Time-varying (matrix) interpolation, deadbeat control and the Löwner case

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1 Abstract

The paper starts out with a short overview of the way in which interpolation problems can be treated in a linear time-varying (LTV) or general matrix context. Although most types of interpolation problems (including the most important Schur-Takagi case) convert smoothly to the more general setting, some problems appear to be tougher to handle. That is the case for the Löwner interpolation problem, which necessitates a different approach than that used for the more traditional cases. It turns out that a combination of ideas from the LTI case (as solved in a number of papers by Antoulas, Ball, Kang and Anderson) on the one hand and the way deadbeat control is handled by Van Dooren on the other (also for the LTI case) provide for the necessary framework when extended to the LTV (or matrix) setting. The paper concludes with some considerations on what the author sees as remaining open problems in LTV interpolation theory.

2 Introduction

Various cases of time-varying or matrix interpolation have been considered in the literature by a number of authors, there is even a comprehensive theory presented in [5]. Although there are many results, there are also substantial gaps. In this paper we try to fill one of these gaps, the extension of Löwner interpolation to the time-varying (matrix) case. In the concluding section we shall also mention some other remaining problems.

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The gist of time-varying (matrix) interpolation is the introduction of a notion of time or causality in matrix computations. This is achieved by thinking of a matrix T as an operator acting on a time-sequence $u = [u_i]_{i=\infty\cdots-\infty}$ to yield a new time sequence $y = Tu$. The time sequences we are considering may be irregular in the sense that the u_i and y_k may belong to different vector spaces (of finite dimensions m_i respect. n_k) so that the elements of the matrix T are blocks of dimension $n_k \times m_i$. Some entries may disappear all together, yielding a dimension zero. At such a position there is then a place holder (not a zero!) that just indicates the presence of an index but nothing more. Blocks of dimension $n \times 0$ or $0 \times m$ are considered empty with a column of n place holders (respect. a row of m place holders) with the convention that their product is actually a zero matrix of dimension $n \times m$, extending the normal matrix calculus to matrices of zero dimensions - a very practical convention meanwhile adopted by Matlab as well. Although the matrix T may have infinite dimensions, finite dimensions can easily be accommodated in the framework by assuming zero dimensions beyond the dimensional range of the matrix (therefore we shall not have to distinguish the finite case from the infinite one in the sequel). The index in the input and output vectors is thought of as 'time'. The product $T_{ii}u_i$ produces a contribution to y_i at time instant i and can hence be viewed as an 'instantaneous computation'. The main diagonal $T_{[0]} = \mathrm{diag}[T_{ii}]$ is then the collection of all instantaneous computations at all time points. Introducing the 'causal time-shift Z' by $[Zu]_i = u_{i-1}$ we can represent the operator T as a sum of shifted diagonals (with $Z^{[i]}$ representing i shifts):

$$T = \sum_{i=-\infty}^{\infty} Z^{[i]} T_{[i]} \tag{1}$$

The matrix representation of Z is a diagonal of unit matrices of appropriate dimensions on the first upper diagonal. We shall endow the input and output spaces with an ℓ_2 Hilbert space structure and require the operator T to be bounded in the corresponding metric (in the finite matrix case this just amounts to a Euclidean metric). When T is bounded in that sense, then so will the $T_{[i]}$ be, but the sum in (2) does not necessarily converge in operator norm (it does induce strong convergence though). Since Z is merely a shift, its inverse Z^{-1} will of course exist. In the Hilbert space setting, this inverse is in fact also an adjoint $Z^{-1} = Z^H$, in which the super index $\cdot^H$ indicates Hermitian conjugation. Upper triangular matrices are 'causal' in the time-domain setting, meaning that an entry u_i in the input sequence only influences entries y_k for $k \geq i$ in the output sequence. Causal systems will be of prime importance in the interpolation problems we wish to consider.

Let us now consider a causal and bounded system

$$T = \sum_{i=0}^{\infty} Z^{[i]} T_{[i]} \tag{2}$$

and let us choose a diagonal operator V with the same input-output dimensions as the shift operator Z (we shall say 'conformal with Z'), and which is such that the operator $(I - VZ^H)$ is boundedly invertible. This will be the case when the spectral radius $\rho(VZ^H) < 1$ (thanks to the Neumann expansion theory: $(I - VZ^H)^{-1} = \sum_{i=0}^{\infty}(VZ^H)^i$). Following classical interpolation ideas, we are interested in representations of the type:

$$T = X + (Z - V)T' \tag{3}$$

in which X is a diagonal operator of appropriate dimensions and T' is again a causal operator. If this expression consisted of scalar entries we would say that T interpolates X at the point V. Such an interpretation actually fits in the present context if all matrices are restricted to be doubly infinite Toeplitz, the operator is replaced by a 'z-transform' and the shift then turns out to be nothing else but multiplication with the complex variable z. Given T in (3), what is X? To obtain a comfortable expression, let us define the diagonal (downward) shift as follows:

$$V^{(k)} = Z^{-k}VZ^k \tag{4}$$

and the continuous product

$$V^{[k]} = V \cdots V^{(k-1)} \tag{5}$$

then we find (at least formally)

$$X = \sum_{i=0}^{\infty} V^{[i]}T_{[i]}. \tag{6}$$

The convergence of this expression has been studied in the literature [1] and it produces what has been termed - because of connections with reproducing kernels going beyond the present paper - the 'W-transform' of T at the (diagonal) point V indicated by

$$T^{\wedge}(V) = \sum_{i=0}^{\infty} T_{[i]}V^{[i]}. \tag{7}$$

For full details of this theory we refer to the literature [6, 1], where this mode of thinking was presented first.

One more ingredient needed for the development of interpolation theory in our matrix environment is the notion of time-varying system realization. A causal time-varying system is characterized by a 'state evolution' and instantaneous maps from input to output, present state to output and input to next state. There is a 'hidden' state sequence $[x_k]$ representing the memory the computational system has from past computations. The instantaneous maps at time point k are then given by a collection of time-varying matrices $\{A_k, B_k, C_k, D_k\}$ called a *realization* and representing the maps

$$\left\{ \begin{array}{rcl} x_{k+1} & = & A_k x_k + B_k u_k \\ y_k & = & C_k x_k + D_k u_k \end{array} \right. \tag{8}$$

The original operator T can be recovered from the realization by collecting the matrices in block diagonal operators $A = \mathrm{diag}_{\infty\cdots-\infty}(A_k), B = \mathrm{diag}_{\infty\cdots-\infty}(B_k)$ etc... as

$$T = D + C(I - ZA)^{-1}ZB \tag{9}$$

and it should be clear that $T_{ij} = 0$ when $i < j$, $T_{ii} = D_i$ and $T_{ij} = C_i A_{i+1} \cdots A_{j-1} B_j$ when $i > j$, representing the effect u_j has on the output y_i through the computational scheme (notice the reversal of the index in the representations of u, y and T, to accommodate various representations used in the literature. A more natural system is used in [5]). The dimension δ_k of the state x_k can be chosen minimal at each time point, resulting in a minimal system overall. How to compute this dimension and resulting realizations is the subject of realization theory - see the literature [5]. State realizations, even minimal ones, are by no means unique. A state transformation at the point k is a non-singular matrix matrix R_k that defines a new state space basis and a transformation $x'_k = x_k R_k$. The corresponding transformation on the realization is then given by

$$\begin{bmatrix} A'_k & B'_k \\ C'_k & D'_k \end{bmatrix} = \begin{bmatrix} R_{k+1}^{-1} A_k R_k & R_{k+1}^{-1} B_k \\ C_k R_k & D_k \end{bmatrix} \tag{10}$$

(in the sequel we shall work exclusively on minimal realizations). Minimal realizations are characterized by the fact that the corresponding reachability and observability operators are non-singular. At time point k the reachability operator describes how the state x_k is reached from (strict) past inputs $u_{k;p}$: $x_k = \mathcal{R}_k u_{k;p}$ with $\mathcal{R}_k = [B_{k-1}, A_{k-1}B_{k-2}, A_{k-1}A_{k-2}B_{k-3}, \cdots]$ while the observability operator describes the contribution of a given state x_k to the present and future outputs $y_{k,f} = \mathcal{O}_k x_k$ (the operator from past inputs to future outputs is known as the Hankel operator H_k at time point k and it factorizes minimally as $H_k = \mathcal{O}_k \mathcal{R}_k$ - this is the basis of realization theory). We shall say that a system is totally realizable if its reachability operator has a bounded right inverse. This will be the case iff the reachability Gramians

$$\Lambda_{rk} = \mathcal{R}_k \mathcal{R}_k^H \tag{11}$$

are uniformily bounded away from zero.

3 Classical interpolation problems in the time-varying setting

We now have the machinery to define and solve classical interpolation problems in the new matrix setting. We only quickly summarize the results, the detailed theory is given in [5]. The generalization of the 'tangential Nevanlinna-Pick problem' takes as input data at time point k:
1. a collection of n_k 'interpolation block diagonals' $(\nu_i)_k$
2. for each i in n_k directional data (vectors or matrices - again block diagonals) $(\xi_i)_k$ and $(\eta_i)_k$.
This data should be 'conformal' in the sense that it can be assembled in diagonal operators

$$V = \mathrm{diag}_k(\mathrm{diag}_i(\nu_i)_k), \xi = \mathrm{diag}_k(\mathrm{diag}_i(\xi_i)_k), \eta = \mathrm{diag}_k(\mathrm{diag}_i(\eta_i)_k) \tag{12}$$

such that

$$\left[\begin{array}{c|cc} V & \xi & \eta \end{array}\right] \tag{13}$$

form conformal matrices of block diagonals (the precise index-varying dimensions are immaterial as long as everything is conformal).

The (generalized) Nevanlinna-Pick problem then asks to find a causal (upper), contractive (time varying) operator S with the property that $(Z-V)^{-1}(\xi S-\eta)$ is again causal (upper).

The problem finds a solution in terms of a so called causal J-inner operator Θ with reachability pair given by $\{V^H, [\xi, -\eta]^H\}$. This completely parallels the classical solution, even though some more complex algebra is involved (and no complex function calculus). A J-unitary (the J-inner being a special case) operator decomposes in four blocks

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \tag{14}$$

each of which is a causal, time-varying operator in its own right such that

$$J_2 = \Theta^H J_1 \Theta, J_1 = \Theta J_2 \Theta^H \tag{15}$$

for conformal sign operators J_1 and J_2 of the type $J_i = \begin{bmatrix} I & \\ & -I \end{bmatrix}$. Such an operator will exist and be J-inner iff the Gramian

$$\Lambda_{r,J} = \mathcal{R}_k^H J \mathcal{R}_k \tag{16}$$

based on the reachability pair is strictly positive definite (the solution can be extended to the semi-positive case through a limiting procedure, but the Θ matrix will not exist any more). Defining $\mathbf{P}_0$ as the operator that projects on the main diagonal of a matrix, this condition spells out as the requirement that the *Pick matrix*

$$\Lambda_{\text{Pick}} = \mathbf{P}_0\left((Z-V)^{-1}[\xi\xi^H - \eta\eta^H](Z^H - V^H)^{-1}\right) >> 0 \tag{17}$$

in which the expansions of the inverses have to be done in the 'Von Neumann' fashion described above. The Nevanlinna-Pick theorem then becomes

Theorem 1. *The Nevanlinna-Pick generalized interpolation problem has a solution iff the Pick matrix (17) is positive definite. In case it is strictly positive definite then all solutions are given by*

$$S = (\Theta_{11} - \Theta_{12} S_L)(\Theta_{21} S_L - \Theta_{22})^{-1} \tag{18}$$

in which S_L is an causal contractive and conformal but otherwise arbitrary operator.

For proof and more information we refer to the literature (oc). Key to the property is, besides the algebraic properties of the Θ matrix, the fact that it maps the interpolation data (which by the Von Neumann expansion method expand to an anti-causal series) in a rather general way to causal:

$$(Z-V)^{-1}\begin{bmatrix} \xi & \eta \end{bmatrix}\Theta \in \text{causal}. \tag{19}$$

Once this basis laid, it is not too difficult to expand in various directions: Hermite-Fejer, Caratheodory-Fejer, Nudel'man types of interpolation. It is even possible in some cases to translate left-sided interpolation problems to right-sided and mix up situations as double sided (leading to Sylvester equations that are very difficult to solve in practice). Details to be found in the literature, for the time-varying case e.g. in [5].

An altogether different type of interpolation, with important applications to model reduction, is the Schur-Takagi case. Here the requirement of strict causality for the interpolating operator is waived but not the norm constraint, allowing a controlled amount of singularity (as described by local state dimensions). The solution is again given in terms of a causal, J-unitary operator, but now the condition 'J-inner' is no longer relevant and replaced by an inertia condition on a modified Pick operator, which is again related to the reachability pair that is characteristic for the interpolation data - see [5] for a comprehensive treatment. A specialization of this technique for the Schur (polynomial) case has been studied in [4] with surprising results. In the remainder of this paper we wish, however, to concentrate on a type of interpolation whose extension to time-varying to the best of our knowledge has not yet been considered for time-varying systems - Löwner or unconstrained interpolation. Again, an equation of the type (19) plays a central role, but now the operator Θ must not be J-unitary anymore, but polynomial in Z . Although it should be possible to develop a theory of polynomial operators for the time-varying case that mimicks the LTI theory, we shall follow another route that has some interest in itself and avoids some pitfalls that already became obvious in the treatment of the singular polynomial Schur case [4].

4 Deadbeat control

Suppose that a system representation as in (8) is given and that we wish to establish a strategy to bring an arbitrary state x_k that may exist at an arbitrary time point k to zero in a minimal number of steps - this is called deadbeat control. It will evidently be possible if for all controllability matrices (for all time points) there exists a finite horizon for which it is non-singular. We shall need a little more: there should be a uniform finite bound on the controllability time delay, more precisely, there is a finite time lag τ such all $\mathcal{R}_{k,\tau}$ for all k are non-singular. We shall be able to show that under such a condition there exists a feedback law F_i which is such that all $(A_i - B_i F_i)Z$ is nilpotent. As a result we shall be able to derive polynomial representations in the shift Z for operators that are relevant to the interpolation theory. This will be done in the next section.

Preliminaries

Let $a : \mathcal{U} \to \mathcal{Y}$ be an operator between (Euclidean or other) spaces. If $\{\xi_i, i = 1 \cdots m\}$ is a collection of vectors in $\mathcal{U}$, let $\xi = [\xi_i, \cdots, \xi_m]$ be a 'stack' of these vectors (it is a row-vector of vectors which at this point do not have a representation yet, although it can be viewed as a matrix if the ξ_i are column vectors). The space spanned by these vector will be indicated by span(ξ). $a\xi = [a\xi_1, \cdots, a\xi_m]$ is then the stack of image vectors. If the $\{\xi_i\}$ form a basis for $\mathcal{U}$ then $\mathbf{x} = \sum_{i=1}^{m} \xi_i x_i$ represents a vector in $\mathcal{U}$, which can also be written as $\mathbf{x} = \xi x$ - or even, in the Einstein notation that sums repeated indices, $\mathbf{x} = \xi_i x_i$. The distinction between $\mathbf{x}$ and x is that the former is an 'abstract' vector, while the latter stacks its components in a given basis as a column vector. If $\mathbf{y} = a\mathbf{x}$, ξ and η are bases for $\mathcal{U}$ and cY respectively, then a matrix A representing a in the given bases is defined (again using Einstein's summing convention) by $a\xi_i = \eta_j A_{ji}$. With $\mathbf{y} = a\mathbf{x}$ we the have $y = Ax$ in which x and y are column vectors and A is a matrix. This notation is common in geometric analysis.

Let now ξ and $\mathbf{u}$ be stacks of vectors such that $\operatorname{span}(a\xi) \subset \operatorname{span}(b\mathbf{u})$ and a and b operators, then it is easy to see that there exists a matrix F such that $a\xi = b\mathbf{u}F$. One just has to express the fact that the individual $a\xi_i = (b\mathbf{u})_j F_{ji}$ can be written as a linear combination of the vectors in $b\mathbf{u}$. F will not be unique in general, but if $b\mathbf{u}$ form a basis it will. This procedure can be stacked: suppose that $\operatorname{span}(a\xi_1) \subset \operatorname{span}(b\mathbf{u})$ and $\operatorname{span}(a\xi_2) \subset \operatorname{span}(b\mathbf{u})$ as well, then $a[\xi_1, \xi_2] = b\mathbf{u}[F_1, F_2]$ for appropriate F_i's. Let ξ, $\mathbf{u}$ and η be complete stacks of basis vectors in their respective spaces and suppose that $a\xi = \eta A$ and $b\mathbf{u} = \eta B$ for matrices A and B, then $\eta A = \eta BF$ and since the components of η form a basis one must have $A = BF$.

The time-varying case

A system can be deadbeat controlled if any state, at any time instant can be brought to zero in finite time (preferably in minimum time even, which makes the problem meaningful also for finite matrices). Let us consider such a system and let us position ourselves at a time instant i. Then the state space $\mathcal{B}_i$ can be partitioned as follows. All the states that can be brought to zero in one-step using an appropriate input at time step i form a subspace which we shall call $\mathcal{S}_{i,1}$, those that can be brought to zero in at most two steps, using appropriate inputs at time points i and $i+1$, we shall call $\mathcal{S}_{i,2}$ etc... and there will be a maximum k_i, consisting of all the states that can be brought to zero in at most k_i time steps. Clearly

$$\mathcal{S}_{i,1} \subset \mathcal{S}_{i,2} \subset \cdots \subset \mathcal{S}_{i,k_i} = \mathcal{B}_i \tag{20}$$

The following main result is a generalization of the original Popov theorem valid for the LTI case [8]. We follow the attractive method proposed by Van Dooren [7], which extends with some effort to the LTV case.

Theorem 2. *Suppose that $\{A_i, B_i\}$ is the controllability pair at time point i of a system that can be deadbeat controlled in finite time k_i at each time point i. Then there exists a block-diagonal matrix F_i such that $A_i^f = A_i + B_i F_i$ is such that for each i the continuous product $A_{i+k_i-1}^f A_{i+k_i-2}^f \cdots A_i^f = 0$.*

In other words: the 'beating to death' can be done by state feedback at each time point.

Proof

We use the notation developed earlier in this section. Let a_i be the state transition operator at time i (corresponding to A_i), and b_i the operator that maps the input at time i to the state at time $i+1$ (corresponding to the matrix B_i). Let $\mathbf{u}_i$ be a stack of base vectors of the input space at time i. Any state $\mathbf{x}$ belonging to the subspace $\mathcal{S}_{i,1}$ can be mapped to zero according to the definition, and hence $a\mathbf{x}$ must belong to span($b\mathbf{u}$). If ξ_1 forms a basis for $\mathcal{S}_{i,1}$, then span$(a\xi_1) \subset$ span$(b\mathbf{u})$ and there exists (by the preliminaries) a matrix $F_{i,1}$ such that $a\xi_1 = b_i\mathbf{u}F_{i,1}$. Let now $[\xi_1, \xi_2, \cdots, \xi_{k_i}], 1 \leq k \leq k_i$ be a stack of base vectors for $\mathcal{B}_i$, partitioned according to the sequence of spaces $\mathcal{S}_{i,k}$ and such that $[\xi_1, \cdots, \xi_k]$ spans $\mathcal{S}_{i,k}$. We then have that for all $1 \leq k \leq k_i$ span$(a\xi_k) \subset \mathcal{S}_{i+1,k-1} +$ span$(b\mathbf{u}_i)$, because if a state at time point i can be controlled in k steps, there must exist an input at time i so that the resulting state at time $i+1$ can be controlled in one time step less, namely in $k-1$ steps. Let $[\eta_1, \eta_2, \cdots, \eta_{k_{i+1}}]$ be a stack of basisvextors of $\mathcal{B}_{i+1}$ which is conformal to the subspaces $\mathcal{S}_{i+1,k}$, then we have that span$(a_i\xi_k) \subset$ span$(\eta_1, \cdots, \eta_{k-1})+$span$(b_i\mathbf{u})$ and there exist matrices $A^f_{i,1k}, A^f_{i,2k}, \cdots, A^f_{i,k-1,k}$ and F_k such that

$$a_i\xi_k = [\eta_1, \cdots, \eta_k] \begin{bmatrix} A^f_{i,1k} \\ \vdots \\ A^f_{i,k-1,k} \end{bmatrix} + [\eta_1, \cdots, \eta_{k_{i+1}}] \begin{bmatrix} B_{i,1} \\ \vdots \\ B_{i,k_{i+1}} \end{bmatrix} F_{i,k} \tag{21}$$

Stacking these for $k = 1, \cdots k_i$ we obtain

$$a_i\xi = \left[\eta_1, \cdots, \eta_{k_{i+1}}\right] (\begin{bmatrix} 0 & A^f_{i,11} & \cdots & A^f_{i,1,k_i} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & A^f_{i,k_i-1,k_i} \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} B_{i,1} \\ \vdots \\ B_{i,k_{i+1}} \end{bmatrix} \cdot [F_{i,1}, \cdots, F_{i,k_i}]) \tag{22}$$

Representing $a_i\xi = \left[\eta_1, \cdots, \eta_{k_{i+1}}\right] A_i$ and assembling the matrices we obtain

$$A_i = A^f_i + B_iF_i. \tag{23}$$

In this expression, A^f (viewed over all time points) maps state vectors onto shrinking subspaces, the shrinkage having the value k_i at time point i.
QED
From the theorem it follows that A is $-BF$ away from a 'nilpotent' state transition operator (actually ZA^f is nilpotent). The theorem yields an additional benefit: a representation of the state transition operator A as the sum of a (special) upper triangular operator and a part that is spanned by B. There is a nice way to compute F, generalizing the schema given by Van Dooren [7], which will be presented in a more comprehensive paper.

5 Application to Löwner interpolation

In the remainder of this paper we wish, however, to concentrate on a type of interpolation whose extension to the time-varying to the best of our knowledge has not yet been considered - the Löwner type case. In the LTI case the problem was solved in a set of remarkable papers, dating back to the late 80's and early 90's (including

MTNS'90) [2, 3]. Again, an equation of the type (19) plays a central role, but now the operator Θ must not be J-unitary, but polynomial in Z. Although it should be possible to develop a theory of polynomial operators for the time-varying case that mimicks the LTI theory, we shall follow another route that has some interest in itself and avoids the many pitfalls that already became obvious in the Schur-Takagi case. We are again given interpolation data that determine a reachability pair as before, but now we ask for interpolants that are merely causal, without norm constraints. To be able to handle the situation with our time-varying formalism, we do require $(I - Z^H V)$ to be invertible in the Von Neumann sense (hence $\rho(Z^H V) < 1$ - this is enough to handle the finite matrix case, since then the condition is always satisfied, but also the 'ISI-case' where the time-varying system is stable LTI near $+\infty$ and $-\infty$ as well as the quasi-periodic case. The development will concentrate on finding a generic Θ that is polynomial in Z and satisfies (19). We shall find Θ - if it exists - via a detour that involves another classical problem in system theory, namely deadbeat control.

Returning to the interpolation problem, let us assume that we are given interpolation data as defined in (12). Our strategy will be, mimicking the LTI case as in [3], to determine operators Q and W which are polynomial in Z and such that

$$(Z - V)^{-1} \begin{bmatrix} \xi & \eta \end{bmatrix} Q = W \tag{24}$$

If Q is sufficiently rich (we require it to be invertible and of minimal state dimensions), it will turn out to yield (hopefully all) solutions to the interpolation problem. Here we concentrate on obtaining such a Q. We do this via a detour inspired by the way other classical interpolation problems are solved in the LTV case. First we determine a minimal causal unitary operator U which is such that $(Z-V)^{-1} \begin{bmatrix} \xi & \eta \end{bmatrix} U$ is a causal (upper) operator, and next we represent $U = QP^{-1}$ in which both Q and P are polynomial in Z, with some additional properties that make the representation essentially unique - namely minimality and P^{-1} causal as well. Note that $(Z - V)^{-1} = Z^H(I - VZ^H)^{-1}$ in the current set up (with the restrictions imposed on V) is essentially anti-causal (upper). Let us say that a pair $[V, B]$ is strictly backward controllable if there exists an ϵ such that for all operators $\mathcal{R}_i = [B_i, V_i B_{i+1}, V_i V_{i+1} B_{i+2}, \cdots]$ it is true that $\mathcal{R}_i \mathcal{R}_i^H > \epsilon$, and uniformly backward controllable in finite time τ if in addition there exists a fixed time delay τ such that for $\mathcal{R}_{i,\tau} = [B_i, V_i B_{i+1}, \cdots, V_i V_{i+1} \cdots V_{i+\tau-1} B_{i+\tau-1}]$ it is true that $\mathcal{R}_{i,\tau} \mathcal{R}_{i,\tau}^H > \epsilon$ independently of i. We have [5]:

Theorem 3. *The unitary operator U will exist if and only if the pair $[V|\xi, \eta]$ is strictly backward controllable. U is then a unitary operator with observability pair* $\left[\begin{array}{c} V^* \\ \hline \xi^* \\ \eta^* \end{array}\right]$.

We have in addition the following easy to prove theorem (the proof goes by considering transition matrices reaching over the uniform controllability or observability period τ).

Theorem 4. *A unitary operator is (uniformly) controllable in finite time iff it is also (uniformly) observable in finite time and vice versa.*

The next step consists in representing U as a ratio of polynomials in the shift Z. The easiest way to do that is to start from a unitary representation for U:

$$U = d + cZ(I - aZ)^{-1}b \tag{25}$$

We then have:

Theorem 5. *Let U be a causal, unitary operator which is controllable in uniform finite time τ. Then there exist polynomial matrices P and Q in Z of order at most τ such that $U = QP^{-1}$. Moreover, their local state space degree can be chosen to be at most equal to the degree of U, and P can be chosen so that also P^{-1} is a causal operator, in which case Q^{-1} will be anticausal.*

Proof

The proof follows from the deadbeat construction. According to the assumptions the pair $[a, b]$ is controllable in uniform finite time and hence there exist a diagonal operator F such that $(a - bF)Z$ is nilpotent. Consider the operator

$$I + FZ(I - aZ)^{-1}b \tag{26}$$

By a standard argument this operator is invertible and its inverse is given by

$$P = I - FZ(I - a^f Z)^{-1}b \tag{27}$$

in which $a^f = a - bF$. As $a^f Z$ is nilpotent, $(I - a^f Z)^{-1}$ is polynomial of order at most τ, again by the deadbeat construction. Let us now compute UP^{-H}:

$$\begin{aligned} UP^{-H} &= \left(d + cZ(I - aZ)^{-1}b\right)\left(I + b^H(I - Z^H a^H)^{-1} Z^H F^H\right) \\ &= (d + cF^H) + cZ(I - Za)^{-1}(b + aF^H) \end{aligned} \tag{28}$$

because of the unitarity of the representation for U. Our candidate for Q^{-1} hence becomes $(d^H + Fc^H) + (b^H + Fa^H)(I - a^H Z^H)^{-1} Z^H c^H$, which by the way is anti-causal. We now remark that its anti-causal transition matrix

$$\begin{bmatrix} a^H & c^H \\ b^H + Fa^H & d^H + Fc^H \end{bmatrix} \tag{29}$$

is locally invertible, yielding, again by a standard argument, a causal system realization for its inverse

$$Q = d + (c - dF)Z(I - a^f Z)^{-1}b \tag{30}$$

and Q becomes indeed polynomial because $a^f Z$ is nilpotent.
QED
The theorem hence yields an explicit polynomial representation for U. We now have

$$(Z - V)^{-1} \begin{bmatrix} \xi & \eta \end{bmatrix} QP^{-1} \in \text{upper} \tag{31}$$

and hence also (because P is upper)

$$(Z-V)^{-1}\left[\begin{array}{cc}\xi & \eta\end{array}\right]Q \in \text{upper} \tag{32}$$

but the latter can only be polynomial because the product anti-causal times polynomial in Z can only produce the sum of an anti-causal term (which now is zero) and a polynomial term hence producing (24). Let $Q=\left[\begin{array}{cc}Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{array}\right]$ be a decomposition of Q conformal with U, then interpolants are produced from the expression $S=(Q_{11}p+Q_{12}q)(Q_{21}p+Q_{22}q)^{-1}$, provided the inverse exists. Actually, with $Q_1 = Q_{11}p+Q_{12}q)$ and $Q_2 = Q_{21}p+Q_{22}q$ we have

$$\xi Q_1^{\wedge}(V)+\eta Q_2^{\wedge}(V)=0 \tag{33}$$

and hence $\xi Q_1^{\wedge}(V)(Q_2^{\wedge}(V))^{-1}=\eta$ provided the inverse exists - a necessary condition. This is actually a weak form of interpolation, because in the time-varying context it is not true that $S^{\wedge}(V)=Q_1^{\wedge}(V)(Q_2^{\wedge}(V))^{-1}$. The 'load' operators p and q can be chosen constant, or causal dynamic systems, e.g. they may partially cancel the anticausal zeros in the $Q_{i,j}$ matrices. If p, q are chosen constant, a necessary (and given some technical conditions sufficient) condition for invertibility is local invertibility (i.e. for every time point i) of the corresponding transition operator

$$\left[\begin{array}{cc}a^f & b_1p+b_2q\\ c_2-(dF)_2 & d_{21}p+d_{22}q\end{array}\right]_i \tag{34}$$

Given the controllability conditions on the existence of U and Q, this can always be achieved - there is a large collection of constant p and q achieving the requested invertibility. Conversely, suppose that an interpolant $S=Q_1Q_2^{-1}$ exists of minimal degree with Q_1 and Q_2 polynomials in Z, the latter invertible. Then from $(Z-V)^{-1}B\left[\begin{array}{c}Q_1\\ -Q_2\end{array}\right]\in$ upper follows $U^*\left[\begin{array}{c}Q_1\\ -Q_2\end{array}\right]=\left[\begin{array}{c}p'\\ q'\end{array}\right]\in$ upper and hence S can be recovered as $S=(Q_{11}p+Q_{12}q)(Q_{21}p+Q_{22}q)^{-1}$ with $\left[\begin{array}{c}p\\ q\end{array}\right]=P^{-1}\left[\begin{array}{c}p'\\ q'\end{array}\right]$. The detailed exploration of these points and especially the possibility of reducing the degree of the interpolant has not been done yet to the best of our knowledge.

6 Discussion

A number of classical interpolation problems carry over nicely to the LTV case, but in some cases the algebra becomes rather involved. Consideration of the Löwner interpolation problem leads to interesting results in deadbeat control (extending the LTI case to LTV), and a new representation of a causal unitary operator as a ratio of polynomials. These may in turn lead to an extension of the classical LTI results to the LTV case, but so far the conditions are rather involved. Another unsolved interpolation problem in the LTV case is (again to the best of our knowledge) the singular care of the Schur-Takagi problem, with important applications to model reduction theory.

Bibliography

[1] D. Alpay, P. Dewilde, and H. Dym. Lossless Inverse Scattering and reproducing kernels for upper triangular operators. In I. Gohberg, editor, *Extension and Interpolation of Linear Operators and Matrix Functions*, volume 47 of *Operator Theory, Advances and Applications*, pages 61–135. Birkhäuser Verlag, 1990.

[2] A.C. Antoulas and B.D.O. Anderson. On the scalar rational interpolation problem. *IMA J. Math. Control Inform.*, 3:61–88, 1986.

[3] A.C. Antoulas, J.A. Ball, J. Kang and J.C. Willems. On the solution of the minimal rational interpolation problem. *Linear Algebra and its Applications*, 137:511–573, 1990.

[4] D. Alpay, P. Dewilde and D. Volok. Interpolation and approximation of quasiseparable systems: the schur-takagi case. *Calcolo*, 42:139–156, 2005.

[5] P. Dewilde and A.-J. van der Veen. *Time-varying Systems and Computations*. Kluwer, 1998.

[6] P.M. Dewilde. A course on the algebraic Schur and Nevanlinna-Pick interpolation problems. In Ed. F. Deprettere and A.J. van der Veen, editors, *Algorithms and Parallel VLSI Architectures*. Elsevier, 1991.

[7] P. Van Dooren. A unitary method for deadbeat control. *Proceedings MTNS*, 1983.

[8] V.M. Popov. Invariant description of linear, time-invariant controllable systems. *SIAM J. on Control*, 10:252–264, 1972.