shows the convergence in probability of the nearest local minima of the finite sample approximation to the local minima of the infinite sample cost function. We also answer a long-standing problem of the mean-squared error (MSE) behavior of the (finite sample) least squares constant modulus algorithm (LS-CMA), namely whether there exist LS-CMA receivers with good MSE performance. We demonstrate how the proposed techniques can be used to determine the required number of samples for LS-CMA to exceed a specified performance. The paper concludes with simulations that validate the results.**

***Index Terms*—Blind source separation, constant modulus algorithm, finite sample analysis.**

I. INTRODUCTION

BLIND equalization and source separation is a wide field of research. Initiated by the works of Sato [2], many authors have followed, e.g., Godard [3], Jutten and Herault [4], Treichler and Agee [5], Shalvi and Weinstein [6], Cardoso [7], and Comon [8]. Many solutions are tied to the optimization of certain cost functions (also known as *contrasts* [8]), e.g., cumulant-based methods [7], mixed second-order/fourth-order methods, augmentations with independence constraints (related to finding beamformers to all sources) [9], characteristic function-based techniques [10], [11], etc. An overview of blind source separation techniques and blind equalization can be found in [12]–[14].

Manuscript received March 8, 2007; revised January 24, 2008. Published August 13, 2008 (projected). The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Philippe Loubaton. This research has been partially supported by the EU-FP6 under contract 506790 and by NWO-STW under the VICI programme (DTC.5893). Parts of this paper were presented at the IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), Istanbul, Turkey, June 2000.

A. Leshem is with the School of Engineering, Bar-Ilan University, Ramat-Gan 52900, Israel (e-mail: leshema@eng.biu.ac.il; leshem@ieee.org).

A.-J. van der Veen is with Department of Electrical Engineering, Delft University of Technology, 2628CD Delft, The Netherlands (e-mail: allejan@cobalt.et.tudelft.nl).

Digital Object Identifier 10.1109/TSP.2008.921721

The analysis of these algorithms has focused on proving the existence of "good" local minima of the cost function (those leading to separation), the absence of undesired local minima (not associated to separation), computational complexity, suitable step sizes in gradient descent implementations, and more recently the effectiveness of natural gradient descent techniques. Although the properties of these cost functions have been well studied, they implicitly assume an infinite number of samples, because they are formulated in terms of expectations.

A question that has not been sufficiently studied yet is, For finite-sample approximations of the blind source separation cost functions, do the local minima converge to the "true" infinite-sample solutions? and second, What is the asymptotic speed of convergence, i.e., how many samples are at least needed to arrive close to the "true" solutions? Partial answers have been provided by Comon *et al.* [15] and Moreau *et al.* [16] for some cumulant-based contrasts in terms of bias and variance of the estimated separator. Also asymptotic weak consistency follows from the general theory of asymptotic statistics [17], [18]. However, no effective results on the number of samples required to obtain a given accuracy with a given probability exist.

In this paper we propose a general framework for analyzing such questions for various cost functions through the use of probabilistic inequalities such as the Chebyshev inequality and inequalities related to higher order moments of the function and its derivatives.

The constant modulus algorithm (CMA) [5] is among the most widely used and analyzed algorithms in this context. The asymptotic behavior of the underlying constant modulus (CM) cost function is now well understood, i.e., the location of the local minima of the CM cost function have been characterized, first in the noiseless case and then in the noisy case [19]–[22]. It was shown that (under conditions) local minima of the CM cost function are close to the minima of the mean square error (MSE) cost function [23]. At the same time, many of the blind source separation cost functions have been shown to belong to the same family [24], and therefore converge to the same receivers.

For finite samples, many cost functions can be reformulated in terms of similar deterministic least squares cost functions, which has led, e.g., to the least squares CMA [25], a fixed window version of LS-CMA [26], and the ACMA [27]. For the fixed window LS-CMA, no finite-sample analysis has been done. For ACMA, a convergence result states that the beamformers converge to the linear minimum MSE (LMMSE or Wiener) beamformers, asymptotically in number of samples N or signal to noise ratio [28], [29]. Other results gauge the finite sample performance in terms of Cramér–Rao bounds [30], [31]. Finally, some of the literature focuses on identifiability: the number of samples necessary for obtaining a unique solution in the noiseless case [32].

The contributions of this paper are as follows:

- we present a new methodology of analyzing finite sample behavior of general blind source separation techniques, in particular a new probabilistic analysis of the rate of convergence in samples;
- we answer a long standing problem of MSE behavior of LS-CMA, namely whether there exist LS-CMA receivers with good MSE performance.

The first result enables a bounding of the number of samples necessary to achieve a certain accuracy of general blind source separation techniques. The bound is in terms of the smallest eigenvalue of the Hessian of the (infinite sample) cost function at the location of the local optimum, and the selected probability region. This enables e.g., to derive the required number of samples for a specific method to reach a specified performance, in terms of the conditioning of the mixing matrix. It should be noted that while we provide the first effective bound on the number of samples required to achieve a given accuracy with a predetermined probability, our bounds are not tight. We foresee that adding assumptions on the existence of the moment generating function of the data, together with Chernoff type bounds will enhance the tightness of the bound. This is out of the scope of the current paper, which provides the general methodology.

The paper is structured as follows. In Section II, we formulate the problem and relate it to various blind source separation techniques. In Section III, the main theorem regarding the location of "finite sample" local minima is formulated and proved. In Section IV we discuss the MSE behavior of the fixed window LS-CMA and show the existence of LS-CMA receivers with good MSE performance. This section also illustrates the theorem by estimating the required number of samples for LS-CMA to ensure a certain signal-to-interference ratio with a given outage probability. Finally, simulations in Section V illustrate the results. We end up with some conclusions and remark on future extensions.

II. Data Model and Problem Formulation

A. Cost Functions

Assume that we measure a noisy mixture of unknown signals,

$$\mathbf{x}_n = \mathbf{A}\mathbf{s}_n + \mathbf{n}_n$$

where $\mathbf{x}_n$ is the received signal vector (p entries) at time n, $\mathbf{A}$ is a "tall" (overdetermined), complex or real mixing matrix for an instantaneous channel, $\mathbf{s}_n$ is a vector of K transmitted signals assumed to be independent and non-Gaussian, and $\mathbf{n}_n$ is the receiver noise vector that is assumed to be spatially and temporally white Gaussian noise with covariance $\sigma^2\mathbf{I}$. This setup is very general and covers various problems from blind separation of narrowband communication signals to the separation of medical signals such as electro-encephalogram (EEG) and magneto-encephalogram (MEG).

An adaptive beamformer $\mathbf{w}$ is basically a linear combiner of the received signal vector, $y_n = \mathbf{w}^H\mathbf{x}_n$ where H denotes the complex conjugate transpose. The goal is to design $\mathbf{w}$ such that y_n approaches one signal out of the mixture, with maximal signal to interference plus noise ratio. When the transmitted signals are unknown but certain statistical properties of the signals are known, the problem is called blind source separation.

Many existing blind source separation techniques are based on the optimization of a cost function $J(\mathbf{w}) = E_{\mathbf{x}}J(\mathbf{w},\mathbf{x})$, where the notation $E_{\mathbf{x}}$ denotes the expectation operator with respect to the random variable $\mathbf{x}$. The separating beamformer is obtained by finding a vector $\mathbf{w}$ which optimizes the cost function J. In practice, the cost function $J(\mathbf{w})$ is unavailable to us and is estimated from the data. When the received signals and noise are stationary ergodic, the estimation phase is reduced to computing sample averages

$$J^N(\mathbf{w},\mathbf{X}) = \frac{1}{N}\sum_{n=1}^{N} J(\mathbf{w},\mathbf{x}_n) \tag{1}$$

where N is the number of samples and $\mathbf{X}$ is a matrix containing the measured vectors,

$$\mathbf{X} = [\mathbf{x}_1,\ldots,\mathbf{x}_N]. \tag{2}$$

There are numerous adaptive algorithms to separate users based on optimizing a stochastic cost function $E_{\mathbf{x}}J(\mathbf{w},\mathbf{x})$. One of the most successful is the CMA, which follows from minimizing the CMA(2,2) cost function

$$J_{CM}(\mathbf{w}) = E_{\mathbf{x}}\left(|\mathbf{w}^H\mathbf{x}|^2 - 1\right)^2. \tag{3}$$

Others include minimizing the CMA(p,q) cost function,

$$J_{p,q}(\mathbf{w}) = E_{\mathbf{x}}\left(|\mathbf{w}^H\mathbf{x}|^p - 1\right)^q,$$

and maximizing the Shalvi–Weinstein contrast

$$J_{sw}(\mathbf{w}) = \kappa_4(\mathbf{w}^H\mathbf{x}).$$

where $\kappa_4(\cdot)$ is a fourth-order cumulant (see [33] for its relation to the CMA cost function). There are many related fourth-order cumulant-based cost functions [7], [8], [15], [34], [35] which also fit here.

With K sources, the preceding cost functions should have K local optima, and there is an issue about finding all of them (e.g., by using multiple initial points and hoping they converge to independent solutions). Alternatively, the multiuser CMA of [9] and [36] combines the CMA cost function with a term that expresses the stochastic independence of the K outputs after correct separation. To put this in our framework, let $\mathbf{w}$ be a vector accumulating all the beamformers of the individual users

$$\mathbf{w} = \left[\mathbf{w}_1^T,\ldots,\mathbf{w}_K^T\right]^T \tag{4}$$

The cost function is given by

$$J(\mathbf{w}) = E_{\mathbf{x}}\left[\sum_{k=1}^{K}\left|\left|\mathbf{w}_k^H\mathbf{x}\right|^2 - 1\right|^2 + \alpha\sum_{i\neq j}\left|\mathbf{w}_i^H\mathbf{x}\mathbf{x}^H\mathbf{w}_j\right|^2\right] \tag{5}$$

where α is a positive constant. Other multi-user-based techniques based on deflation (e.g., [37]) do not immediately fit in our framework and are not considered here.

B. Problem Formulation

We will study the relation of the local minima of a general cost function

$$J^{\infty}(\mathbf{w}) = E_{\mathbf{x}} J(\mathbf{w}, \mathbf{x}) \tag{6}$$

to the local minima of its finite sample approximation $J^N(\mathbf{w}, \mathbf{X})$. We will assume that we are given a set of N i.i.d. realizations of the channel output, $\mathbf{X} = [\mathbf{x}_1, \cdots, \mathbf{x}_N]$, and minimize $J^N(\mathbf{w}, \mathbf{X})$ defined by (1).

Let $\mathbf{w}^o$ be a local minimum of $J^{\infty}(\mathbf{w})$. We would like to bound the distance of $\mathbf{w}^o$ to the closest local minimum of $J^N(\mathbf{w}, \mathbf{X})$, in terms of the statistical properties of the cost function and its derivatives. Obviously, since our definition is based on realizations of a random process, we cannot expect to obtain a deterministic result. Therefore, we consider, given a probability ε and a radius r, whether the probability that a local minimum of $J^N(\mathbf{w}, \mathbf{X})$ has a distance at most r from $\mathbf{w}^o$ is greater than $1 - \varepsilon$. More specifically, we would like to know that the probability of not having a local minima within a given radius converges to zero as N tends to infinity. In the next sections we solve this problem, and also provide bounds on the rate of convergence of the local minima of $J^N(\mathbf{w}, \mathbf{X})$ to $\mathbf{w}^o$, for sufficiently large N.

III. The Location of Local Minima

A. Main Result

We will prove a general result on the location of local minima of finite sample approximations to cost functions, and show convergence in probability of the local minima of the approximation to the local minima of $J^{\infty}(\mathbf{w})$. In our main Theorem 3.1, the gradient and Hessian are defined assuming that $\mathbf{w}$ is a real vector. If it is complex, we can apply the theorem to $\mathbf{r} = [\mathrm{Re}(\mathbf{w})^T, \mathrm{Im}(\mathbf{w})^T]^T$. When the function depends on a matrix $\mathbf{W}$ we can replace it by $\mathbf{w} = \mathrm{vec}(\mathbf{W})$, where vec is the operation of converting a matrix into a vector of the elements.

We begin with some notation that simplifies the treatment of multidimensional Taylor series.

Definition 3.1: Let $\mathbf{m} = [m_1, \ldots, m_p]^T \in N^p$, $\mathbf{w} = [w_1, \ldots, w_p]^T \in R^p$. Let $f(\mathbf{w}) : R^p \to R$,

$$D^{\mathbf{m}} f = \frac{\partial^{m_1} \ldots \partial^{m_p} f(\mathbf{w})}{\partial w_1^{m_1} \ldots \partial w_p^{m_p}}, \tag{7}$$

and

$$\mathbf{w}^{\mathbf{m}} = \prod_{i=1}^{p} w_i^{m_i}. \tag{8}$$

The Taylor series expansion is given by

$$f(\mathbf{w}) = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\{\mathbf{m}: m_1 + \ldots + m_p = m\}} \binom{m}{m_1, \ldots, m_p} \times D^{\mathbf{m}} f(\mathbf{w}^o)(\mathbf{w} - \mathbf{w}^o)^{\mathbf{m}} \tag{9}$$

where $\binom{m}{m_1, \ldots, m_p}$ is the multinomial coefficient.

We define now the mth order tensor of the partial derivatives.

Definition 3.2:

$$\nabla^m f(\mathbf{w}) = \left[D^{[m,0,\ldots,0]} f(\mathbf{w}), \ldots, D^{[0,\ldots,0,m]} f(\mathbf{w}) \right]^T. \tag{10}$$

In the proof, we will assume several technical regularity conditions.

A1) The Hessian of $J^{\infty}(\mathbf{w})$ is (strictly) positive definite at $\mathbf{w}^o$.

A2) For all $\mathbf{w}$

$$\lim_{\|\mathbf{x}\| \to \infty} |J(\mathbf{w}, \mathbf{x}))| = \infty.$$

A3) All the third-order derivatives of $J(\mathbf{w}, \mathbf{x})$ with respect to $\mathbf{w}$ are continuous and bounded at $\mathbf{w}^o$, i.e.,

$$\left| \frac{\partial^3 J(\mathbf{w}^o, \mathbf{x})}{\partial w_k \partial w_l \partial w_m} \right| \leq M(\mathbf{x}), \quad k, l, m = 1, \cdots, p$$

for some integrable function $M(\mathbf{x})$ for which $E_{\mathbf{x}}(M(\mathbf{x})) < \infty$ and $E_{\mathbf{x}}(M(\mathbf{x})^2) < \infty$.

A4) For each $\mathbf{w}$, let

$$D_3(\mathbf{w}, \mathbf{x}) = 8 \sum_{m=3}^{\infty} \frac{p^m}{2^m m!} \|\nabla^m J(\mathbf{w}, \mathbf{x})\|_1 \tag{11}$$

where $\nabla^m J$ is the tensor of mth-order partial derivatives of $J(\mathbf{w}, \mathbf{x})$. We assume that $D_3(\mathbf{w}^o, \mathbf{x})$ is a random variable with a finite variance

$$V_D(\mathbf{w}^o) = \mathrm{var}_{\mathbf{x}} \left(D_3(\mathbf{w}^o, \mathbf{x}) \right) < \infty. \tag{12}$$

A5) The probability $\lim_{r \to \infty} P(\|\mathbf{x}\| > r) = 0$.

Assumption A1) holds in many cases of interest. For example, for the cost function (5), it is shown in [9] that (when all sources have kurtosis less than 2) the Hessian is positive definite at a stationary point only if it corresponds to a desired weight vector (i.e., one that separates the sources). This also implies that the Hessian of the ordinary CMA(2,2) cost function is positive definite at the local minima. In particular, we assume that the Hessian matrix is nonsingular. In the noiseless case this implies that the number of signals is equal to the number of sensors.[1] This limitation is artificial and follows from the fact that if there are more sensors than sources, there are no true local minima since adding to $\mathbf{w}$ a component in the direction of the noise subspace (the subspace orthogonal to the column span of $\mathbf{A}$) will not change the value of J^{∞}. To overcome it one should note that by the results of [21] the CM receivers are all in the signal subspace (the column span of $\mathbf{A}$) even in the noisy case. Hence, a sensible first step would be to project the sensors data onto the signal subspace, e.g., as estimated from the second order statistics of the data $\mathbf{X}$.

Assumptions A2)–A5) are mild and used to ensure that the second-order approximation of $J(\mathbf{w}, \mathbf{x})$ has a uniformly bounded error provided we throw away certain realizations of $\mathbf{x}$ that have arbitrarily small probability of occurrence. Note that for $A2$ to hold, we need to limit ourselves to receivers in the signal subspace. However, as discussed above, this does

[1] And that a suitable phase normalization has been used, since without it, beamformers are unique only up to a unimodular scalar.

not pose any limitation on the applicability of the results. Furthermore, its only use, is to simplify some arguments and it might be dropped at the price of somewhat more complicated proof. Since we are mainly concerned with local minima we can always bound $\mathbf{w}$ to a compact sphere around $\mathbf{w}^o$, so that without loss of generality Assumption 3 will hold with respect to $\mathbf{w}$. Regarding the moments of $M(\mathbf{X})$ we should notice that the assumption is very mild. For example, if $J(\mathbf{w},\mathbf{x})$ is a polynomial function (e.g., originating from cumulants) and the moment generating function of $\mathbf{x}$ exists then by the Chernoff bound the tail of the probability density function (pdf) of $\mathbf{x}$ decays exponentially and all the relevant moments exist. Actually existence of the variance of the third-order derivatives at $\mathbf{w}^o$ will suffice. A4) holds for many cases of interest. An important case where A4) holds is when $J(\mathbf{w},\mathbf{x}) = J(y)$ where $y = \mathbf{w}^T\mathbf{x}$, $J(y)$ is a polynomial of degree s and the moments of $\mathbf{x}$ of order $2s$ exist the assumption holds. This covers, e.g., all the interesting cumulant-based source separation techniques. For this assumption to hold it is sufficient that the variance of the norm of the mth-order derivatives vector grows sub-exponentially with m, since $p^m/m!$ decays to 0. This holds for example whenever each component grows sub-exponentially. A5) always holds when $\mathbf{x}$ is a random variable which has a finite first-order moment.

Theorem 3.1: Let $\mathbf{w}^N$ be the local minimum of $J^N(\mathbf{w},\mathbf{X})$ closest to $\mathbf{w}^o$, and assume that $J^N(\mathbf{w},\mathbf{X})$ and $\mathbf{x}$ satisfy Assumptions A1)–A5). Then, for every $r > 0$ and any $\varepsilon > 0$, there is $N = N(r,\varepsilon)$ such that $P(\|\mathbf{w}^N - \mathbf{w}^o\| < r) > 1-\varepsilon$. Moreover, for all $r < 1/2$ and ε, we can choose[2]

$$N(r,\varepsilon) = \max\left\{\frac{10E_{\mathbf{x}}\left(\|\nabla J(\mathbf{w}^o,\mathbf{x})\|^2\right)}{\lambda_{\min}^2 r^2\varepsilon}, N_0(\varepsilon), N_\delta(\varepsilon)\right\}$$

where $\nabla J(\mathbf{w}^o,\mathbf{x})$ is the gradient of $J(\mathbf{w},\mathbf{x})$ evaluated at $\mathbf{w}^o$, $\lambda_{\min}$ is the smallest eigenvalue of the Hessian $\mathbf{H}^\infty(\mathbf{w})$ of $J^\infty(\mathbf{w})$ evaluated at $\mathbf{w}^o$ and $N_0(\varepsilon) = 2560V_D(\mathbf{w}^o)/\lambda_{\min}^2\varepsilon$ depends only on ε but not on r, where $V_D(\mathbf{w}^o)$ is defined in (12). $N_\delta(\varepsilon) = 40V_H(\mathbf{w}^o)/\varepsilon\lambda_{\min}^2$, where

$$V_H(\mathbf{w}^o) = \sum_{i=1}^{p}\sum_{j=1}^{p} \mathrm{var}_{\mathbf{x}}\left(\mathbf{H}_{ij}(\mathbf{w}^o,\mathbf{x})\right)$$

and $\mathbf{H}_{ij}(\mathbf{w},\mathbf{x})$ is the i,jth element of the Hessian $\mathbf{H}(\mathbf{w},\mathbf{x})$ of $J(\mathbf{w},\mathbf{x})$.

In contrast to existing asymptotic analysis our bounds provide simply computable bounds. Before we continue, we explain the role of the constants $N_0(\varepsilon), N_\delta(\varepsilon)$. These constants depend only on ε and not on r so they do not affect the asymptotic behavior with respect to r, and for small r the first term dominates. $N_0(\varepsilon)$ provides the number of samples required to ensure that the probability that the lowest eigenvalue of the sample Hessian $\mathbf{H}^N(\mathbf{w},\mathbf{X})$ is above $\lambda_{\min}/2$ with probability at least $1-(\varepsilon/10)$ while $N_\delta(\varepsilon)$ is the number of samples required to bound the error in the second-order Taylor expansion of $J^N(\mathbf{w},\mathbf{X})$ around $\mathbf{w}^o$ by a third-order polynomial $(\lambda_{\min}/16)\|\mathbf{w}-\mathbf{w}^o\|^3$ with probability at least $1-(\varepsilon/10)$. To simplify our analysis, we will always be interested in the case where $r \le 1/2$. This does not limit the scope of the theorem since for $1/2 < r$ we can use the number of samples required when $r = 1/2$.

[2]This expression corrects an error in [1].

B. Proof of Theorem 3.1

The proof of Theorem 3.1 is divided into two parts. First, we study the location of local minima of $J_2^N(\mathbf{w},\mathbf{X})$ which is the second-order approximation to $J^N(\mathbf{w},\mathbf{X})$ around $\mathbf{w}^o$. Then, we show that if $r < 1/2$, the existence of a local minimum of J_2^N in a sphere of radius r ensures the existence of a local minimum of J^N in the same sphere, and combine these two results.

The proof uses a union bound type of argument. First, we exclude two small sets of possible values of $\mathbf{X}$; $\mathcal{E}_1, \mathcal{E}_2$. We will show that given that $N \ge \max\{N_0(\varepsilon), N_\delta(\varepsilon)\}$ each of these sets has probability smaller than $\varepsilon/10$. Then, we will show that given that $N \ge 10E_{\mathbf{x}}(\|\nabla J(\mathbf{w}^o,\mathbf{x})\|^2)/\lambda_{\min}^2 r^2\varepsilon$ and $r < 1/2$ the probability that $J^N(\mathbf{w},\mathbf{X})$ does not have a local minimum within distance r from $\mathbf{w}^o$ is lower than $8\varepsilon/10$ *provided that* $\mathbf{X} \notin \mathcal{E}_1 \cup \mathcal{E}_2$. The proof will be finished since the probability that there is no local minimum within distance r is less than

$$P(\mathbf{X}\in\mathcal{E}_1) + P(\mathbf{X}\in\mathcal{E}_2) + P(\nexists \text{ a local minimum within radius } r \wedge \mathbf{X}\notin \mathcal{E}1\cup\mathcal{E}_2) \le \frac{\varepsilon}{10}+\frac{\varepsilon}{10}+\frac{8\varepsilon}{10} = \varepsilon. \quad (13)$$

Let $J_2^N(\mathbf{w},\mathbf{X})$ be the second-order Taylor approximation of $J^N(\mathbf{w},\mathbf{X})$ around $\mathbf{w}^o$

$$J_2^N(\mathbf{w},\mathbf{X}) = J^N(\mathbf{w}^o,\mathbf{X}) + \left[\nabla J^N(\mathbf{w}^o,\mathbf{X})\right]^T(\mathbf{w}-\mathbf{w}^o) + \frac{1}{2}(\mathbf{w}-\mathbf{w}^o)^T\mathbf{H}^N(\mathbf{w}^o,\mathbf{X})(\mathbf{w}-\mathbf{w}^o) \quad (14)$$

where ∇J^N is the gradient vector and $\mathbf{H}^N$ the Hessian matrix of J^N. It follows that $J_2^N(\mathbf{w},\mathbf{X})$ has a local minimum at

$$\mathbf{w}_2^N = \mathbf{w}^o - \left[\mathbf{H}^N(\mathbf{w}^o,\mathbf{X})\right]^{-1}\nabla J^N(\mathbf{w}^o,\mathbf{X}) \quad (15)$$

provided that $\mathbf{H}^N(\mathbf{w}^o,\mathbf{X})$ is positive definite. By assumption, $\mathbf{H}^\infty(\mathbf{w}^o)$ is positive definite. Therefore, for sufficiently large N, $\mathbf{H}^N(\mathbf{w}^o,\mathbf{X})$ is also positive definite with high probability. This follows from the convergence in probability of the eigenvalues of $\mathbf{H}^N(\mathbf{w}^o,\mathbf{X})$ to those of $\mathbf{H}^\infty(\mathbf{w}^o)$, which holds for any ergodic source $\mathbf{X}$. To better quantify the probability that $\mathbf{H}^N(\mathbf{w}^o,\mathbf{X})$ is also positive definite in the i.i.d. case, we will bound the probability that $|\lambda_{\min}^N - \lambda_{\min}| < (\lambda_{\min}/2)$.

Lemma 3.2: Let $N_\delta(\varepsilon) \ge 40\sum_{i,j=1}^{p}\mathrm{var}_{\mathbf{x}}(\mathbf{H}_{ij}(\mathbf{w}^o,\mathbf{x}))/\varepsilon\lambda_{\min}^2$. Then

$$P\left(\left|\lambda_{\min}^N - \lambda_{\min}\right| > \frac{\lambda_{\min}}{2}\right) < \frac{\varepsilon}{10}.$$

To prove the lemma, we will use the following lemma which is an immediate consequence of Weyl's theorem [38], using the symmetry of the Hessian and the sample Hessian (remember that $J(\mathbf{w},\mathbf{x})$ has continuous second-order derivatives).

Lemma 3.3: For every N, let $\lambda_{\min}, \lambda_{\min}^N$ be the minimal eigenvalues of $\mathbf{H}^\infty(\mathbf{w}^o)$ and $\mathbf{H}^N(\mathbf{w}^o,\mathbf{x})$, respectively. Then

$$\left|\lambda_{\min}^N - \lambda_{\min}\right| \le \left\|\mathbf{H}^N(\mathbf{w}^o,\mathbf{x}) - \mathbf{H}^\infty(\mathbf{w}^o)\right\|_2. \quad (16)$$

Proof (Lemma 3.2): Using Lemma 3.3, we obtain that

$$P\left(\left|\lambda_{\min}^N - \lambda_{\min}\right| > \frac{\lambda_{\min}}{2}\right) \leq P\left(\left\|\mathbf{H}^N(\mathbf{w}^o,\mathbf{x}) - \mathbf{H}^\infty(\mathbf{w}^o)\right\|_2 > \frac{\lambda_{\min}}{2}\right). \quad (17)$$

Therefore

$$P\left(\left|\lambda_{\min}^N - \lambda_{\min}\right| > \frac{\lambda_{\min}}{2}\right) \leq P\left(\left\|\mathbf{H}^N(\mathbf{w}^o,\mathbf{x}) - \mathbf{H}^\infty(\mathbf{w}^o)\right\|_2^2 > \frac{\lambda_{\min}^2}{4}\right). \quad (18)$$

Since for every matrix $\mathbf{A}$ we have $\|\mathbf{A}\|_2^2 \leq \|\mathbf{A}\|_F^2$ and when there are two nonzero singular values strict inequality holds (which holds in our case by positive definiteness) we obtain that

$$P\left(\left|\lambda_{\min}^N - \lambda_{\min}\right| > \frac{\lambda_{\min}}{2}\right) < P\left(\left\|\mathbf{H}^N(\mathbf{w}^o,\mathbf{x}) - \mathbf{H}^\infty(\mathbf{w}^o)\right\|_F^2 > \frac{\lambda_{\min}^2}{4}\right). \quad (19)$$

Note that

$$\left\|\mathbf{H}^N(\mathbf{w}^o,\mathbf{x}) - \mathbf{H}^\infty(\mathbf{w}^o)\right\|_F^2 = \sum_{i,j=1}^{p}\left(\mathbf{H}_{ij}^N(\mathbf{w}^o,\mathbf{x}) - \mathbf{H}_{ij}^\infty(\mathbf{w}^o)\right)^2.$$

Since $E(\mathbf{H}_{ij}(\mathbf{w},\mathbf{x})) = \mathbf{H}_{ij}^\infty(\mathbf{w})$ and using Markov's inequality applied to the right side, we obtain that

$$P\left(\left|\lambda_{\min}^N - \lambda_{\min}\right| > \frac{\lambda_{\min}}{2}\right) < \frac{4\sum_{i,j=1}^{p}\sum_{n=1}^{N}\operatorname{var}_{\mathbf{x}_n}\left(\mathbf{H}_{ij}(\mathbf{w}^o,\mathbf{x}_n)\right)}{N^2\lambda_{\min}^2} \quad (20)$$

By the i.i.d assumption on $\mathbf{x}_n$ we obtain that

$$P\left(\left|\lambda_{\min}^N - \lambda_{\min}\right| > \frac{\lambda_{\min}}{2}\right) < \frac{4\sum_{i,j=1}^{p}\operatorname{var}_{\mathbf{x}}\left(\mathbf{H}_{ij}(\mathbf{w}^o,\mathbf{x})\right)}{N\lambda_{\min}^2} \quad (21)$$

Choosing $N_\delta(\varepsilon) \geq (40\sum_{i,j=1}^{p}\operatorname{var}_{\mathbf{x}}(\mathbf{H}_{ij}(\mathbf{w}^o,\mathbf{x}))/\varepsilon\lambda_{\min}^2)$ and substituting into (21) yields the conclusion of Lemma 3.2. □

By Lemma 3.2, we obtain that for all $N > N_\delta(\varepsilon)$ and with probability higher than $1-\varepsilon/10$

$$\frac{\lambda_{\min}}{2} < \lambda_{\min}^N < \frac{3\lambda_{\min}}{2}. \quad (22)$$

Let

$$\mathcal{E}_1 = \left\{\mathbf{X} : \lambda_{\min}^N < \frac{\lambda_{\min}}{2}\right\} = \left\{\mathbf{X} : \frac{1}{\lambda_{\min}^N} > \frac{2}{\lambda_{\min}}\right\}$$

then $P(\mathcal{E}_1) < \varepsilon/10$. From now on we assume that $\mathbf{X} \notin \mathcal{E}_1$.

To say something about $\mathbf{w}_2^N - \mathbf{w}^o$ based on (15), let $\lambda_{\min}^N$ be the smallest eigenvalue of $\mathbf{H}^N(\mathbf{w}^o,\mathbf{X})$. In this case, the maximal eigenvalue of $[\mathbf{H}^N(\mathbf{w}^o,\mathbf{X})]^{-1}$ is given by $1/\lambda_{\min}^N$. Since $\lambda_{\min}(\mathbf{A}) = \lambda_{\max}(\mathbf{A}^{-1})$ the same value of $N_\delta(\varepsilon)$ ensures that with probability less than $\varepsilon/10$

$$\lambda_{\max}\left(\left(\mathbf{H}^N(\mathbf{w}^o,\mathbf{x})\right)^{-1}\right) < 2\lambda_{\max}\left(\left(\mathbf{H}^\infty(\mathbf{w}^o)\right)^{-1}\right).$$

Substituting into (15) we obtain (for $\mathbf{X} \notin \mathcal{E}_1$)

$$\left\|\mathbf{w}_2^N - \mathbf{w}^o\right\|^2 < \frac{2\left\|\nabla J^N(\mathbf{w}^o,\mathbf{X})\right\|^2}{\lambda_{\min}^2} \quad (23)$$

so that

$$P\left(\left\|\mathbf{w}_2^N - \mathbf{w}^o\right\| > r \wedge \mathbf{X} \notin \mathcal{E}_1\right) < P\left(\frac{\sqrt{2}\left\|\nabla J^N(\mathbf{w}^o,\mathbf{X})\right\|}{\lambda_{\min}} > r\right).$$

Assuming that the $\mathbf{x}_k$ are i.i.d., and using the fact that $\mathbf{w}^o$ is a local minimum of $J^\infty(\mathbf{w})$, we obtain that for all k

$$E_{\mathbf{x}_k}\left(\nabla J(\mathbf{w}^o,\mathbf{x}_k)\right) = 0.$$

Therefore (using the fact that $\mathbf{x}_k$ are i.i.d)

$$E_{\mathbf{X}}\left(\left\|\nabla J^N(\mathbf{w}^o,\mathbf{X})\right\|^2\right) = \frac{1}{N^2}\sum_{k=1}^{N}E_{\mathbf{x}_k}\left(\|\nabla J(\mathbf{w}^o,\mathbf{x}_k)\|^2\right) = \frac{1}{N}E_{\mathbf{x}}\left(\|\nabla J(\mathbf{w}^o,\mathbf{x})\|^2\right). \quad (24)$$

Using Markov's inequality and the positivity of $\|\nabla J^N(\mathbf{w}^o,\mathbf{x})\|$ we conclude that

$$P\left(\left\|\mathbf{w}_2^N - \mathbf{w}^o\right\| > r \wedge \mathbf{X} \notin \mathcal{E}_1\right) < \frac{2E_{\mathbf{x}}\left(\|\nabla J(\mathbf{w}^o,\mathbf{x})\|^2\right)}{\lambda_{\min}^2 N r^2}. \quad (25)$$

For given (ϵ', r), setting N such that

$$N \geq \frac{2E_{\mathbf{x}}\left(\|\nabla J(\mathbf{w}^o,\mathbf{x})\|^2\right)}{\lambda_{\min}^2\varepsilon' r^2} \quad (26)$$

yields the desired result for the local minima of J_2^N, i.e., $P(\|\mathbf{w}_2^N - \mathbf{w}^o\| > r \wedge \mathbf{X} \notin \mathcal{E}_1) < \varepsilon'$. Moreover, $P(\|\mathbf{w}_2^N - \mathbf{w}^o\| > r) < \varepsilon' + \varepsilon/10$.

The second part of the proof is to transfer this result to J^N: To show that for $r < 1/2$ there is a local minimum $\mathbf{w}^N$ of $J^N(\mathbf{w},\mathbf{X})$ sufficiently close to $\mathbf{w}^o$, i.e., within the sphere $\|\mathbf{w} - \mathbf{w}^o\| < r$ with probability better than $1-\epsilon$. To this end, we will apply (25) to a radius $r/2$, i.e., choose N such that $\|\mathbf{w}_2^N - \mathbf{w}^o\| < r/2$ (with probability better than $1-\epsilon$), and simultaneously consider all $\mathbf{w}$ such that $\|\mathbf{w} - \mathbf{w}_2^N\| = r/2$. It then follows that $\|\mathbf{w} - \mathbf{w}^o\| \leq r$ and it only remains to show that there is a local minimum of J^N inside this sphere.

Thus let $\varepsilon' = 8\varepsilon/10$ and $r < (1/2)$. Choose N large enough such that [cf. (25)]

$$P\left(\left\|\mathbf{w}_2^N - \mathbf{w}^o\right\| > \frac{r}{2} \wedge \mathbf{X} \notin \mathcal{E}_1\right) < \frac{8E_{\mathbf{x}}\|\nabla J(\mathbf{w}^o,\mathbf{x})\|^2}{\lambda_{\min}^2 N r^2} = \varepsilon' = \frac{8\varepsilon}{10}.$$

The argument is based on two inequalities. For the first, note that the Taylor approximation (14) implies that

$$J^N(\mathbf{w},\mathbf{X}) = J_2^N(\mathbf{w},\mathbf{X}) + O\left(\|\mathbf{w} - \mathbf{w}^o\|^3\right). \quad (27)$$

Equivalently, (for $\mathbf{X} \notin \mathcal{E}_1$) there is a constant $c(\mathbf{X})$ such that for every $\mathbf{w}$ we have

$$\left| J^N(\mathbf{w}, \mathbf{X}) - J_2^N(\mathbf{w}, \mathbf{X}) \right| < c(\mathbf{X}) \|\mathbf{w} - \mathbf{w}^o\|^3. \tag{28}$$

This follows from the fact that we are interested in a compact ball around $\mathbf{w}^0$ (since $\|\mathbf{w} - \mathbf{w}^o\| < 1/2$) and using the Lagrange bound on the tail of the Taylor expansion, together with Assumption A3). The following lemma proves that by eliminating arbitrarily small subset of $\mathbf{X}$'s and choosing N sufficiently large we can assume without loss of generality that $c(\mathbf{X})$ can be chosen independently of r and $\mathbf{X}$.

Lemma 3.4: Let $\Delta^N(\mathbf{w}, \mathbf{X}) = |J^N(\mathbf{w}, \mathbf{X}) - J_2^N(\mathbf{w}, \mathbf{X})|$. Let $\zeta > 0$, $\|\mathbf{w} - \mathbf{w}^o\| < 1/2$. Let c be given. Then there exists $N_0(\zeta, c)$ such that for all $N > N_0(\zeta, c)$

$$P\left(\Delta^N(\mathbf{w}, \mathbf{X}) > c\|\mathbf{w} - \mathbf{w}^o\|^3\right) < \zeta.$$

Furthermore, we can choose $N_0(\zeta, c) = V_D(\mathbf{w}^o)/c^2\zeta$.

The proof uses the boundedness and continuity of the third and higher order derivatives of $J(\mathbf{w}, \mathbf{x})$ (Assumptions A2)–A4)). The details are given in Appendix I.

To use the lemma, let

$$c = \frac{\lambda_{\min}}{16} \tag{29}$$

and $\zeta = \varepsilon/10$. By the lemma we obtain that

$$P\left(\Delta^N(\mathbf{w}, \mathbf{X}) > \frac{\lambda_{\min}}{16}\|\mathbf{w} - \mathbf{w}^o\|^3\right) < \varepsilon/10 \tag{30}$$

for all $N \geq N_0(\varepsilon) = 2560 V_D(\mathbf{w}^o)/\lambda_{\min}^2 \varepsilon$.

Hence, by requiring $N_0(\varepsilon) < N$, we can ignore a small set of realizations of $\mathbf{X}$, $\mathcal{E}_2$, such that $P(\mathcal{E}_2) < \varepsilon/10$. For every $\mathbf{X} \notin \mathcal{E}_2$, we have $\Delta^N(\mathbf{w}, \mathbf{X}) < c\|\mathbf{w} - \mathbf{w}^o\|^3$. Let $\mathcal{E}_3 = \mathcal{E}_1 \cup \mathcal{E}_2$. Then $P(\mathcal{E}_3) < 2\varepsilon/10$. As before, from now on we assume that $\mathbf{X} \notin \mathcal{E}_3$. Since $\mathbf{X} \notin \mathcal{E}_3$, $\Delta^N(\mathbf{w}, \mathbf{X}) < c\|\mathbf{w} - \mathbf{w}^o\|^3 < cr^3$ for all $\mathbf{w}$ such that $\|\mathbf{w} - \mathbf{w}^o\| \leq r \leq 1/2$. The situation (outside $\mathcal{E}_3$) is illustrated by the large circle in Fig. 1.

For a second inequality, we consider the expression (14) for J_2^N, but centered at its minimum $\mathbf{w}_2^N$, which gives

$$J_2^N(\mathbf{w}, \mathbf{X}) = J_2^N\left(\mathbf{w}_2^N, \mathbf{X}\right) + \frac{1}{2}\left(\mathbf{w} - \mathbf{w}_2^N\right)^T \mathbf{H}^N \left(\mathbf{w} - \mathbf{w}_2^N\right)$$

so that

$$J_2^N(\mathbf{w}, \mathbf{X}) - J_2^N\left(\mathbf{w}_2^N, \mathbf{X}\right) \geq \frac{1}{2}\lambda_{\min}^N \left\|\mathbf{w} - \mathbf{w}_2^N\right\|^2. \tag{31}$$

This is illustrated by the second largest circle in Fig. 1.

We now consider all $\mathbf{w}$ on the sphere $\|\mathbf{w} - \mathbf{w}_2^N\| = r/2$. With probability $1-\epsilon$, we also have $\|\mathbf{w}_2^N - \mathbf{w}^o\| \leq (r/2)$, and hence $\|\mathbf{w} - \mathbf{w}^o\| \leq r$ with probability $1 - \varepsilon$. For $\mathbf{X} \notin \mathcal{E}_3$ both (28), (31) hold, and hence we obtain (for $\mathbf{X} \notin \mathcal{E}_3$)

$$\begin{aligned}
&J^N(\mathbf{w}, \mathbf{X}) - J^N\left(\mathbf{w}_2^N, \mathbf{X}\right) \\
&\quad = J^N(\mathbf{w}, \mathbf{X}) - J_2^N(\mathbf{w}, \mathbf{X}) + J_2^N(\mathbf{w}, \mathbf{X}) - J^N\left(\mathbf{w}_2^N, \mathbf{X}\right) \\
&\quad \geq J_2^N(\mathbf{w}, \mathbf{X}) - J^N\left(\mathbf{w}_2^N, \mathbf{X}\right) - \left|J^N(\mathbf{w}, \mathbf{X}) - J_2^N(\mathbf{w}, \mathbf{X})\right| \\
&\quad \geq J_2^N(\mathbf{w}, \mathbf{X}) - J^N\left(\mathbf{w}_2^N, \mathbf{X}\right) - cr^3 \\
&\quad \geq J_2^N(\mathbf{w}, \mathbf{X}) - J_2^N\left(\mathbf{w}_2^N, \mathbf{X}\right) \\
&\quad\quad + J_2^N\left(\mathbf{w}_2^N, \mathbf{X}\right) - J^N\left(\mathbf{w}_2^N, \mathbf{X}\right) - cr^3 \\
&\quad \geq J_2^N(\mathbf{w}, \mathbf{X}) - J_2^N\left(\mathbf{w}_2^N, \mathbf{X}\right) \\
&\quad\quad - \left|J_2^N\left(\mathbf{w}_2^N, \mathbf{X}\right) - J^N\left(\mathbf{w}_2^N, \mathbf{X}\right)\right| - cr^3 \\
&\quad \geq \frac{1}{8}\lambda_{\min}^N r^2 - 2cr^3 \\
&\quad \geq \frac{1}{16}\lambda_{\min} r^2 - 2cr^3
\end{aligned} \tag{32}$$

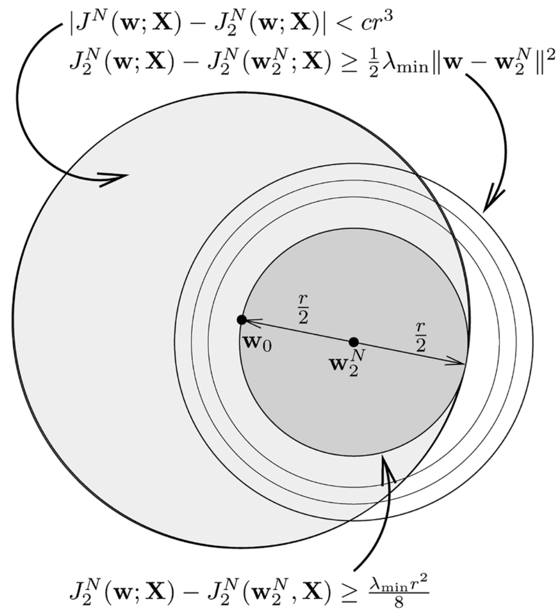

Fig. 1. Local behavior of J^N and J_2^N around $\mathbf{w}^o$.

where the last inequality follows from the assumption that $\mathbf{X} \notin \mathcal{E}_1$. This is positive for all $r < 1/2$, by the choice of c in (29) or equivalently, for all $N \geq N(r, \varepsilon)$. Hence, $J^N(\mathbf{w}_2^N, \mathbf{X})$ is smaller than $J^N(\mathbf{w}, \mathbf{X})$ for any $\mathbf{X} \notin \mathcal{E}_3$ and $\mathbf{w}$ such that $\|\mathbf{w} - \mathbf{w}_2^N\| = r/2$. Therefore, the sphere of radius $r/2$ around $\mathbf{w}_2^N$, $B(\mathbf{w}_2^N, r/2) = \{\mathbf{w} : \|\mathbf{w} - \mathbf{w}_2^N\| \leq r/2\}$ contains a point where J^N is smaller than all values on the boundary. Hence, there must be a local minimum $(\mathbf{w}^N)$ of $J^N(\mathbf{w}, \mathbf{X})$ inside this sphere.

By our choice of N, r we have $\|\mathbf{w}_2^N - \mathbf{w}^o\| \leq r/2$ for all $\mathbf{X} \notin \mathcal{E}_3$. The triangle inequality now implies that this local minima $\mathbf{w}^N$ is within distance r from $\mathbf{w}^o$ with probability better than $1 - \epsilon'$ (assuming $\mathbf{X} \notin \mathcal{E}_3$). Finally, we need to take into account the fact that ε' was chosen ignoring $\mathcal{E}_3$ which has probability $2\varepsilon/10$, so substituting $\varepsilon' = 8\varepsilon/10$ or $\varepsilon = 10\varepsilon'/8$ and using a union bound (13) to take into account the probability that $\mathbf{X} \in \mathcal{E}_3$ finishes the proof. Note that this constant could have been improved by choosing higher values of N_δ and N_0 and reducing $\mathcal{E}_1$ and $\mathcal{E}_2$.

IV. MSE Properties of LS-CMA Receivers

A. The Location of Local Minima

The theorem in the previous section is a general result for a wide class of cost functions. We will now specialize this to the LS-CMA, and combine Theorem 3.1 with the results of Zeng *et al.* [20], [23] to obtain some information about the MSE properties of LS-CMA receivers. More explicitly, Theorem 1 of [23]

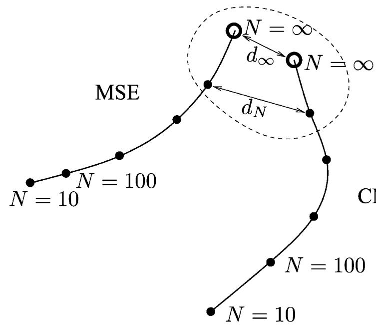

Fig. 2. Convergence towards the MSE and CMA local minima.

provides sufficient conditions for the existence of a CMA receiver within a small neighborhood of the Wiener receiver. Furthermore it was shown that the excess MSE of the CMA receiver is a continuous function of the parameters of the problem (i.e., noise variance, coefficients of the mixing matrices, etc.).

Let $\mathbf{w}_{\text{wiener}}$ be the Wiener solution for the source separation problem, i.e.,

$$\mathbf{w}_{\text{wiener}} = \arg\min_{\mathbf{w}} J^{\infty}_{\text{MSE}}(\mathbf{w}) = \arg\min_{\mathbf{w}} \sum_{n=1}^{N} \|\mathbf{w}^H\mathbf{x}_n - \mathbf{s}_n\|^2 \tag{33}$$

and let $\mathbf{w}^o$ be the local minimum of the CMA cost function closest to $\mathbf{w}_{\text{wiener}}$. By Theorem 1 of [23], we know that there is a small positive number δ_U such that

$$J^{\infty}_{\text{MSE}}(\mathbf{w}^o) < J^{\infty}_{\text{MSE}}(\mathbf{w}_{\text{wiener}}) + \delta_U. \tag{34}$$

Furthermore, by our main theorem we know that for every $\varepsilon > 0$ and $r > 0$ there is an $N(r,\varepsilon)$ such that for all $N > N(r,\varepsilon)$

$$P\left(\exists \mathbf{w} : \|\mathbf{w}-\mathbf{w}^o\| < r \text{ and } \mathbf{w} \text{ is a local minimum of } J^N(\mathbf{w},\mathbf{X})\right) > 1-\varepsilon.$$

By continuity of $J^{\infty}_{\text{MSE}}(\mathbf{w})$, for every $\delta > 0$ we can choose r_δ sufficiently small such that if $\|\mathbf{w}^o - \mathbf{w}\| < r_\delta$ then

$$|J^{\infty}_{\text{MSE}}(\mathbf{w}) - J^{\infty}_{\text{MSE}}(\mathbf{w}^o)| < \delta.$$

Hence for every ε, $\delta > 0$ there is r_δ and $N(r_\delta,\varepsilon)$ such that for all $N > N(r_\delta,\varepsilon)$

$$P\left(\exists \mathbf{w}_N : J^{\infty}_{\text{MSE}}(\mathbf{w}_N) < \delta_U + \delta\right) > 1-\varepsilon. \tag{35}$$

This shows that for sufficiently large N, with arbitrarily high probability the local minima of $J^N_{\text{CMA}}(\mathbf{w})$ have good MSE properties.

Fig. 2 provides some insight into this result. We have high probability of obtaining good local minima of the finite cost function J^N in the vicinity of the local minima of J^∞. Hence, these local minima have good MSE properties.

B. The Required Number of Samples

We further illustrate Theorem 3.1 with a design example, where we compute the required number of samples N to reach a specified performance for the CMA$(2,2,N)$ least squares cost function. The computation is rather involved and to save space we will only demonstrate the main term which is dominating for any fixed ε as r becomes small. We assume that the sources are constant modulus, complex, circularly symmetric, with variance normalized to 1 (since a scaling can be absorbed in $\mathbf{A}$).

Thus let $J = J_{CM}$ be the CMA(2,2) cost function mentioned in (3). Since $\mathbf{w}$ is complex, it is convenient to use a complex gradient and Hessian, defined as follows [39]: let $\mathbf{c} = [\mathbf{w}^T, \bar{\mathbf{w}}^T]^T$, then

$$\nabla_{\mathbf{c}} J = \left(\frac{\partial J}{\partial \mathbf{c}}\right)^H = \begin{bmatrix} \left(\frac{\partial J}{\partial \mathbf{w}}\right)^H \\ \left(\frac{\partial J}{\partial \bar{\mathbf{w}}}\right)^H \end{bmatrix}$$

and

$$\mathbf{H}_{\mathbf{cc}} = \frac{\partial}{\partial \mathbf{c}}\left(\frac{\partial J}{\partial \mathbf{c}}\right)^H = \begin{bmatrix} \mathbf{H}_{\mathbf{ww}} & \mathbf{H}_{\bar{\mathbf{w}}\mathbf{w}} \\ \mathbf{H}_{\mathbf{w}\bar{\mathbf{w}}} & \mathbf{H}_{\bar{\mathbf{w}}\bar{\mathbf{w}}} \end{bmatrix},$$
$$\mathbf{H}_{\mathbf{ww}} = \frac{\partial}{\partial \mathbf{w}}\left(\frac{\partial J}{\partial \mathbf{w}}\right)^H, \quad \mathbf{H}_{\bar{\mathbf{w}}\mathbf{w}} = \frac{\partial}{\partial \bar{\mathbf{w}}}\left(\frac{\partial J}{\partial \mathbf{w}}\right)^H.$$

Here, we use the convention that the partial derivative with respect to a vector is a row vector, and we use Brandwood's conventions for the derivative with respect to a complex number [40]. The relations between the complex vector $\mathbf{c}$ and the real vector $\mathbf{r} = [\text{Re}(\mathbf{w})^T, \text{Im}(\mathbf{w})^T]^T$ as used in Theorem 3.1 are such that $\|\mathbf{c}\|^2 = 2\|\mathbf{r}\|^2$, and $\|\nabla J_{\mathbf{c}}\|^2 = (1/2)\|\nabla J_{\mathbf{r}}\|^2$. The relation between $\mathbf{H}_{\mathbf{cc}}$ and $\mathbf{H}_{\mathbf{rr}}$ is a two-sided linear transformation, such that the eigenvalues of $\mathbf{H}_{\mathbf{cc}}$ are one-half those of $\mathbf{H}_{\mathbf{rr}}$ [41].

In the case of CMA(2,2), the block entries of the complex gradient and Hessian are

$$\nabla J_{\mathbf{w}}(\mathbf{w},\mathbf{x}_k) = 2\left(|y_k|^2 - 1\right)\bar{y}_k\mathbf{x}_k \tag{36}$$
$$\mathbf{H}_{\mathbf{ww}}(\mathbf{w}) = 2\text{E}_{\mathbf{x}}\left(\left[2|y_k|^2 - 1\right]\mathbf{x}_k\mathbf{x}_k^H\right) \tag{37}$$
$$\mathbf{H}_{\bar{\mathbf{w}}\mathbf{w}}(\mathbf{w}) = 2\text{E}_{\mathbf{x}}\left(\bar{y}_k^2\mathbf{x}_k\mathbf{x}_k^T\right) \tag{38}$$

where $y_k = \mathbf{w}^H\mathbf{x}_k$ is the output of the beamformer. (The Hessian is evaluated for infinite samples, whereas the gradient is a random vector corresponding to the cost function for a single sample.) In Appendix II, we derive that, evaluated at $\mathbf{w}^o$,

$$\text{E}\left(\|\nabla J_{\mathbf{w}}(\mathbf{w},\mathbf{x}_k)\|^2\right) = 8\sigma^2\|\mathbf{w}^o\|^2\|\mathbf{A}\|_F^2 + O(\sigma^4) \tag{39}$$

where σ^2 is the noise power (for simplicity, $\mathbf{w}^o$ was approximated by the MMSE beamformer, which is known to be close to the CMA optimum). It follows that, in the context of Theorem 3.1,

$$\begin{aligned}\text{E}_{\mathbf{x}}\left(\|\nabla J_{\mathbf{r}}(\mathbf{w}^o,\mathbf{x}_k)\|^2\right) &= 2\text{E}_{\mathbf{x}}\left(\|\nabla J_{\mathbf{c}}(\mathbf{w}^o,\mathbf{x}_k)\|^2\right) \\ &= 4\text{E}_{\mathbf{x}}\left(\|\nabla J_{\mathbf{w}}(\mathbf{w}^o,\mathbf{x}_k)\|^2\right) \\ &= 32\sigma^2\|\mathbf{w}^o\|^2\|\mathbf{A}\|_F^2 \\ &\le 32\,K\sigma^2\|\mathbf{w}^o\|^2\sigma^2_{\max}\end{aligned} \tag{40}$$

where $\sigma_{\max}$ is the largest singular value of $\mathbf{A}$ and K is the number of sources.

Furthermore, in Appendix II it is shown that

$$\mathbf{H}_{\mathbf{ww}}(\mathbf{w}^o) = 2\mathbf{A}\mathbf{A}^H \tag{41}$$
$$\mathbf{H}_{\bar{\mathbf{w}}\mathbf{w}}(\mathbf{w}^o) = 2\mathbf{a}\mathbf{a}^T \tag{42}$$

where $\mathbf{a}$ is the column of $\mathbf{A}$ corresponding to the source selected by $\mathbf{w}^o$. Thus, the complex Hessian is

$$\mathbf{H_{cc}} = 2\begin{bmatrix} \mathbf{AA}^H & \mathbf{aa}^T \\ \bar{\mathbf{a}}\bar{\mathbf{a}}^T & \bar{\mathbf{A}}\mathbf{A}^T \end{bmatrix} = 2\begin{bmatrix} \mathbf{a} \\ \bar{\mathbf{a}} \end{bmatrix}[\mathbf{a}^H, \mathbf{a}^T] + 2\begin{bmatrix} \mathbf{A}'\mathbf{A}'^H & \\ & \bar{\mathbf{A}}'\mathbf{A}'^T \end{bmatrix} \quad (43)$$

where $\mathbf{A}'$ is $\mathbf{A}$ with column $\mathbf{a}$ taken away. The latter expression shows that $\mathbf{H_{cc}}$ has a 1-D null space. The explanation for this is that the optimal beamformer for CMA is not unique, we can always multiply $\mathbf{w}^o$ with a unimodular scalar α. The null space can be removed by placing a suitable phase normalization on $\mathbf{w}$. Assuming this has been done, the second smallest eigenvalue, $\lambda_2(\mathbf{H_{cc}})$ becomes relevant. Applying Weyl's theorem [38] to (43), it can be shown that

$$\lambda_2(\mathbf{H_{cc}}) \geq 2\lambda_1(\mathbf{AA}^H)$$

Thus, in the context of Theorem 3.1, for the smallest nonzero eigenvalue $\lambda_{\min}$ of $\mathbf{H_{rr}}(\mathbf{w}^o)$ we have

$$\lambda_{\min} = 2\lambda_2(\mathbf{H_{cc}}) \geq 4\sigma_{\min}^2$$

where $\sigma_{\min}$ is the smallest singular value of $\mathbf{A}$.

Let

$$L(\mathbf{w}) = \left\|\mathbf{w}^H\mathbf{A} - (\mathbf{w}^o)^H\mathbf{A}\right\|^2.$$

Since $\mathbf{w}^H\mathbf{a} \approx 1$, L can be interpreted as the loss in Signal to Interference Ratio (SIR) due to a finite-sample beamformer as opposed to the optimal infinite-sample beamformer. Note that

$$L(\mathbf{w}) \leq \|\mathbf{w} - \mathbf{w}^o\|^2\|\mathbf{A}\|^2$$

so that

$$P(L < L_{\max}) \geq P\left(\|\mathbf{w} - \mathbf{w}^o\|^2 < \frac{L_{\max}}{\sigma_{\max}^2}\right).$$

For a given $\mathbf{A}$, the design is now as follows. Choose an acceptable loss in SIR (i.e., $L_{\max}$), and an acceptable probability that this loss will be exceeded (ϵ); set $r^2 = L_{\max}/\sigma_{\max}^2$. According to Theorem 3.1, the required number of samples is

$$N(L_{\max}, \epsilon) = \frac{10E_{\mathbf{x}}\left(\|\nabla J(\mathbf{w}^o, \mathbf{x})\|^2\right)}{\lambda_{\min}^2 r^2\epsilon} \leq \frac{10 \cdot 32\, K\sigma^2\|\mathbf{w}^o\|^2\sigma_{\max}^4}{16\sigma_{\min}^4 L_{\max}\epsilon}.$$

Further define the Signal to Noise Ratio (SNR) at the input of the receiver as

$$\text{SNR} = \frac{\mathbf{a}^H\mathbf{a}}{\sigma^2}.$$

Note that $\mathbf{a}^H\mathbf{a} \leq \sigma_{\max}^2$. Also note that $\|\mathbf{w}^o\|^2 = [(\mathbf{A}^H\mathbf{A})^{-1}]_{1,1} + O(\sigma^4) \leq \sigma_{\min}^{-2}$, so that

$$\|\mathbf{w}^o\|^2\sigma^2 \leq \sigma_{\min}^{-2}\frac{\sigma_{\max}^2}{\text{SNR}}.$$

Thus, to reach the desired performance it is sufficient to set

$$N = \frac{20\, K}{\text{SNR} L_{\max}\epsilon} \cdot \left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^6.$$

Note that $\sigma_{\max}/\sigma_{\min}$ is equal to the condition number of $\mathbf{A}$. In conclusion, the conditioning of $\mathbf{A}$ very strongly determines the minimum number of samples to use, according to this technique.

V. Simulations for the CM Cost Function

To demonstrate the finite sample behavior of cost functions, we show the results of simulations for the CMA$(2, 2, N)$ approximation of the CM cost function.

In the first experiment we have mixed two random phase CM signals using the unitary matrix

$$\mathbf{A} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

and $\theta = 60^\circ$. We have added random Gaussian noise with signal to noise ratio (SNR) of 30 dB. For logarithmically spaced number of samples N between 100 and 10^5, we have initialized a minimization of the CM cost function with one of the zero forcing solutions (a column of $\mathbf{A}^{-H}$), and let it converge to the nearest local minimum using a Broyden–Fletcher–Goldfarb–Shanno (BFGS) quasi-Newton algorithm. The justification for the initialization is that for good SNR the CM beamformer is expected to be near the zero-forcing solution. For each value of N we have repeated the experiment 1000 times. The true (infinite sample) CM beamformer was estimated by averaging the locations of all the experiments with 10^5 many samples.

Fig. 3(a)–(d) presents the deviations of the estimated CM beamformers from the true CM beamformer, for various values of N. Each dot in the figure represents the two coordinates of the difference vector $\mathbf{w} - \mathbf{w}^o$ for a single experiment. It is seen that the locations of the local minima tend to concentrate around the true solution, as expected from Theorem 3.1, and become more accurate for increasing N.

Next we have turned to estimate the distribution of the local minima as a function of the number of samples N. Let r be the radius which ensures probability $1 - \varepsilon$ of finding a local minimum of the CMA$(2, 2, N)$ cost function close to the CM minimum. We expect to find a linear connection between the logarithm of r and the logarithm of N. To test this we have used the order statistics of r. Fig. 4 presents the tenth, fiftieth, and ninetieth percentiles of $-\log_{10}(r^2)$ as a function of $\log_{10} N$. We can clearly see that the lines are parallel and are linear. To verify the linearity we also computed the least squares fit, which are presented by the continuous lines.

To demonstrate the effect of the eigenvalue spread of the mixing matrix (which directly influences $\lambda_{\min}$ of the Hessian matrix of the cost function), we have repeated the above experiment with a mixing matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

This matrix has singular values 5.47, 0.37 and condition number 14.9. Fig. 5 presents the locations of the local minima. We can clearly see that the minima are now spread along an elongated

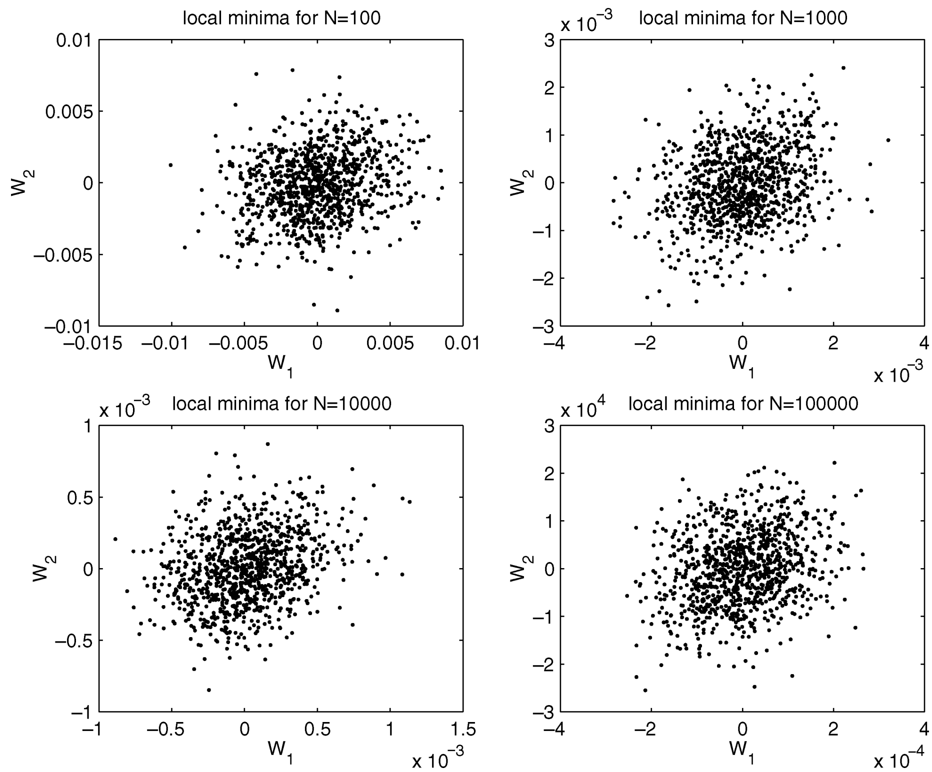

Fig. 3. Unitary mixing: Distribution of local minima for various values of N: (a) $N = 100$, (b) $N = 1000$, (c) $N = 10^4$, and (d) $N = 10^5$.

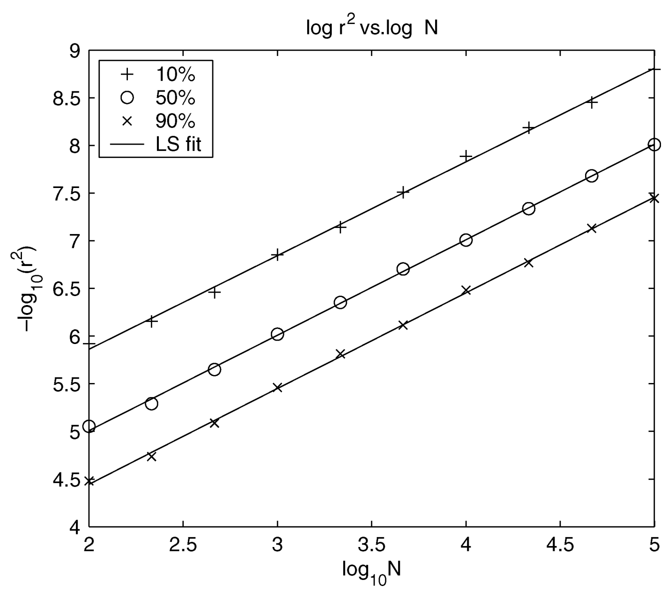

Fig. 4. Unitary mixing: Order statistics of the log distance to the CM beamformer as a function of N. The symbols "$+$", $\circ$, $*$ show the tenth, fiftieth, and ninetieth percentiles as function of $\log_{10} N$, respectively. The continuous lines are the LS fits to the estimated percentiles.

ellipsoid, due to the spread in singular values. Fig. 6 presents the order statistics, which again are linear.

Finally, to test the dependence between N and r for any given ε we have computed the coefficients a, b of the LS fit of $-\log_{10}(r^2) = a\log_{10}(N) + b$, as a function of the percentile ε. Fig. 7(a) and (b) describes the results for each simulation respectively. It is seen that the dependence is of the form $N = b(\varepsilon)/r^2$, i.e., the a coefficient is extremely close to 1 in all cases, and independent of ϵ. This completely agrees with the analysis of the relation between r^2 and N presented in the previous sections. The dependence of b on ε is not entirely as predicted. We expect that the accuracy of the theorem can be improved either using the central limit theorem or Chernoff type bounds. However, a complete characterization of the dependence on ε is beyond the scope of this paper. Note that for small values of ε b grows as $1/\varepsilon$. Finally, we comment on the two constants $N_0(\varepsilon), N_\delta(\varepsilon)$. Since we see that even for $N = 100$ the $1/r^2$ fits well for various values of ε, we conclude, that in reality these constants are not necessary even for moderate values of N. It is a shortcoming of our bounding technique, that requires these constants. Better optimization of these constants can probably be achieved using Chernoff bounds, assuming that the moment generating function of $\mathbf{x}$ exists.

VI. Conclusion

In this paper, we have discussed the location of local minima of finite sample approximations of general source separation cost functions. We derived the rate of convergence of the local minima to the local minima of the infinite sample cost function. While these results are not optimal, we are the first that provide effective bounds in the nonasymptotic case. Simulation results suggest that the main term $N(r, \varepsilon)$ is the leading term. We have also demonstrated how this can be used to evaluate the convergence behavior of the Least Squares CMA algorithm, which has been an open question for a long time. For this specific case we have shown the explicit dependence of the rate of convergence on the condition number of the mixing matrix. Finally, we have demonstrated the dependence on the condition number through simulated experiments.

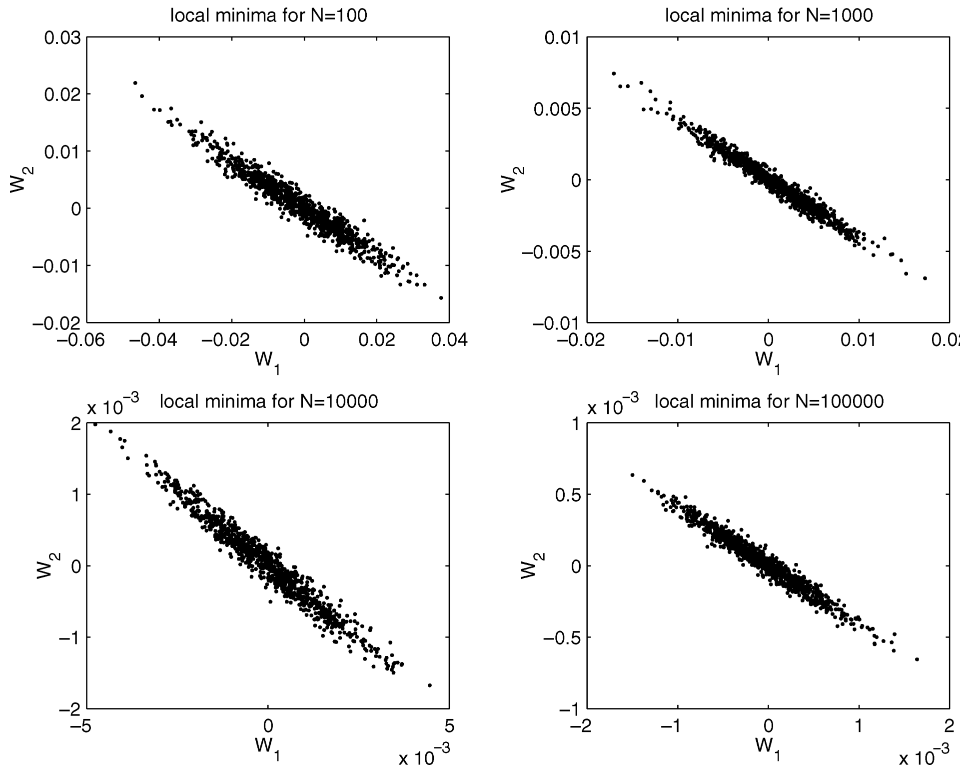

Fig. 5. Nonunitary mixing: Distribution of local minima for various values of N: (a) $N=100$, (b) $N=1000$, (c) $N=10^4$, and (d) $N=10^5$.

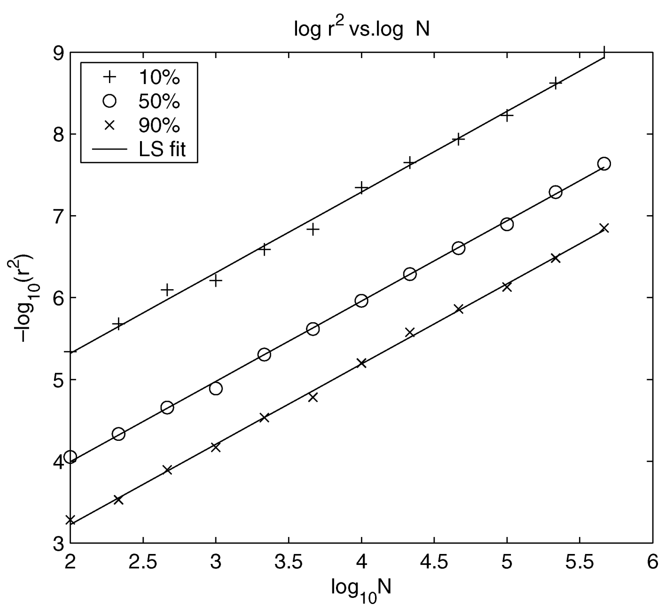

Fig. 6. Nonunitary mixing: Order statistics of the log distance to the CM beamformer as a function of N.

Appendix I

Bounding $\Delta^N(\mathbf{w},\mathbf{X})$

In this appendix, we prove Lemma 3.4.

Proof: The proof uses Assumption A4) that $D_3(\mathbf{w},\mathbf{x})$ has a finite variance. To simplify notation let $r=\|\mathbf{w}-\mathbf{w}^o\|$. Assume $r<1/2$. By definition the Taylor series expansion of $J(\mathbf{w},\mathbf{x})$ around $\mathbf{w}^o$ is given by

$$\begin{aligned}
&\Delta^N(\mathbf{w},\mathbf{X})\\
&\quad\le\Bigg|\frac{1}{N}\sum_{n=1}^{N}\sum_{m=3}^{\infty}\frac{1}{m!}\sum_{\{\mathbf{m}:m_1+,\ldots,+m_p=m\}}\frac{m!}{m_1!\ldots m_p!}\\
&\qquad\times D^{\mathbf{m}}J(\mathbf{w}^o)(\mathbf{w}^o,\mathbf{x}_n)(\mathbf{w}-\mathbf{w}^o)^{\mathbf{m}}\Bigg|\\
&\quad\le\frac{1}{N}\sum_{n=1}^{N}\sum_{m=3}^{\infty}\frac{1}{m!}r^m\sum_{\{\mathbf{m}:m_1+,\ldots,+m_p=m\}}\frac{m!}{m_1!\ldots m_p!}\\
&\quad\times\left|D^{\mathbf{m}}J(\mathbf{w}^o,\mathbf{x}_n)\right|\\
&\quad\le\frac{1}{N}\sum_{n=1}^{N}\sum_{m=3}^{\infty}8\frac{p^m}{2^m m!}\left\|\nabla^m J(\mathbf{w}^o,\mathbf{x}_n)\right\|_1 r^3.
\end{aligned}\tag{44}$$

To observe the first note that for every $\mathbf{m}$ such that $m_1+\ldots+m_p=m$, $|(\mathbf{w}-\mathbf{w}^o)^{\mathbf{m}}|\le r^m$. For the second inequality we need the following computations: $r^m\le r^3 2^{-(m-3)}=(8/2^m)r^3$ using the assumption that $r<1/2$. Let $a_{\mathbf{m}}=m!/m_1!\ldots m_p!$, $b_{\mathbf{m}}=\mathbf{D}^{\mathbf{m}}J(\mathbf{w}^o,\mathbf{x}_n)$. We have

$$\begin{aligned}
\sum_{\{\mathbf{m}:m_1+,\ldots,+m_p=m\}}|a_{\mathbf{m}}|&=\sum_{\{\mathbf{m}:m_1+,\ldots,+m_p=m\}}\frac{m!}{m_1!\ldots m_p!}\\
&=p^m,\\
\sum_{\{\mathbf{m}:m_1+,\ldots,+m_p=m\}}|b_{\mathbf{m}}|&=\sum_{\{\mathbf{m}:m_1+,\ldots,+m_p=m\}}\left|D^{\mathbf{m}}J(\mathbf{w}^o,\mathbf{x}_n)\right|\\
&=\left\|\nabla^m J(\mathbf{w}^o,\mathbf{x}_n)\right\|_1.
\end{aligned}$$

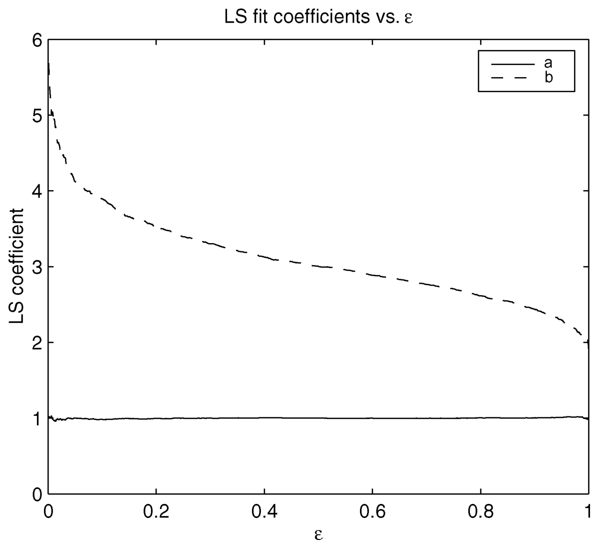

Fig. 7. Regression coefficients as a function of percentile. (a) Unitary mixing, (b) nonunitary mixing.

Using the fact that

$$\sum |a_{\mathbf{m}}||b_{\mathbf{m}}| \leq \left(\sum |a_{\mathbf{m}}|\right)\left(\sum |b_{\mathbf{m}}|\right)$$

and substituting into the second line of (44) we obtain the desired bound. By (11) we obtain that

$$\Delta^N(\mathbf{w}^o, \mathbf{X}) \leq \frac{1}{N}\sum_{n=1}^{N} D_3(\mathbf{w}^o, \mathbf{x}_n) r^3. \qquad (45)$$

Hence for every c

$$P\left(\Delta^N(\mathbf{w}, \mathbf{X}) > cr^3\right) \leq P\left(\frac{1}{N}\sum_{n=1}^{N} D_3(\mathbf{w}^o, \mathbf{x}_n) r^3 > cr^3\right). \qquad (46)$$

Therefore

$$P\left(\Delta^N(\mathbf{w}, \mathbf{X}) > cr^3\right) \leq P\left(\frac{1}{N}\sum_{n=1}^{N} D_3(\mathbf{w}^o, \mathbf{x}_n) > c\right). \qquad (47)$$

Using the i.i.d. property of $\mathbf{x}_i$ and the Chebyshev inequality we obtain

$$P\left(\Delta^N(\mathbf{w}, \mathbf{X}) > cr^3\right) \leq \frac{\mathrm{var}\left(D_3(\mathbf{w}^o, \mathbf{x})\right)}{Nc^2} = \frac{V_D(\mathbf{w}^o)}{Nc^2}. \qquad (48)$$

By definition of r this ends the proof of Lemma 3.4.

Finally we comment that when $J(\mathbf{w}, \mathbf{x}) = J(y)$ where $y = \mathbf{w}^T\mathbf{x}$ we can obtain tighter bounds using the fact that

$$\nabla^m J(\mathbf{w}^T\mathbf{x}) = (\mathbf{x} \otimes \cdots \otimes \mathbf{x}) \frac{d^m J}{dy^m}$$

where $\otimes$ denotes the Kronecker product. Hence, whenever the moment generating function of $J(y)$ exists, and Chernoff type bounds for $D_3(\mathbf{w}, \mathbf{x})$ can be derived. Furthermore we can clearly see that if $J(y)$ is a polynomial of degree s in y the infinite series become finite and the variance trivially exists as long as the signals and the noise have moments up to order $2s$.

Appendix II
Derivation of Expression (41) for the Hessian of the LS-CMA Cost Function

Using $|y_k|^2 = \mathbf{w}^H\mathbf{x}\mathbf{x}^H\mathbf{w} = (\bar{\mathbf{x}} \otimes \mathbf{x})^H(\bar{\mathbf{w}} \otimes \mathbf{w})$ and other properties of Kronecker products, the expression for the Hessian (37) can be written as

$$\mathrm{vec}\left(\mathbf{H}_{\mathbf{ww}}(\mathbf{w})\right) = 4\mathrm{E}(\bar{\mathbf{x}} \otimes \mathbf{x})(\bar{\mathbf{x}} \otimes \mathbf{x})^H(\bar{\mathbf{w}} \otimes \mathbf{w}) - 2\mathrm{E}(\bar{\mathbf{x}} \otimes \mathbf{x}).$$

For the model $\mathbf{x} = \mathbf{A}\mathbf{s} + \mathbf{n}$, it is known that [28]

$$\begin{aligned} \mathrm{E}(\mathbf{x}\mathbf{x}^H) &= \mathbf{R} = \mathbf{A}\mathbf{A}^H + \sigma^2\mathbf{I} \\ \mathrm{E}(\bar{\mathbf{x}} \otimes \mathbf{x})(\bar{\mathbf{x}} \otimes \mathbf{x})^H &= \mathbf{K} + \mathrm{E}(\bar{\mathbf{x}} \otimes \mathbf{x})\mathrm{E}(\bar{\mathbf{x}} \otimes \mathbf{x})^H \\ &\quad + \mathrm{E}(\bar{\mathbf{x}}\bar{\mathbf{x}}^H) \otimes \mathrm{E}(\mathbf{x}\mathbf{x}^H) \end{aligned} \qquad (49)$$

where $\mathrm{E}(\bar{\mathbf{x}} \otimes \mathbf{x}) = \mathrm{vec}(\mathbf{R}) = (\bar{\mathbf{A}} \circ \mathbf{A})\mathbf{1} + \sigma^2\mathrm{vec}(\mathbf{I})$, $\mathbf{1}$ is a vector of ones, $\circ$ is the Khatri–Rao product (column-wise Kronecker product), and $\mathbf{K}$ is the fourth-order cumulant of $\mathbf{x}$, given by [28]

$$\mathbf{K} = -(\bar{\mathbf{A}} \circ \mathbf{A})(\bar{\mathbf{A}} \circ \mathbf{A})^H.$$

It follows that

$$\begin{aligned} \mathrm{vec}\left(\mathbf{H}_{\mathbf{ww}}(\mathbf{w})\right) = 4 &\left[-(\bar{\mathbf{A}} \circ \mathbf{A})(\bar{\mathbf{A}} \circ \mathbf{A})^H \right. \\ &\left. + \mathrm{vec}(\mathbf{R})\mathrm{vec}(\mathbf{R})^H + \bar{\mathbf{R}} \otimes \mathbf{R}\right](\bar{\mathbf{w}} \otimes \mathbf{w}) - 2\mathrm{vec}(\mathbf{R}) \end{aligned}$$

with $\mathbf{R}$ given in (49). This has to be evaluated at the minimum of the cost function.

The treatment becomes feasible in the noise-free case: in that case $\mathbf{w}^o = \mathbf{A}^{-H}\mathbf{e}$, where $\mathbf{e}$ is one of the columns of the identity matrix, corresponding to the selected source. Note that

$$(\bar{\mathbf{A}} \circ \mathbf{A})^H(\bar{\mathbf{w}}^o \otimes \mathbf{w}^o) = \mathbf{e} \odot \mathbf{e} = \mathbf{e}$$

so that

$$\begin{aligned} (\bar{\mathbf{A}} \circ \mathbf{A})(\bar{\mathbf{A}} \circ \mathbf{A})^H(\bar{\mathbf{w}}^o \otimes \mathbf{w}^o) &= \bar{\mathbf{a}} \otimes \mathbf{a} \\ (\bar{\mathbf{A}} \circ \mathbf{A})\mathbf{1}\mathbf{1}^H(\bar{\mathbf{A}} \circ \mathbf{A})^H(\bar{\mathbf{w}}^o \otimes \mathbf{w}^o) &= (\bar{\mathbf{A}} \circ \mathbf{A})\mathbf{1} \\ (\mathbf{A}\mathbf{A}^H \otimes \mathbf{A}\mathbf{A}^H)(\bar{\mathbf{w}}^o \otimes \mathbf{w}^o) &= \bar{\mathbf{a}} \otimes \mathbf{a}. \end{aligned}$$

Thus, without noise, it follows that

$$\begin{aligned}\mathrm{vec}\,(\mathbf{H}_{\mathbf{ww}}(\mathbf{w}^o)) &= 4\left(-\bar{\mathbf{a}}\otimes\mathbf{a}+(\bar{\mathbf{A}}\circ\mathbf{A})\mathbf{1}+\bar{\mathbf{a}}\otimes\mathbf{a}\right)\\ &\quad -2(\bar{\mathbf{A}}\circ\mathbf{A})\mathbf{1}\\ &= 2(\bar{\mathbf{A}}\circ\mathbf{A})\mathbf{1}\end{aligned}$$

so that we finally obtain

$$\mathbf{H}_{\mathbf{ww}}(\mathbf{w}^o) = 2\mathbf{A}\mathbf{A}^H.$$

In a similar way, starting from (38) it can be shown that $\mathbf{H}_{\bar{\mathbf{w}}\mathbf{w}}(\mathbf{w}^o) = 2\mathbf{a}\mathbf{a}^T$.

Appendix III
Derivation of Expression (39) for the Norm of the Gradient of the LS-CMA Cost Function

The expected value of the norm of the gradient at $\mathbf{w}^o$ involves eighth-order statistics. It turns out to be equal to zero in the noise-free case. Assuming small but nonzero noise, we will retain only terms up to order σ^2, and, as before, evaluate at $\mathbf{w} = \mathbf{R}^{-1}\mathbf{a}$ rather than at the true minimum of the cost function. The complete derivation is very long and therefore we only present the main steps.

Let $\mathbf{g}(\mathbf{w}) = \nabla_{\mathbf{w}} J(\mathbf{w},\mathbf{x})$ and recall that

$$\mathbf{g}(\mathbf{w}) = 2\left(|y|^2-1\right)\bar{y}\mathbf{x}), \quad y = \mathbf{w}^H\mathbf{x}.$$

The desired result is an expression for $\mathrm{E}(\mathbf{g}^H\mathbf{g}) = \mathrm{vec}(\mathbf{I})^H\mathrm{E}(\bar{\mathbf{g}}\otimes\mathbf{g})$. Define $\mathbf{p} = \bar{\mathbf{x}}\otimes\mathbf{x}$ and $\mathbf{y} = \bar{\mathbf{w}}\otimes\mathbf{w}$, then

$$\begin{aligned}\bar{\mathbf{g}}\otimes\mathbf{g} &= 4(\bar{\mathbf{x}}\otimes\mathbf{x})(\mathbf{w}^H\mathbf{x}\mathbf{x}^H\mathbf{w}-1)^2\mathbf{w}^H\mathbf{x}\mathbf{x}^H\mathbf{w}\\ &= 4\mathbf{p}(\mathbf{p}^H\mathbf{y}-1)^2\mathbf{p}^H\mathbf{y}\\ &= 4\left\{\mathbf{p}\mathbf{p}^H\mathbf{y}-2\mathbf{p}(\bar{\mathbf{p}}\otimes\mathbf{p})^H(\bar{\mathbf{y}}\otimes\mathbf{y})\right.\\ &\qquad \left.+\mathbf{p}(\bar{\mathbf{p}}\otimes\mathbf{p}\otimes\bar{\mathbf{p}})^H(\bar{\mathbf{y}}\otimes\mathbf{y}\otimes\bar{\mathbf{y}})\right\}. \end{aligned}\tag{50}$$

We will need expressions for the statistics of $\mathbf{p}$ up to fourth order. To simplify this, we write as model for $\mathbf{p}$ (cf. [28])

$$\mathbf{p} = \underset{\sim}{\mathbf{A}}\,\underset{\sim}{\mathbf{s}} + \underset{\sim}{\mathbf{n}} + \mathbf{r}$$

where

$$\begin{aligned}\underset{\sim}{\mathbf{A}} &:= \bar{\mathbf{A}}\otimes\mathbf{A}\\ \underset{\sim}{\mathbf{s}} &:= \bar{\mathbf{s}}\otimes\mathbf{s}-\mathrm{vec}(\mathbf{I})\\ \underset{\sim}{\mathbf{n}} &:= \bar{\mathbf{A}}\bar{\mathbf{s}}\otimes\mathbf{n}+\bar{\mathbf{n}}\otimes\mathbf{A}\mathbf{s}+\left(\bar{\mathbf{n}}\otimes\mathbf{n}-\mathrm{vec}(\sigma^2\mathbf{I})\right)\\ \mathbf{r} &:= \mathrm{E}(\mathbf{p}) = \mathrm{vec}(\mathbf{R}) = \mathrm{vec}(\mathbf{A}\mathbf{A}^H+\sigma^2\mathbf{I}).\end{aligned}$$

The "source vectors" $\underset{\sim}{\mathbf{s}}$ and $\underset{\sim}{\mathbf{n}}$ have the following properties [28]:

$$\begin{aligned}&\mathrm{E}(\underset{\sim}{\mathbf{s}}) = 0, \quad \mathrm{E}(\underset{\sim}{\mathbf{n}}) = 0\\ &\mathrm{E}(\underset{\sim}{\mathbf{s}}\,\underset{\sim}{\mathbf{n}}^H) = 0\\ \underset{\sim}{\mathbf{R}}_s &:= \mathrm{E}(\underset{\sim}{\mathbf{s}}\,\underset{\sim}{\mathbf{s}}^H) = \mathbf{I}-(\mathbf{I}\circ\mathbf{I})(\mathbf{I}\circ\mathbf{I})^H\\ \underset{\sim}{\mathbf{R}}_n &:= \mathrm{E}(\underset{\sim}{\mathbf{n}}\,\underset{\sim}{\mathbf{n}}^H) = \sigma^2(\bar{\mathbf{A}}\bar{\mathbf{A}}^H\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{A}\mathbf{A}^H)+O(\sigma^4).\end{aligned}$$

If we define the permutation operator $\mathbf{J}$ such that $\mathbf{J}(\mathbf{a}\otimes\mathbf{b}) = \mathbf{b}\otimes\mathbf{a}$, then we further have

$$\begin{aligned}\mathrm{E}(\underset{\sim}{\mathbf{s}}\,\underset{\sim}{\mathbf{s}}^T) &= \mathbf{J}\,\underset{\sim}{\mathbf{R}}_s\\ \mathrm{E}(\underset{\sim}{\mathbf{n}}\,\underset{\sim}{\mathbf{n}}^T) &= \mathbf{J}\,\underset{\sim}{\mathbf{R}}_n.\end{aligned}$$

For higher-order expectations, we regard $\underset{\sim}{\mathbf{s}}$ and $\underset{\sim}{\mathbf{n}}$ as independent, and use

$$\begin{aligned}\mathrm{E}(\bar{\underset{\sim}{\mathbf{s}}}\otimes\underset{\sim}{\mathbf{s}})(\bar{\underset{\sim}{\mathbf{s}}}\otimes\underset{\sim}{\mathbf{s}})^H &= \mathrm{vec}(\underset{\sim}{\mathbf{R}}_s)\mathrm{vec}(\underset{\sim}{\mathbf{R}}_s)^H+\bar{\underset{\sim}{\mathbf{R}}}_s\otimes\underset{\sim}{\mathbf{R}}_s\\ &\quad -(\underset{\sim}{\mathbf{R}}_s\circ\underset{\sim}{\mathbf{R}}_s)(\bar{\underset{\sim}{\mathbf{R}}}_s\circ\underset{\sim}{\mathbf{R}}_s)^H.\end{aligned}$$

The above list of properties is used to derive that, for the first term in (50),

$$\mathrm{E}(\mathbf{p}\mathbf{p}^H)\mathbf{y} = (\underset{\sim}{\mathbf{A}}\,\underset{\sim}{\mathbf{R}}_s\underset{\sim}{\mathbf{A}}^H+\underset{\sim}{\mathbf{R}}_n+\mathbf{r}\mathbf{r}^H)\mathbf{y}.$$

This has to be evaluated for $\mathbf{y} = \bar{\mathbf{w}}\otimes\mathbf{w}$ with $\mathbf{w} = \mathbf{R}^{-1}\mathbf{a}$. To this end, we can derive that

$$\begin{aligned}\mathbf{r}^{\mathrm{H}}\mathbf{y} &= \mathbf{a}^H\mathbf{w} = 1-\sigma^2\|\mathbf{w}\|^2+O(\sigma^4)\\ \underset{\sim}{\mathbf{R}}_n\mathbf{y} &= \sigma^2(\bar{\mathbf{w}}\otimes\mathbf{a}+\bar{\mathbf{a}}\otimes\mathbf{w})+O(\sigma^4)\\ \underset{\sim}{\mathbf{A}}\,\underset{\sim}{\mathbf{R}}_s\underset{\sim}{\mathbf{A}}^H\mathbf{y} &= 2\sigma^2\|\mathbf{w}\|^2\,\underset{\sim}{\mathbf{a}}-\sigma^2(\bar{\mathbf{w}}\otimes\mathbf{a}+\bar{\mathbf{a}}\otimes\mathbf{w})\end{aligned}$$

where $\underset{\sim}{\mathbf{a}} = \bar{\mathbf{a}}\otimes\mathbf{a}$, so that

$$\mathrm{E}(\mathbf{p}\mathbf{p}^H)\mathbf{y} = \mathbf{r}+\sigma^2\|\mathbf{w}\|^2(2\,\underset{\sim}{\mathbf{a}}-\mathbf{r})+O(\sigma^4).$$

The second term in (50) requires

$$\begin{aligned}&\mathrm{E}\left[\mathbf{p}(\bar{\mathbf{p}}\otimes\mathbf{p})^H\right]\\ &= \mathrm{E}\left[(\underset{\sim}{\mathbf{A}}\,\underset{\sim}{\mathbf{s}}+\underset{\sim}{\mathbf{n}}+\mathbf{r})\left((\bar{\underset{\sim}{\mathbf{A}}}\,\bar{\underset{\sim}{\mathbf{s}}}+\bar{\underset{\sim}{\mathbf{n}}}+\bar{\mathbf{r}})\otimes(\underset{\sim}{\mathbf{A}}\,\underset{\sim}{\mathbf{s}}+\underset{\sim}{\mathbf{n}}+\mathbf{r})\right)^H\right]\\ &= \underset{\sim}{\mathbf{A}}\,\underset{\sim}{\mathbf{R}}_s(\bar{\mathbf{r}}\otimes\underset{\sim}{\mathbf{A}})^H+\underset{\sim}{\mathbf{A}}\,\mathbf{J}\,\underset{\sim}{\mathbf{R}}_s(\bar{\underset{\sim}{\mathbf{A}}}\otimes\mathbf{r})^H+\underset{\sim}{\mathbf{R}}_n(\bar{\mathbf{r}}\otimes\mathbf{I})^H\\ &\quad +\mathbf{J}\,\bar{\underset{\sim}{\mathbf{R}}}_n(\mathbf{I}\otimes\mathbf{r})^H+\mathbf{r}(\bar{\mathbf{r}}\otimes\mathbf{r})^H+\mathbf{r}\mathrm{vec}(\underset{\sim}{\mathbf{R}}_s)^H(\bar{\underset{\sim}{\mathbf{A}}}\otimes\underset{\sim}{\mathbf{A}})^H\\ &\quad +\mathbf{r}\mathrm{vec}(\underset{\sim}{\mathbf{R}}_n)^H.\end{aligned}$$

This evaluates to

$$\mathrm{E}\left[\mathbf{p}(\bar{\mathbf{p}}\otimes\mathbf{p})^H\right](\bar{\mathbf{y}}\otimes\mathbf{y}) = \mathbf{r}+4\sigma^2\|\mathbf{w}\|^2\,\underset{\sim}{\mathbf{a}}+O(\sigma^4).$$

Without further details, we mention that a lengthy but otherwise mechanical derivation shows that the third term in (50) evaluates to a similar expression, and that we finally obtain

$$\mathrm{E}(\bar{\mathbf{g}}\otimes\mathbf{g}) = 8\sigma^2\|\mathbf{w}\|^2\mathbf{r}+O(\sigma^4).$$

This gives the result. Fig. 8 gives evidence of the validity of the expression, by comparing the model for $\mathrm{E}(\|\mathbf{g}\|^2)$ to the result of a simulation, for the case of $M = 5$ antennas in a uniform

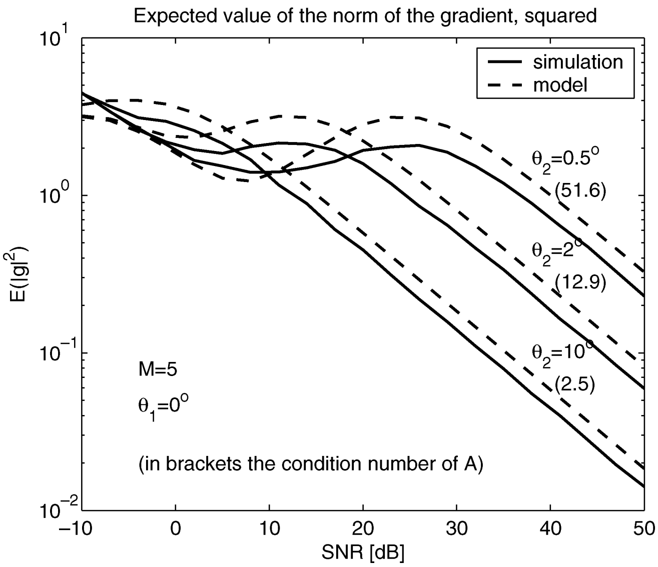

Fig. 8. Expected value of the norm of the gradient, for varying SNR and DOAs.

linear array (half-wavelength spacing), and $K = 2$ equipowered sources with varying separations. It is seen that the model is reasonably accurate over a wide range of SNRs.

REFERENCES

[1] A. Leshem and A.-J. van der Veen, "On the finite sample behavior of the constant modulus cost function," in *Proc. IEEE Int. Conf. Acoustics, Speech, Signal Processing (ICASSP)*, Istanbul, Turkey, Jun. 2000, vol. 5, pp. 2537–2540.

[2] Y. Sato, "A method of self-recovering equalization for multilevel amplitude-modulation systems," *IEEE Trans. Commun.*, vol. 23, no. 6, pp. 679–682, Jun. 1975.

[3] D. Godard, "Self-recovering equalization and carrier tracking in two-dimensional data communication systems," *IEEE Trans. Commun.*, vol. 28, no. 11, pp. 1867–1875, Nov. 1980.

[4] C. Jutten and J. Herault, "Blind separation of sources I. An adaptive algorithm based on neuromimetic architecture," *Signal Process.*, vol. 24, pp. 1–10, Jul. 1991.

[5] J. Treichler and B. Agee, "A new approach to multipath correction of constant modulus signals," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 31, no. 2, pp. 459–471, Apr. 1983.

[6] O. Shalvi and E. Weinstein, "New criteria for deconvolution of non-minimum phase systems," *IEEE Trans. Inf. Theory*, vol. 36, pp. 312–320, Mar. 1990.

[7] J.-F. Cardoso and A. Souloumiac, "Blind beamforming for non-Gaussian signals," *Proc. Inst. Electr. Eng.—F, Radar Signal Process.*, vol. 140, pp. 362–370, Dec. 1993.

[8] P. Comon, "Independent component analysis—A new concept?," *Signal Process.*, vol. 36, pp. 287–314, Apr. 1994.

[9] L. Castedo, C. Escudero, and A. Dapena, "A blind source separation method for multiuser communications," *IEEE Trans. Signal Process.*, vol. 45, no. 5, pp. 1343–1348, May 1997.

[10] A. Taleb, "An algorithm for the blind identification of n independent signal with 2 sensors," in *Proc. 16th Symp. Signal Processing Its Applications (ISSPA)*, Kuala-Lumpur, Malaysia, Aug. 2001, pp. 5–8.

[11] A. Yeredor, "Blind channel estimation using first and second derivatives of the characteristic function," *Signal Process. Lett.*, vol. 9, no. 3, pp. 100–103, Mar. 2002.

[12] *Unsupervised Adaptive Filtering Volume I: Blind Source Seperation*, S. Haykin, Ed. New York: Wiley-Interscience, 1993.

[13] *Unsupervised Adaptive Filtering Volume II: Blind Deconvolution*, S. Haykin, Ed. New York: Wiley-Interscience, 1993.

[14] A. Hyvarinen, J. Karhunen, and P. Oja, *Independent Component Analysis*. New York: Wiley-Interscience, 2001.

[15] P. Comon, P. Chevalier, and V. Capdevielle, "Performance of contrast-based blind source separation," in *Proc. IEEE Signal Processing Advances in Wireless Communications (SPAWC)*, Paris, France, Apr. 1997, pp. 345–348.

[16] E. Moreau, J.-C. Pesquet, and N. Thirion-Moreau, "Convolutive blind signal separation based on asymmetrical contrast functions," *IEEE Trans. Signal Process.*, vol. 55, no. 1, pp. 356–371, Jan. 2007.

[17] Z. Bai and Y. Wu, "General M-estimation," *J. Multivar. Anal.*, vol. 63, pp. 119–135, 1997.

[18] A. van der Vaart, *Asymptotic Statistics*. Cambridge, U.K.: Cambridge Univ. Press, 1997.

[19] I. Fijalkow, A. Touzni, and J. Treichler, "Fractionally spaced equalization using CMA:Robustness to channel noise and lack of disparity," *IEEE Trans. Signal Process.*, vol. 45, no. 1, pp. 56–66, Jan. 1997.

[20] H. Zeng, L. Tong, and C. Johnson, "Relationships between the constant modulus and Wiener receivers," *IEEE Trans. Inf. Theory*, vol. 44, no. 4, pp. 1523–1538, 1998.

[21] D. Liu and L. Tong, "An analysis of constant modulus algorithm for array signal processing," *Signal Process.*, vol. 73, pp. 81–104, Jan. 1999.

[22] P. Schniter and C. Johnson, "Bounds for the MSE performance of constant modulus estimators," *IEEE Trans. Inf. Theory*, vol. 46, pp. 2544–2560, Jul. 2000.

[23] H. Zeng, L. Tong, and C. Johnson, "An analysis of constant modulus receivers," *IEEE Trans. Signal Process.*, vol. 47, no. 11, pp. 2990–2999, Nov. 1999.

[24] P.-A. Regalia, "On the equivalence between the Godard and Shalvi–Weinstein schemes of blind equalization," *Signal Process.*, vol. 73, pp. 185–190, Feb. 1999.

[25] B. Agee, "The least-squares CMA: A new technique for rapid correction of constant modulus signals," in *Proc. IEEE Int. Conf. Acoustics, Speech, Signal Processing (ICASSP)*, Tokyo, Japan, 1986, pp. 953–956.

[26] X. Zhuang and A. Swindlehurst, "Fixed window constant modulus algorithm," in *Proc. IEEE Int. Conf. Acoustics, Speech, Signal Processing (ICASSP)*, 1999, vol. 5, pp. 2623–2626.

[27] A.-J. van der Veen and A. Paulraj, "An analytical constant modulus algorithm," *IEEE Trans. Signal Process.*, vol. 44, no. 5, pp. 1136–1155, May 1996.

[28] A.-J. van der Veen, "Asymptotic properties of the algebraic constant modulus algorithm," *IEEE Trans. Signal Process.*, vol. 49, no. 8, pp. 1796–1807, Aug. 2001.

[29] A.-J. van der Veen, "Statistical performance analysis of the algebraic constant modulus algorithm," *IEEE Trans. Signal Process.*, vol. 50, no. 12, pp. 3083–3097, Dec. 2002.

[30] A. Leshem and A.-J. van der Veen, "Direction of arrival estimation for constant modulus signals," *IEEE Trans. Signal Process.*, vol. 47, no. 11, pp. 3125–3129, Nov. 1999.

[31] B. Sadler, R. Kozick, and T. Moore, "On the performance of source separation with constant modulus signals," in *Proc. IEEE Int. Conf. Acoustics, Speech, Signal Processing (ICASSP)*, May 2002, pp. 2373–2376.

[32] A. Leshem, N. Petrochilos, and A.-J. van der Veen, "Finite sample identifiability of multiple constant modulus signals," *IEEE Trans. Inf. Theory*, vol. 49, pp. 2314–2319, Sep. 2003.

[33] L. Tong, M. Gu, and S. Kung, "A geometrical approach to blind signal estimation," in *Signal Processing Advances in Wireless and Mobile Communications*, G. Giannakis, Ed. *et al.* Englewood Cliffs, NJ: Prentice-Hall, 2000, vol. 1, ch. 8.

[34] J.-F. Cardoso, "High-order contrasts for independent component analysis," *Neural Comput.*, vol. 11, pp. 157–192, Jan. 1999.

[35] P. Comon, "From source separation to blind equalization, contrast-based approaches," in *Int. Conf. Image and Signal Processing (ICISP)*, Agadir, Morocco, May 2001, pp. 20–32.

[36] C. Papadias and A. Paulraj, "A space-time constant modulus algorithm for SDMA systems," in *Proc. IEEE 46th Vehicular Technology Conf.*, May 1996, vol. 1, pp. 86–90.

[37] N. Delfosse and P. Loubaton, "Adaptive blind separation of independent sources: A deflation approach," *Signal Process.*, vol. 45, pp. 59–83, 1995.

[38] R. Horn and C. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1985.

[39] K. Kreutz-Delgado, "The complex gradient operator and the CR-Calculus," Electr. and Comput. Eng. Dept., Univ. of California San Diego, Tech. Rep. Course Lecture Supplement No. ECE275CG-F05v1.2b, Sep.–Dec. 2005 [Online]. Available: http://dsp.ucsd.edu/~kreutz/PEI05.html

[40] D. Brandwood, "A complex gradient operator and its application in adaptive array theory," *Proc. Inst. Electr. Eng. F—Commun., Radar, Signal Process.*, vol. 130, pp. 11–16, Feb. 1983.

[41] A. van den Bos, "Complex gradient and Hessian," *Proc. Inst. Electr. Eng.—Vis. Image Signal Process.*, vol. 141, pp. 380–382, Dec. 1994.

Amir Leshem (M'98–SM'06) received the B.Sc. (*cum laude*) degree in mathematics and physics, the M.Sc. (*cum laude*) degree in mathematics, and the Ph.D. in mathematics, all from the Hebrew University, Jerusalem, Israel, in 1986, 1990, and 1998, respectively.

From 1998 to 2000, he was with the Faculty of Information Technology and Systems, Delft University of Technology, Delft, The Netherlands, as a Postdoctoral Fellow working on algorithms for the reduction of terrestrial electromagnetic interference in radio-astronomical radio-telescope antenna arrays and signal processing for communication. From 2000 to 2003, he was Director of Advanced Technologies with Metalink Broadband, where he was responsible for research and development of new DSL and wireless MIMO modem technologies and served as a member of ITU-T SG15, ETSI TM06, NIPP-NAI, and IEEE 802.3 and 802.11. From 2000 to 2002, he was also a Visiting Researcher at Delft University of Technology. He is one of the founders of the new school of electrical and computer engineering at Bar-Ilan University, Ramat-Gan, Israel, where he is currently an Associate Professor and head of the Signal Processing track. From 2003 to 2005, he was also the Technical Manager of the U-BROAD consortium developing technologies to provide 100 Mb/s and beyond over copper lines. His main research interests include multichannel wireless and wireline communication, applications of game theory to dynamic and adaptive spectrum management of communication networks, array and statistical signal processing with applications to multiple-element sensor arrays and networks in radio-astronomy, brain research, wireless communications and radio-astronomical imaging, set theory, logic, and foundations of mathematics.

Alle-Jan van der Veen (M'95–SM'03–F'05) was born in The Netherlands in 1966. He received the Ph.D. (*cum laude*) degree from Delft University of Technology (TU Delft), Delft, The Netherlands, in 1993.

Throughout 1994, he was a Postdoctoral Scholar at Stanford University, Stanford, CA. Currently, he is a Full Professor in signal processing at TU Delft. His research interests are in the general area of system theory applied to signal processing, and in particular algebraic methods for array signal processing, with applications to wireless communications and radio astronomy.

Dr. van der Veen is the recipient of a 1994 and 1997 IEEE Signal Processing Society (SPS) Young Author paper award. He was an Associate Editor for IEEE TRANSACTIONS ON SIGNAL PROCESSING from 1998 to 2001, Chairman of IEEE SPS Signal Processing for Communications Technical Committee from 2002 to 2004, and Editor-in-Chief of the IEEE SIGNAL PROCESSING LETTERS from 2002 to 2005. He currently is Editor-in-Chief of the IEEE TRANSACTIONS ON SIGNAL PROCESSING, and member-at-large of the Board of Governors of IEEE SPS.