

# A Low Complexity Blind Estimator of Narrowband Polynomial Phase Signals

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**Abstract**—Consider the problem of estimating the parameters of multiple polynomial phase signals observed by a sensor array. In practice, it is difficult to maintain a precisely calibrated array. The array manifold is then assumed to be unknown, and the estimation is referred to as blind estimation. To date, only an approximated maximum likelihood estimator (AMLE) was suggested for blindly estimating the polynomial coefficients of each signal. However, this estimator requires a multidimensional search over the entire coefficient space. Instead, we propose an estimation approach which is based on two steps. First, the signals are separated using a blind source separation technique, which exploits the constant modulus property of the signals. Then, the coefficients of each polynomial are estimated using a least squares method applied to the unwrapped phase of the estimated signal. This estimator does not involve any search in the coefficient spaces. The computational complexity of the proposed estimator increases linearly with respect to the polynomial order, whereas that of the AMLE increases exponentially. Simulation results show that the proposed estimator achieves the Cramér-Rao lower bound at moderate or high signal to noise ratio.

**Index Terms**—Algebraic constant modulus algorithm, blind estimation, maximum likelihood estimation, polynomial phase signals, synthetic aperture radar.

## I. INTRODUCTION

**P**OLYNOMIAL phase signals (PPSs) are commonly used in synthetic aperture radar (SAR) imaging and radio communications applications. A recent example is the time and frequency synchronization in a multiple-input-multiple-output (MIMO) long term evolution (LTE) with universal mobile telecommunications system (UMTS), as defined by the third-generation partnership project (3GPP) standard. In these applications the phase of the observed signal is usually modulated as a polynomial function of time. This signal modeling is motivated by Weierstrass's theorem which implies that an arbitrary time varying phase observed over a finite observation time can be well approximated by a sufficiently higher order polynomial.

The problem of estimating the coefficients of the polynomial phases has been researched for signals observed by a single sensor [1]–[5] and by a sensor array [6]–[13]. The estima-

tion techniques in [6]–[12] assume that the array is perfectly calibrated and the spatial signature of the array is completely known. In practice, it is difficult to maintain a precisely calibrated array due to several reasons including temperature, pressure, humidity, mechanical vibrations, and objects in the near field of the array. Furthermore, calibration is an expensive process. Alternatively, the spatial signature of the array is assumed to be unknown, and blind<sup>1</sup> estimation techniques are applied [14]. To date, the paper of Zeira and Friedlander [13] is the only published work on blind estimation of the polynomial coefficients of each of the signals impinging on a sensor array. The focus of our paper is on this blind estimation of PPSs.

An interesting application of parameter estimation of PPSs is surveillance systems where the imaging and identification of moving targets is performed by SAR [15]. The main challenge is to separate the moving targets and then to estimate the motion parameters of each target (velocity and acceleration). These parameters are embedded in the coefficients of each polynomial. Another possible application occurs when a ground active radar receives the reflected signal from a moving target in a multipath environment. The output of the antenna array is a sum of PPSs, where each PPS has a polynomial with different coefficients. In both cases the estimation should be performed even if the array is uncalibrated.

The exact maximum likelihood estimator (MLE) of the coefficients of each of the polynomials is computationally intensive. It requires a  $PQ$ -dimensional search in the polynomial coefficients space where  $P$  is the order of each polynomial and  $Q$  is the number of PPSs. Instead, in [13] Zeira and Friedlander proposed an approximated MLE (AMLE) which reduces the computation of the exact MLE to a single  $P$ -dimensional search in the polynomial coefficients space. The AMLE is based on the assumption that PPSs tend to be orthogonal to each other, in other words, that their parameters are not nearly identical. Then, the AMLE is obtained by decoupling the estimation of one PPS from the others. Given the data, we perform a  $P$ -dimensional search in the polynomial coefficients space, and evaluate the cost function of the AMLE for each point in the space. The PPSs are estimated as the  $Q$  local peaks of the cost function.

Herein, we suggest an estimation method which we term as SEparate-ESTimate (SEES) estimator. First, we exploit the fact that the waveforms of the signals are constant modulus, and separate the signals using the zero-forcing algebraic constant modulus algorithm (ZF-ACMA) [16], [17]. The ZF-ACMA estimates all signals simultaneously using linear algebraic operations. Then, we estimate the polynomial coefficients of each

<sup>1</sup>The term “blind” usually refers to an estimation method which assumes that the spatial signature of the array is unknown.

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signal from the wrapped phase of the estimated signal. We suggest to first perform phase unwrapping as proposed in [2] and then estimate the polynomial coefficients given the unwrapped phases using a least squares (LS) method. The last step can be implemented with other approaches. For example, the method in [18] integrates the phase unwrapping and the polynomial coefficients estimation into one process using a recursive LS technique. In [19] and [20] a similar two-step approach was proposed for the maximum likelihood separation and direction of arrival (DOA) estimation of unstructured constant modulus signals using a sensor array, where the array response is assumed to be known. We show that the complexity of the proposed SEES technique increases linearly with respect to (w.r.t.) the polynomial order and the number of samples, whereas that of the AMLE increases exponentially. This complexity reduction is important in real-time systems, for example. We also derive the asymptotic bias and covariance of the polynomial coefficients estimated by the SEES method. The estimates are shown to be asymptotically unbiased. For the case of two second-order PPSs (also known as linear frequency-modulation (FM) signals, or simply chirps) with closely spaced DOAs impinging on a uniform linear array (ULA), we further derive approximated closed-form expressions for the variances of the estimates of the initial frequency (first coefficient) and of the frequency rate (twice the value of the second coefficient) of each signal. It is shown that both variances decrease as  $1/(\text{SNR} \cdot M^3 \cdot (\Delta\theta)^2)$  where SNR stands for the signal to noise ratio,  $M$  is the number of array elements, and  $\Delta\theta$  is the separation between the DOAs of the signals. However, the variance of the initial frequency decreases as  $\mathcal{O}(1/N^3)$  while the variance of the estimate of the frequency rate decreases as  $\mathcal{O}(1/N^5)$ .

The performance of the proposed SEES estimator is also demonstrated in simulations. The performance criterion is defined as the root mean square error (RMSE) between the estimated polynomial coefficients and their true values. We consider several types of PPSs including: chirp signals, quadratic FM signals, and fourth-order PPSs. We compare the RMSE of the SEES estimator with the RMSE of the AMLE as a function of: the SNR, the DOAs separation, the number of samples, the separation between the initial frequencies, and the separation between the frequency rates of the signals. The Cramer-Rao lower bound (CRLB) derived in [13] is used as a performance benchmark for the Monte-Carlo trials, together with the theoretical RMSE of the proposed estimator. The main conclusions obtained from the simulation results are: 1) The RMSE of the SEES estimator achieves the CRLB at moderate or high SNR; 2) Compared to the AMLE, the RMSE of the SEES estimator is more sensitive to a small DOA separation. However, for large separation, the performance of both estimators is equivalent; 3) The RMSE of the SEES estimator is equivalent to that of the AMLE w.r.t. the number of samples; 4) Both the AMLE and the SEES are not sensitive to the separation between the initial frequencies of the signals, and to the separation between the frequency rates of the signals; 5) The processing time of the SEES algorithm is much smaller than that of the AMLE.

The rest of the paper is organized as follows: In Section II we present the problem formulation, in Section III we derive the exact MLE and the AMLE, in Section IV we develop the

SEES estimator, in Section V we derive the asymptotic bias and covariance of the estimated polynomial coefficients of each signal, and develop closed-form expressions for the case of signals with closely spaced DOAs, in Section VI we discuss the computational complexity of the AMLE and the proposed estimator, in Section VII we present numerical results, and finally in Section VIII we conclude the paper.

The following notation is used in the paper: uppercase bold and lowercase bold fonts denote matrices and vectors, respectively. The superscripts  $(\cdot)^*$ ,  $(\cdot)^T$ ,  $(\cdot)^H$ ,  $(\cdot)^{-1}$  stand for conjugate, transpose, conjugate transpose, and inverse, respectively.  $\mathbf{I}_N$  is the  $N \times N$  identity matrix.  $\mathbf{1}_N$  (or  $\mathbf{0}_N$ ) is the  $N \times 1$  vector with all elements equal to one (or zero).  $[\mathbf{X}]_{n,m}$  stands for the  $(n, m)$ th element of the matrix  $\mathbf{X}$ .  $\Re\{\mathbf{x}\}$  and  $\Im\{\mathbf{x}\}$  are the real and imaginary parts of the complex vector  $\mathbf{x}$ , respectively.  $\mathcal{O}(n)$  indicates that the complexity of an algorithm is in the order of  $n$  real multiplication operations.  $\text{vec}(\mathbf{X})$  stands for the stacking of the columns of the matrix  $\mathbf{X}$ .  $\|\mathbf{x}\|$  is the 2-norm of  $\mathbf{x}$ .  $|x|$  is the absolute value of  $x$ .  $\arg(x) \in [0, 2\pi)$  is the phase of  $x$ .  $\otimes$  is the Kronecker product. The bias and the covariance of the estimate  $\hat{\mathbf{x}}$  of a real random vector  $\mathbf{x}$  are defined as  $\text{bias}(\hat{\mathbf{x}}) \triangleq E[\hat{\mathbf{x}}] - \mathbf{x}$ , and  $\text{cov}(\hat{\mathbf{x}}) = E[(\hat{\mathbf{x}} - E[\hat{\mathbf{x}}])(\hat{\mathbf{x}} - E[\hat{\mathbf{x}}])^T]$ , respectively.

## II. PROBLEM FORMULATION

Consider a sensor array composed of  $M$  sensors, and  $Q$  transmitting sources. Each source transmits a narrowband PPS. The  $M \times 1$  noisy sampled signal vector at the array output is given by [10], [11], [13]

$$\begin{aligned} \mathbf{x}(n) &= \sum_{q=1}^Q \alpha_q \mathbf{a}(\theta_q) s_q(n) + \mathbf{e}(n) \\ &= \mathbf{A} \mathbf{s}(n) + \mathbf{e}(n), \quad n = 0, \dots, N-1 \end{aligned} \quad (1)$$

where  $N$  is the number of samples, and the  $M \times Q$  matrix  $\mathbf{A}$ , the  $Q \times 1$  vector  $\mathbf{s}(n)$  and the  $M \times 1$  vector  $\mathbf{e}(n)$  are defined as

$$\mathbf{x}(n) \triangleq [x_1(n), \dots, x_M(n)]^T \quad (2)$$

$$\mathbf{A} \triangleq [\alpha_1 \mathbf{a}(\theta_1), \dots, \alpha_Q \mathbf{a}(\theta_Q)] \quad (3)$$

$$\mathbf{s}(n) \triangleq [s_1(n), \dots, s_Q(n)]^T \quad (4)$$

$$\mathbf{e}(n) \triangleq [e_1(n), \dots, e_M(n)]^T \quad (5)$$

where  $x_m(n)$ ,  $m = 1, \dots, M$  is the output at the  $m$ th element of the array,  $e_m(n)$ ,  $m = 1, \dots, M$  is the additive noise at the output of the  $m$ th element of the array,  $s_q(n)$  is the  $q$ th PPS,  $\alpha_q$  is the unknown complex amplitude of the  $q$ th PPS, and finally, the  $M \times 1$  vector  $\mathbf{a}(\theta_q)$  is the array response to the  $q$ th signal transmitted from DOA denoted by  $\theta_q$ . We assume that  $\mathbf{a}(\theta)$  (the array manifold) is unknown as occurs, for example, when the array is uncalibrated [13], [14]. Observe that the unknown complex amplitude of the  $q$ th signal,  $\alpha_q$ , is then absorbed in  $\mathbf{a}(\theta_q)$ . The  $q$ th PPS is given as

$$s_q(n) \triangleq e^{j\phi_q(n)} \quad (6)$$

where  $\phi_q(n)$  is the instantaneous phase of the signal

$$\phi_q(n) \triangleq \mathbf{u}^T(n) \mathbf{b}_q \quad (7)$$

and  $\mathbf{u}(n)$  is a  $P \times 1$  vector defined as

$$\mathbf{u}(n) \triangleq [nT_s, \dots, (nT_s)^P]^T \quad (8)$$

where  $T_s$  is the sampling time,  $P$  is the known order of the polynomial, and  $\mathbf{b}_q$  is a  $P \times 1$  vector of polynomial coefficients of the  $q$ th PPS defined as

$$\mathbf{b}_q \triangleq [b_{q,1}, \dots, b_{q,P}]^T. \quad (9)$$

The vector  $\mathbf{e}(n)$  is assumed to be spatially and temporally white Gaussian complex random vector with zero mean and covariance matrix  $\sigma_n^2 \mathbf{I}_M$ , where  $\sigma_n^2$  is unknown.

The unknown parameter vector of the model is then expressed as

$$\boldsymbol{\psi} \triangleq [\mathbf{b}^T, \boldsymbol{\rho}^T, \sigma_n^2]^T \quad (10)$$

where<sup>2</sup>

$$\mathbf{b} \triangleq [\mathbf{b}_1^T, \dots, \mathbf{b}_Q^T]^T \quad (11)$$

$$\boldsymbol{\rho} \triangleq [\Re\{\alpha_1 \mathbf{a}_1^T\}, \Im\{\alpha_1 \mathbf{a}_1^T\}, \dots, \Re\{\alpha_Q \mathbf{a}_Q^T\}, \Im\{\alpha_Q \mathbf{a}_Q^T\}]^T \quad (12)$$

The problem discussed herein is: Given the observations  $\{\mathbf{x}(n)\}_{n=0}^{N-1}$ , estimate the polynomial coefficients  $\{\mathbf{b}_q\}_{q=1}^Q$ , assuming that the array manifold is unknown.

*Remark:* We note that the signals can be considered as narrowband if  $\forall q: (1/\omega_c) |d\phi_q(n)/dn| \ll W/\omega_c$  where  $\omega_c$  and  $W$  are the carrier frequency and bandwidth of the signal, respectively [9, p. 342]. This condition sets restrictions on the possible polynomial coefficients, such as  $|b_{q,1}| < W$ ,  $|b_{q,2}| < W/N$ , etc. For many applications these restrictions might not matter since the sources are not moving too fast. Detailed identifiability conditions on the estimation of PPSs are also presented in [21].

Clearly, the unknown parameter vector is not uniquely estimated for any number of transmitted PPSs. We therefore determine an upper bound on the maximum number of PPSs that their parameters can be uniquely estimated.

#### A. A Necessary Condition for Unique Estimation

A necessary condition to ensure unique estimation is known as the unknowns-measurements condition [16], [22]. This condition requires that the dimension of the set containing the unknowns is equal to or smaller than the dimension of the set containing the measurements (a complex unknown and a complex measurement in these sets are described by their real and imaginary parts). The dimension of the unknown set of the current model is the sum of:  $PQ$  (associated with  $\mathbf{b}$ ),  $2MQ$  (associated with  $\boldsymbol{\rho}$ ), and 1 (associated with  $\sigma_n^2$ ). The dimension of the measurements set of the current model is  $2MN$  (associated with  $\{x_m(n)\}_{m=1, n=0}^{M, N-1}$ ). A unique estimation of  $\boldsymbol{\psi}$  may exist if  $PQ + 2MQ + 1 \leq 2NM$ , that is

$$Q \leq \frac{2NM - 1}{2M + P}. \quad (13)$$

<sup>2</sup>The definition of  $\mathbf{b}$  used here is different from that used in [13] to derive the CRLB. In [13, Eq. (3)] the vector which contains the unknown polynomial coefficients is defined as  $\mathbf{b}' \triangleq \sum_{q=1}^Q \mathbf{b}_q \otimes \mathbf{e}_q$  where  $\mathbf{e}_q$  is the  $q$ th column of  $\mathbf{I}_Q$ .

Assume that  $2M \gg P$  which implies that the number of array elements must be much greater than the order of the PPS. We then conclude that  $Q < N - 1$ . In practice,  $N$  is large and thus this necessary condition holds.

This upper bound on  $Q$  holds for any estimation method. However, to apply the ZF-ACMA we require that [16]: i)  $Q \leq M$ , and ii)  $Q^2 \leq N$ , which further restrict the maximum number of PPSs. We emphasize that the condition in (13) holds under the assumption that all PPSs have the same polynomial order, and that the modification to the general case, where each PPS has a different polynomial order, is straightforward.

### III. THE EXACT AND APPROXIMATED MLES

We present detailed derivations of the exact MLE and the AMLE. The former was not presented in [13], and the latter was not fully explained there.

#### A. The Exact MLE

The exact MLE estimates  $\boldsymbol{\psi}$  given the observations in (1) as follows. Define the  $MN \times 1$  vector  $\mathbf{x}$ , the  $MN \times MQ$  matrix  $\mathbf{S}(\mathbf{b})$  and the  $MQ \times 1$  vector  $\mathbf{a}$  as follows:

$$\mathbf{x} \triangleq [\mathbf{x}^T(0), \dots, \mathbf{x}^T(N-1)]^T \quad (14)$$

$$\mathbf{S}(\mathbf{b}) \triangleq [\mathbf{s}(0), \dots, \mathbf{s}(N-1)]^T \otimes \mathbf{I}_M \quad (15)$$

$$\mathbf{a} \triangleq \text{vec}(\mathbf{A}). \quad (16)$$

Using (14)–(16), the negative log likelihood function of the observations in (1) is given by

$$L(\mathbf{x}; \boldsymbol{\psi}) = \frac{1}{\sigma_n^2} \|\mathbf{x} - \mathbf{S}\mathbf{a}\|^2 + NM \log(\sigma_n^2) + N \log(\pi). \quad (17)$$

Differentiating (17) with respect to  $\sigma_n^2$  and equating the result to zero yields the MLE of  $\sigma_n^2$

$$\hat{\sigma}_n^2 = \frac{1}{MN} \|\mathbf{x} - \mathbf{S}(\hat{\mathbf{b}})\hat{\mathbf{a}}\|^2 \quad (18)$$

where  $\hat{\mathbf{b}}$ ,  $\hat{\mathbf{a}}$  are the MLEs of  $\mathbf{b}$ ,  $\mathbf{a}$ , respectively. Substituting (18) into (17) yields that  $\mathbf{a}$  and  $\mathbf{S}(\mathbf{b})$  minimize the cost function  $\|\mathbf{x} - \mathbf{S}(\mathbf{b})\mathbf{a}\|^2$ . Differentiating w.r.t  $\mathbf{a}$  leads to

$$\hat{\mathbf{a}} = \left( \mathbf{S}^H(\hat{\mathbf{b}})\mathbf{S}(\hat{\mathbf{b}}) \right)^{-1} \mathbf{S}^H(\hat{\mathbf{b}})\mathbf{x}. \quad (19)$$

Let  $\mathbf{b}' \triangleq [\mathbf{b}'_1^T, \dots, \mathbf{b}'_Q^T]^T$  be an arbitrary  $PQ \times 1$  vector in the polynomial coefficients space of all the signals. By substituting (19) into the last cost function, we obtain that the exact MLE of  $\mathbf{b}$ , denoted by  $\hat{\mathbf{b}}$ , is given as

$$\hat{\mathbf{b}} = \arg \max_{\mathbf{b}'} L_e(\mathbf{b}') \quad (20)$$

$$L_e(\mathbf{b}') = \|\mathbf{P}_{\mathbf{S}(\mathbf{b}')} \mathbf{x}\|^2 \quad (21)$$

where  $\mathbf{P}_{\mathbf{S}(\mathbf{b}')}$  is the  $MN \times MN$  matrix defined as

$$\mathbf{P}_{\mathbf{S}(\mathbf{b}')} \triangleq \mathbf{S}(\mathbf{b}') (\mathbf{S}^H(\mathbf{b}')\mathbf{S}(\mathbf{b}'))^{-1} \mathbf{S}^H(\mathbf{b}'). \quad (22)$$

This estimator requires a  $PQ$ -dimensional search over the space of all the polynomial coefficients of all signals.

### B. The AMLE

In [13] the AMLE was proposed in order to reduce the computational load involved with the exact MLE. The AMLE is based on the assumption that PPSs tend to be orthogonal to each other, unless their parameters are nearly identical, that is

$$\frac{1}{N} \sum_{n=0}^{N-1} \mathbf{s}(n) \mathbf{s}^H(n) \cong \mathbf{I}_Q. \quad (23)$$

Using (23) we get that

$$\begin{aligned} \frac{1}{N} \mathbf{S}^H(\mathbf{b}') \mathbf{S}(\mathbf{b}') &= \frac{1}{N} \left[ [\mathbf{s}(0), \dots, \mathbf{s}(N-1)]^* \otimes \mathbf{I}_M \right] \\ &\quad \times \left[ [\mathbf{s}(0), \dots, \mathbf{s}(N-1)]^T \otimes \mathbf{I}_M \right] \\ &= \frac{1}{N} [\mathbf{s}(0), \dots, \mathbf{s}(N-1)]^* \\ &\quad \times [\mathbf{s}(0), \dots, \mathbf{s}(N-1)]^T \otimes \mathbf{I}_M \\ &= \frac{1}{N} \left[ \sum_{n=0}^{N-1} \mathbf{s}(n) \mathbf{s}^H(n) \right]^* \otimes \mathbf{I}_M \\ &\cong \frac{1}{N} \mathbf{I}_{MQ} \end{aligned} \quad (24)$$

where in the second passing we use the result that  $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$ . Substituting (24) into (21) yields that (21) is equivalent to

$$\begin{aligned} L' &= \left\| \mathbf{S}^H(\mathbf{b}') \mathbf{x} \right\|^2 \\ &= \sum_{q=1}^Q \left\| \sum_{n=0}^{N-1} \mathbf{x}(n) e^{-j\mathbf{b}'_q^T \mathbf{u}(n)} \right\|^2 \\ &= \sum_{q=1}^Q \sum_{m=1}^M \left| \sum_{n=0}^{N-1} x_m(n) e^{-j\mathbf{b}'_q^T \mathbf{u}(n)} \right|^2. \end{aligned} \quad (25)$$

Note that in the second passing we used the expression for  $\mathbf{S}(\mathbf{b})$  given in (15), where instead of  $\mathbf{s}(n)$ , we substitute the definition given in (4), and each element of  $\mathbf{s}(n)$  is expressed using (6) and (7).

We conclude that the estimation of the polynomial coefficients of each signal is decoupled from the estimation of the other PPSs. In this sense we can estimate the signals by performing a single search as follows. Let  $\tilde{\mathbf{b}} \triangleq [b_1, \dots, b_P]^T$  be an arbitrary  $P \times 1$  vector in the polynomial coefficients space. The AMLE cost function is then given as

$$L_a(\tilde{\mathbf{b}}) = \sum_{m=1}^M \left| \sum_{n=0}^{N-1} x_m(n) e^{-j\tilde{\mathbf{b}}^T \mathbf{u}(n)} \right|^2. \quad (26)$$

The estimated polynomial coefficients of the  $Q$  PPSs are associated with the  $Q$  highest peaks of  $L_a(\tilde{\mathbf{b}})$ . Notice that the AMLE reduces the  $PQ$ -dimensional search of the exact MLE to a  $P$ -dimensional search in the polynomial coefficients space.

## IV. THE PROPOSED SEES METHOD

In this section, we propose a low complexity method to estimate the polynomial coefficients of each signal, which we term

as SEparate-ESTimate (SEES) method. As we show in the numerical results section, this estimator is not only computationally efficient but also achieves the CRLB at moderate or high SNR. The algorithm is based on two main steps:

- 1) Separate the signals using a blind source separation technique. Herein, we consider the ZF-ACMA [16], [17]. At the end of this step we obtain the estimates of  $\{s_q(n)\}_{q=1, n=0}^{Q, N-1}$ , denoted by  $\{\hat{s}_q(n)\}_{q=1, n=0}^{Q, N-1}$ . The ZF-ACMA uses only the constant modulus property of the signals, without exploiting the polynomial phase structure.
- 2) Estimate the polynomial coefficients of the  $q$ th signal given the unwrapped phases of the  $q$ th estimated signal.

This two-step approach is similar to [19], [20] for the estimation of DOAs of constant modulus signals using a sensor array, assuming that the array response is perfectly known.

Based on these two steps we develop a model which linearly depends on the polynomial coefficients of the  $q$ th signal with the presence of small additive noises. This model will be used to obtain a LS estimate of the polynomial coefficients. According to [17] the output of the ZF-ACMA in the presence of asymptotically small measurement noises or large number of samples is

$$\hat{\mathbf{s}}(n) = \mathbf{W}^H \mathbf{x}(n) \quad (27)$$

where  $\mathbf{W}$  is the  $M \times Q$  zero-forcing beamforming matrix

$$\mathbf{W} = (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H. \quad (28)$$

Note that due to the phase ambiguity of the ZF-ACMA, the phase of each signal at the output of the ZF-ACMA is obtained up to an unknown constant phase shift, which is represented by  $c_q$ ,  $q = 1, \dots, Q$  where  $c_q \in [-\pi, \pi)$ . We collect these phase shifts in a  $Q \times Q$  matrix  $\mathbf{D}$

$$\mathbf{D} \triangleq \text{diag}(e^{jc_1}, \dots, e^{jc_Q}). \quad (29)$$

The output of the ZF-ACMA is then expressed as

$$\hat{\mathbf{s}}(n) = \mathbf{D} \mathbf{s}(n) + \boldsymbol{\varepsilon}(n) \quad (30)$$

where  $\boldsymbol{\varepsilon}(n)$  is the  $Q \times 1$  vector

$$\boldsymbol{\varepsilon}(n) \triangleq [\varepsilon_1(n), \dots, \varepsilon_Q(n)]^T = (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{e}(n). \quad (31)$$

Observe that the  $q$ th signal at the output of the ZF-ACMA is

$$\hat{s}_q(n) = |\hat{s}_q(n)| e^{j\hat{\phi}_q(n)}. \quad (32)$$

Following (30) the magnitude  $|\hat{s}_q(n)|$  is given as

$$\begin{aligned} |\hat{s}_q(n)| &= |\cos(\phi_q(n) + c_q) + \Re\{\varepsilon_q(n)\} \\ &\quad + j(\sin(\phi_q(n) + c_q) + \Im\{\varepsilon_q(n)\})| \\ &= [1 + 2\Re\{\varepsilon_q(n)\} \cos(\phi_q(n) + c_q) \\ &\quad + 2\Im\{\varepsilon_q(n)\} \sin(\phi_q(n) + c_q) + |\varepsilon_q(n)|^2]^{1/2} \end{aligned} \quad (33)$$

Since we assume that small errors are present, that is  $|\varepsilon_q(n)| \ll 1$ ,  $q = 1, \dots, Q$ , we neglect the second, third, and fourth terms under the square root, and thus we get that  $|\hat{s}_q(n)| \cong 1$ . This means that under the assumption of small errors, each signal at the output of the ZF-ACMA is (approximately) a constant

modulus signal. Consider next the phase  $\hat{\phi}_q(n)$  which is given as

$$\hat{\phi}_q(n) = \arctan \left( \frac{\sin(\phi_q(n) + c_q) + \Im\{\varepsilon_q(n)\}}{\cos(\phi_q(n) + c_q) + \Re\{\varepsilon_q(n)\}} \right). \quad (34)$$

By assuming that  $\Re\{\varepsilon_q(n)\}$  and  $\Im\{\varepsilon_q(n)\}$  are small, we can use a first-order Taylor series to approximate this phase as

$$\begin{aligned} \hat{\phi}_q(n) &\cong \arctan \left( \frac{\sin(\phi_q(n) + c_q)}{\cos(\phi_q(n) + c_q)} \right) - \sin(\phi_q(n) + c_q) \\ &\quad \times \Re\{\varepsilon_q(n)\} + \cos(\phi_q(n) + c_q) \Im\{\varepsilon_q(n)\} \\ &= \phi_q(n) + c_q - \sin(\phi_q(n) + c_q) \Re\{\varepsilon_q(n)\} \\ &\quad + \cos(\phi_q(n) + c_q) \Im\{\varepsilon_q(n)\}. \end{aligned} \quad (35)$$

The information on the polynomial coefficients is hidden in the phases  $\{\hat{\phi}_q(n)\}_{q=1, n=0}^{Q, N-1}$ . The wrapped phases are simply computed by considering the arguments of  $\{\hat{s}_q(n)\}_{n=0, q=1}^{N-1, Q}$ . However, to estimate the polynomial coefficients we are interested in the unwrapped version of the phase, which is obtained from  $\{\hat{s}_q(n)\}_{n=0, q=1}^{N-1, Q}$  with an unwrapping procedure. There are several techniques to perform phase unwrapping for PPSs (e.g., [2] and [18]). Herein, we use the procedure presented in [2]. The unwrapped phases of  $\{\hat{\phi}_q(n)\}_{n=0}^{N-1}$ , denoted by  $\{\tilde{\phi}_q(n)\}_{n=0}^{N-1}$ , are given by (see [2, p. 2120] and also [2])

$$\begin{aligned} \tilde{\phi}_q(0) &= \hat{\phi}_q(0) \quad \text{mod } 2\pi \\ \tilde{\phi}_q(1) &= \left( \left( \hat{\phi}_q(1) - \hat{\phi}_q(0) \right) \quad \text{mod } 2\pi \right) + \tilde{\phi}_q(0) \\ \tilde{\phi}_q(n) &= \delta_q(n) + 2\tilde{\phi}_q(n-1) - \tilde{\phi}_q(n-2) \\ &\quad n = 2, \dots, N-1 \end{aligned} \quad (36)$$

where

$$\delta_q(n) \triangleq \left( \hat{\phi}_q(n) - 2\hat{\phi}_q(n-1) + \hat{\phi}_q(n-2) \right) \quad \text{mod } 2\pi. \quad (37)$$

We further assume that the noises are small enough such that they do not cause any  $2\pi$  jumps in the unwrapping procedure. Thus, the unwrapped phase at the end of this step is

$$\begin{aligned} \tilde{\phi}_q(n) &= \hat{\phi}_q(n) \\ &\cong c_q + \sum_{q=1}^Q n^p b_{q,p} - \sin(\phi_q(n) + c_q) \Re\{\varepsilon_q(n)\} \\ &\quad + \cos(\phi_q(n) + c_q) \Im\{\varepsilon_q(n)\}. \end{aligned} \quad (38)$$

Note that  $\Re\{\varepsilon(n)\}$  and  $\Im\{\varepsilon(n)\}$  are zero mean Gaussian random vectors. This serves as a motivation for applying the LS method. By neglecting the additive noise part we obtain an approximated linear model for the parameters of interest (polynomial coefficients) given the measurements (unwrapped phase). Define the  $N \times 1$  vector

$$\tilde{\phi}_q \triangleq \left[ \tilde{\phi}_q(0), \dots, \tilde{\phi}_q(N-1) \right]^T. \quad (39)$$

We then rewrite (38) in a vector form as

$$\tilde{\phi}_q \cong c_q \mathbf{1} + \mathbf{U}^T \mathbf{b}_q = \tilde{\mathbf{U}}^T \mathbf{h}_q \quad (40)$$

where  $\tilde{\mathbf{U}}$  is the  $(P+1) \times N$  Vandermonde matrix

$$\tilde{\mathbf{U}} \triangleq [\mathbf{1}_N \mathbf{U}^T]^T \quad (41)$$

$$\mathbf{U} \triangleq [\mathbf{u}(0), \dots, \mathbf{u}(N-1)] \quad (42)$$

and  $\mathbf{u}(n)$  is given by (8), and also  $\mathbf{h}_q \triangleq [c_q, \mathbf{b}_q^T]^T$  is a  $(P+1) \times 1$  vector. The unknown parameters  $\hat{c}_q, \hat{\mathbf{b}}_q$  are estimated using a LS method as follows (a similar LS estimation was used in [12] without considering the estimation of the arbitrary initial phase)

$$\left[ \hat{c}_q, \hat{\mathbf{b}}_q^T \right] = \arg \min_{\mathbf{h}_q} \|\tilde{\phi}_q - \tilde{\mathbf{U}}^T \mathbf{h}_q\|^2 = (\tilde{\mathbf{U}} \tilde{\mathbf{U}}^T)^{-1} \tilde{\mathbf{U}} \tilde{\phi}_q. \quad (43)$$

Note that the first entry of the estimated vector is the estimate of the nuisance parameter  $c_q$ . The estimate of the polynomial coefficients vector is given after a few mathematical steps as

$$\hat{\mathbf{b}}_q = (\mathbf{U} \mathbf{U}^T)^{-1} \mathbf{U} \left( \mathbf{I}_N - \frac{1}{\kappa} \mathbf{1}_N \mathbf{1}_N^T (\mathbf{I}_N - \mathbf{U} (\mathbf{U} \mathbf{U}^T)^{-1} \mathbf{U}) \right) \tilde{\phi}_q \quad (44)$$

where  $\kappa \triangleq N - \mathbf{1}_N^T \mathbf{U}^T (\mathbf{U} \mathbf{U}^T)^{-1} \mathbf{U} \mathbf{1}_N$ . Using the matrix inversion lemma we obtain that (44) can be written more compactly as

$$\hat{\mathbf{b}}_q = (\mathbf{U} \mathbf{P} \mathbf{U}^T)^{-1} \mathbf{U} \mathbf{P} \tilde{\phi}_q \quad (45)$$

where  $\mathbf{P}$  is the  $N \times N$  projection matrix

$$\mathbf{P} \triangleq \mathbf{I}_N - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^T. \quad (46)$$

We note that the above LS solution can also be obtained by exploiting the Vandermonde structure of  $\mathbf{U}$  [23]–[25] which leads to computational savings. This completes the derivation of the proposed estimator.

## V. PERFORMANCE ANALYSIS

### A. Analysis

We evaluate the bias and covariance of the polynomial coefficients estimate  $\hat{\mathbf{b}}_q$  given in (45) when it is estimated in the presence of noise. Note that the real and imaginary parts,  $\Re\{\mathbf{e}(n)\}$  and  $\Im\{\mathbf{e}(n)\}$  of the noise in (1) are zero mean uncorrelated Gaussian random vectors with covariance matrices of  $(\sigma_n^2/2)\mathbf{I}_M$ . Note that we can also write (31) as

$$\begin{aligned} \boldsymbol{\varepsilon}(n) &= \Re\{\boldsymbol{\varepsilon}(n)\} + j\Im\{\boldsymbol{\varepsilon}(n)\} \\ \Re\{\boldsymbol{\varepsilon}(n)\} &\triangleq \Re\left\{ (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \right\} \Re\{\mathbf{e}(n)\} \\ &\quad - \Im\left\{ (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \right\} \Im\{\mathbf{e}(n)\} \\ \Im\{\boldsymbol{\varepsilon}(n)\} &\triangleq \Re\left\{ (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \right\} \Im\{\mathbf{e}(n)\} \\ &\quad + \Im\left\{ (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \right\} \Re\{\mathbf{e}(n)\}. \end{aligned} \quad (47)$$

Therefore,  $\Re\{\boldsymbol{\varepsilon}(n)\}$  and  $\Im\{\boldsymbol{\varepsilon}(n)\}$  are zero mean Gaussian random vectors with covariance matrices

$$\begin{aligned} \mathbf{C}_1 &\triangleq E \left[ \Re\{\boldsymbol{\varepsilon}(n)\} \Re\{\boldsymbol{\varepsilon}(n)\}^T \right] = E \left[ \Im\{\boldsymbol{\varepsilon}(n)\} \Im\{\boldsymbol{\varepsilon}(n)\}^T \right] \\ &= \frac{\sigma_n^2}{2} \Re\left\{ (\mathbf{A}^H \mathbf{A})^{-1} \right\} \end{aligned} \quad (48)$$

$$\mathbf{C}_2 \triangleq E \left[ \Re\{\boldsymbol{\varepsilon}(n)\} \Im\{\boldsymbol{\varepsilon}(n)\}^T \right] = -\frac{\sigma_n^2}{2} \Im\left\{ (\mathbf{A}^H \mathbf{A})^{-1} \right\}. \quad (49)$$

Observe that the  $(q, q)$ th element of  $\mathbf{C}_2$ , which is the cross-correlation between  $\Re\{\varepsilon_q(n)\}$  and  $\Im\{\varepsilon_q(n)\}$ , is equal to zero. We denote the variance of  $\Re\{\varepsilon_q(n)\}$  and  $\Im\{\varepsilon_q(n)\}$  by

$$\eta_q^2 \triangleq [\mathbf{C}_1]_{q,q} = \frac{\sigma_n^2}{2} \left[ \Re \left\{ (\mathbf{A}^H \mathbf{A})^{-1} \right\} \right]_{q,q}. \quad (50)$$

Recall that we assume that the noises are small enough such that the unwrapped phase is  $\tilde{\phi}_q(n) \cong \hat{\phi}_q(n)$  [i.e., ignoring the noise term in (35)]. However, by taking the noise in (35) into account, we get that the bias and covariance of the vector  $\tilde{\phi}_q$ , defined in (39), are therefore,  $\text{bias}(\tilde{\phi}_q) = \mathbf{0}$  and  $\text{cov}(\tilde{\phi}_q) = \eta_q^2 \mathbf{I}_N$ . Finally, according to (45) we get that the bias of  $\hat{\mathbf{b}}_q$  is

$$\begin{aligned} \text{bias}(\hat{\mathbf{b}}_q) &= (\mathbf{U}\mathbf{P}\mathbf{U}^T)^{-1} \mathbf{U}\mathbf{P}\mathbf{E} \left[ \tilde{\phi}_q(n) \right] - \mathbf{b}_q \\ &= (\mathbf{U}\mathbf{P}\mathbf{U}^T)^{-1} \mathbf{U}\tilde{\mathbf{P}}\tilde{\mathbf{U}}^T \mathbf{h}_q - \mathbf{b}_q \\ &= (\mathbf{U}\mathbf{P}\mathbf{U}^T)^{-1} \mathbf{U}(\mathbf{P}\mathbf{I}_{Nc_q} + \mathbf{P}\mathbf{U}^T \mathbf{b}_q) - \mathbf{b}_q \\ &= (\mathbf{U}\mathbf{P}\mathbf{U}^T)^{-1} \mathbf{U}\mathbf{P}\mathbf{U}^T \mathbf{b}_q - \mathbf{b}_q \\ &= \mathbf{0}_N \end{aligned} \quad (51)$$

where in the second passing we use the result that  $\tilde{\phi}_q(n)$  is unbiased and its mean is given by (40), in the third passing we substitute  $\mathbf{h}_q$  and the matrix  $\tilde{\mathbf{U}}$  [refer to (41)], and in the fourth passing we use the result that  $\mathbf{P}\mathbf{I}_N = \mathbf{0}$  according to (46).

Similarly, the covariance of  $\hat{\mathbf{b}}_q$  is

$$\begin{aligned} \text{cov}(\hat{\mathbf{b}}_q) &= (\mathbf{U}\mathbf{P}\mathbf{U}^T)^{-1} \mathbf{U}\mathbf{P}\text{cov}(\tilde{\phi}_q(n)) \mathbf{P}\mathbf{U}^T (\mathbf{U}\mathbf{P}\mathbf{U}^T)^{-1} \\ &= (\mathbf{U}\mathbf{P}\mathbf{U}^T)^{-1} \mathbf{U}\mathbf{P}\eta_q^2 \mathbf{I}_N \mathbf{P}\mathbf{U}^T (\mathbf{U}\mathbf{P}\mathbf{U}^T)^{-1} \\ &= \eta_q^2 (\mathbf{U}\mathbf{P}\mathbf{U}^T)^{-1} \end{aligned} \quad (52)$$

where in the second passing we substitute the covariance of  $\tilde{\phi}_q(n)$ .

We conclude that asymptotically the SEES estimator is unbiased. We emphasize that in the simulation results the asymptotic covariance in (52) is close to the CRLB. However, this covariance result only holds for large  $N$  and small noise. Due to the complicated structure of the CRLB, it is very cumbersome to show the specific relation of this expression with the CRLB on the estimation of  $\mathbf{b}_q$ . To get a further insight on the result in (52) we consider the case of two chirps observed by a ULA.

### B. Two Closely Spaced Chirps Observed by a ULA

Consider the case of two chirps observed by a ULA with an interelement spacing of half wavelength. Assume that the complex amplitudes of the signals are  $\alpha_q = e^{j\phi_{q,0}}$ ,  $q = 1, 2$  where  $\phi_{q,0} \in [-\pi, \pi)$ . In this case the array manifold is given as  $\mathbf{a}(\theta) = [1, \dots, e^{j\pi \sin(\theta)(M-1)}]^T$ , where  $\theta \in [-\pi/2, \pi/2)$ . By combining (3) with (50) we get

$$\begin{aligned} \eta_1^2 = \eta_2^2 &= \frac{\sigma_n^2}{2} \frac{1}{M} \cdot \left( 1 - \frac{1}{M^2} |\mathbf{a}(\theta_1)^H \mathbf{a}(\theta_2)|^2 \right)^{-1} \\ &= \frac{\sigma_n^2}{2} \frac{1}{M} \\ &\cdot \left( 1 - \frac{1}{M^2} \left( \frac{\sin(\frac{\pi M}{2} (\sin(\theta_2) - \sin(\theta_1)))}{\sin(\frac{\pi}{2} (\sin(\theta_2) - \sin(\theta_1)))} \right)^2 \right)^{-1}. \end{aligned} \quad (53)$$

Note that we can express the two DOAs as  $\theta_1 = \theta_0 - \Delta/2$  and  $\theta_2 = \theta_0 + \Delta/2$  where  $\theta_0 = (\theta_1 + \theta_2)/2$  and  $\Delta = \theta_2 - \theta_1$ . Assume that  $\Delta$  is small. By approximating  $\sin(\theta_1)$  and  $\sin(\theta_2)$  with a first-order Taylor series around  $\theta_0$  we get that

$$\eta_1^2 = \eta_2^2 \cong \frac{\sigma_n^2}{2} \frac{1}{M} \cdot \left( 1 - \frac{1}{M^2} \left( \frac{\sin(\frac{\pi M}{2} (\cos(\theta_0)\Delta\theta))}{\sin(\frac{\pi}{2} (\cos(\theta_0)\Delta\theta))} \right)^2 \right)^{-1}. \quad (54)$$

We further approximate the sine function in the numerator in (54) with a third-order Taylor series, and also approximate the sine function in the denominator in (54) with a first-order Taylor series. This results in

$$\eta_1^2 = \eta_2^2 \cong \frac{6\sigma_n^2}{\pi^2 M^3} \cdot \frac{1}{\cos^2(\theta_0)} \cdot \frac{1}{(\Delta\theta)^2}. \quad (55)$$

Combining (8), (42), (45), (46), (55), and (52) it can be shown that the variances of the polynomial coefficients are

$$\text{var}(\hat{b}_{q,1}) \cong \frac{6\sigma_n^2}{\pi^2 M^3 T_s^2} \cdot \frac{1}{\cos^2(\theta_0)} \cdot \frac{1}{(\Delta\theta)^2} \frac{1}{n_1} \frac{1}{1 - \frac{n_2^2}{n_1 n_3}} \quad (56)$$

$$\text{var}(\hat{b}_{q,2}) \cong \frac{6\sigma_n^2}{\pi^2 M^3 T_s^4} \cdot \frac{1}{\cos^2(\theta_0)} \cdot \frac{1}{(\Delta\theta)^2} \frac{1}{n_3} \frac{1}{1 - \frac{n_2^2}{n_1 n_3}} \quad (57)$$

where  $n_1 \triangleq \sum_{n=0}^{N-1} (n - n_c)^2 = (1/12)N(N-1)(N+1)$ ,  $n_2 \triangleq \sum_{n=0}^{N-1} (n - n_c)(n^2 - n_s) = (1/12)N(N-1)^2(N+1)$ , and  $n_3 \triangleq \sum_{n=0}^{N-1} (n^2 - n_s)^2 = (1/180)(N-1)N(2N-1)(8N^2 - 3N - 11)$ , with  $n_c \triangleq (1/N) \sum_{n=0}^{N-1} n = (1/2)(N-1)$ , and  $n_s \triangleq (1/N) \sum_{n=0}^{N-1} n^2 = (1/6)(N-1)(2N-1)$ . For large number of samples,  $n_c \cong N/2$  and  $n_s \cong N^2/3$ ,  $n_1 \cong N^3/12$ ,  $n_2 \cong N^4/12$ , and  $n_3 \cong (4/45)N^5$ . By substituting these approximated results in (56) and (57) we get the variances of  $\hat{b}_{q,1}$  and  $\hat{b}_{q,2}$  as

$$\text{var}(\hat{b}_{q,1}) \cong \frac{1152\sigma_n^2}{\pi^2 M^3 N T^2} \cdot \frac{1}{\cos^2(\theta_0)} \cdot \frac{1}{(\Delta\theta)^2} \quad (58)$$

$$\text{var}(\hat{b}_{q,2}) \cong \frac{1080\sigma_n^2}{\pi^2 M^3 N T^4} \cdot \frac{1}{\cos^2(\theta_0)} \cdot \frac{1}{(\Delta\theta)^2} \quad (59)$$

where  $T \triangleq N T_s$  is the observation interval. We thus observe that the variances of the initial frequency and the frequency rate estimates decrease w.r.t. to  $M^3 N (\Delta\theta)^2$ . However, the variance of the initial frequency estimate decreases w.r.t.  $T^2$  while the variance of the frequency rate decreases w.r.t.  $T^4$ . Notice that for a fixed number of samples, the variances in (58) and (59) cannot be improved by increasing the sampling interval without limit, since then the bandwidth of the signal also increases and the Nyquist sampling condition might not hold anymore.

## VI. COMPUTATIONAL COMPLEXITY

We evaluate the computational complexity of the AMLE and that of the proposed SEES method by calculating the number of real multiplications involved in each method.

Consider first the cost function of the AMLE in (26). Assume that the possible range of values of each coefficient  $b_p$  is given by  $D$ . Hence, we need to search over  $D^P$  possible points. For each

point  $\tilde{\mathbf{b}}$  we first calculate  $\tilde{\mathbf{b}}^T \mathbf{u}(n)$  which involves  $P$  real multiplications. Then, we need to calculate  $\{x_m(n)e^{-j\tilde{\mathbf{b}}^T \mathbf{u}(n)}\}_{n=0}^{N-1}$  which involves  $2N$  real multiplications, and finally we need to compute the absolute value of the result which requires 2 real multiplications. This is repeated  $M$  times. The total number of real multiplications which is performed for each point in the grid is  $\mathcal{O}(M(P+N+2))$ . Therefore, the total number of real multiplications for all points in the grid is  $\mathcal{O}(M(P+N+2)D^P)$ . Since in practice  $N \gg P+2$ , we conclude that the computation load is approximately  $\mathcal{O}(MND^P)$ . The complexity increases exponentially w.r.t. the polynomial order  $P$ .

Consider next the SEES method. The first part of this method uses the ZF-ACMA algorithm. The dominant part in the complexity of the ZF-ACMA is  $\mathcal{O}(Q^4N)$  [16]. The second part of the method involves the LS method in (43). If we assume constant sampling rate then  $\tilde{\mathbf{U}}$  and thus  $\mathbf{U}$  are known *a priori* and are therefore computed off-line and do not have to be included in the complexity calculations. To obtain the estimate of the coefficients vector we only need to multiply  $(\mathbf{U}\mathbf{P}\mathbf{U}^T)^{-1}\mathbf{U}\mathbf{P}$  with  $\tilde{\phi}_q$  which requires  $PN$  real multiplications. This is done  $Q$  times for all the signals and thus the complexity of this LS step is  $PQN$ . However, in practice the sampling rate may change due to carrier offset and timing inaccuracy and we need to recompute the matrix  $\tilde{\mathbf{U}}$ . In this case we can exploit its Vandermonde structure to obtain the LS solution [25]. We first perform a QR decomposition on  $\tilde{\mathbf{U}}$  as suggested in [25] which results in an equation set given by  $\mathbf{Q}\mathbf{R}\mathbf{h}_q = \tilde{\phi}_q$  where  $\mathbf{Q}$  is an  $N \times N$  matrix and  $\mathbf{R} \triangleq [\tilde{\mathbf{R}}^T \mathbf{0}]^T$  is an  $N \times (P+1)$  matrix where  $\tilde{\mathbf{R}}$  is a  $(P+1) \times (P+1)$  upper triangle matrix. The complexity of this step is  $5N(P+1) + 7(P+1)^2/2$  and it is done once. Since  $\mathbf{Q}$  is an orthogonal matrix we obtain that  $\mathbf{h}_q$  solves  $\tilde{\mathbf{R}}\mathbf{h}_q = [\mathbf{Q}^T \tilde{\phi}_q]_{P+1}$  where the right hand side represents the  $P+1$  rows of the result. The complexity of the multiplication of  $[\mathbf{Q}^T \tilde{\phi}_q]_{P+1}$  is  $(P+1)N$ . The complexity of solving this set of equations using the triangular structure of  $\tilde{\mathbf{R}}$  is  $(P+1)^2/2$ . The last procedure is performed  $Q$  times to obtain the coefficient vectors of all signals. Therefore, the total number of real multiplications required for obtaining a set of coefficients of one signal is  $\mathcal{O}(Q^4N + 5N(P+1) + 7(P+1)^2/2 + Q(P+1)N + Q(P+1)^2/2)$  (or  $\mathcal{O}(Q^4N + PQN)$  if the sampling rate is constant). We note that the complexity increases linearly w.r.t.  $N$ . In practice,  $Q \ll N$  and  $P \ll N$ , and therefore we get that the complexity is approximately  $\mathcal{O}(Q^4N + 5N(P+1) + Q(P+1)N)$ . Contrary to the AMLE, the complexity of the SEES estimator increases linearly w.r.t. the polynomial order  $P$ .

## VII. NUMERICAL RESULTS

To demonstrate the performance of the proposed method, we present the results of simulated experiments. The noise power  $\sigma_n^2$  is adjusted to give the desired SNR defined as  $\text{SNR} = -10 \log_{10} \sigma_n^2$  [dB]. In each of the simulation examples we evaluated the RMSE on the estimate polynomial coefficients. We defined the RMSE on the estimation of the  $p$ th coefficient  $b_p$  of the  $q$ th signal as  $\text{RMSE}_q(b_p) \triangleq \sqrt{(1/N_{exp}) \sum_{n=1}^{N_{exp}} (\hat{b}_{q,p,n} - b_{q,p})^2}$  where

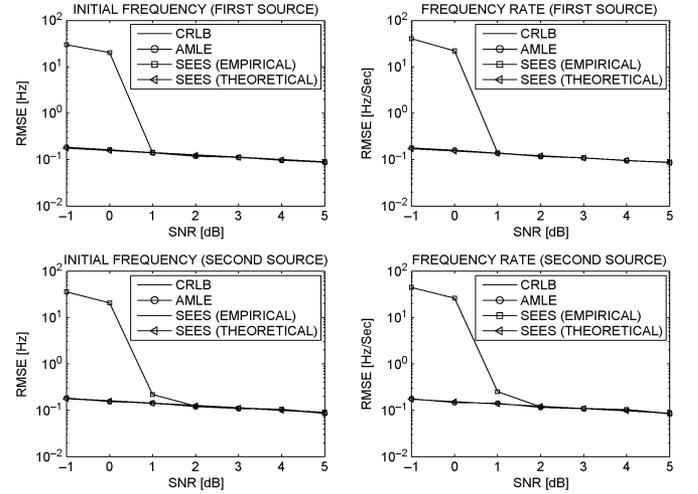


Fig. 1. RMSEs of the estimated initial frequency and the frequency rate with their associated CRLB versus the SNR.

$\hat{b}_{q,p,n}$  is the estimate of  $b_p$  of the  $q$ th signal at the  $n$ th trial, and  $N_{exp} = 500$  is the number of Monte Carlo (MC) independent trials. For comparison, in each simulation we added the theoretical covariances of the estimators according to (52), and also compared the results with the associated CRLB [13, Section II].

Unless otherwise stated, we use a  $M = 8$  element ULA with half wavelength spacing. We consider two PPSs of order two ( $P = 2$ ), that is, chirp signals, sampled with a sampling frequency of  $f_s = 1/T_s = 60$  [Hz], and the number of samples is  $N = 120$  (similar parameters were used in [13]). The continuous time signals are given by  $s_1(t) = e^{j2\pi\xi f_s t(0.8-0.3t/T)}$ ,  $s_2(t) = e^{j2\pi\xi f_s t(0.2-0.6t/T)}$ , where  $\xi = 0.25$ , and  $T = N/f_s$ .

1) *RMSE Versus SNR*: We compare the RMSE of the estimated initial frequency ( $b_1$ ) and the estimated frequency rate ( $2b_2$ ) versus the SNR. We considered SNR values from  $-1$  to  $5$  [dB] with a step of  $1$  [dB]. The DOAs of the signals are  $\theta_1 = -10$  [deg] and  $\theta_2 = 20$  [deg]. The RMSE results are presented in Fig. 1. As can be seen, for low SNR the performance of the AMLE is superior. However, as the SNR increases, the RMSE of the proposed SEES method improves and approaches the CRLB. Also, observe that the theoretical variances are similar to those obtained by the CRLB.

2) *RMSE Versus the Number of Samples*: We compare the RMSE of the estimated initial frequency ( $b_1$ ) and the estimated frequency rate ( $2b_2$ ) versus the number of samples,  $N$ . The DOAs of the signals are  $\theta_1 = -10$  [deg] and  $\theta_2 = 20$  [deg]. The SNR is  $5$  [dB]. We varied the number of samples from  $50$  to  $200$  with a step of  $25$ . The RMSE result of the first signal are presented in Fig. 2 (the results of the second signal are similar). As can be observed, both estimators have similar performance for any value of  $N$ . We also compared the averaged time required for each method to obtain each point in Fig. 2, which is an equivalent measure for the complexity of each method. We use standard time measurements of the software MATLAB to calculate the processing time used by each method for each of the MC trials, and then averaged the results over the total number of trials. In Table I we show the comparison of the averaged processing time of each method versus the number of samples. It is

TABLE I  
AVERAGED PROCESSING TIME OF THE AMLE AND THE SEES METHOD VERSUS THE NUMBER OF SAMPLES

Avg. Proc. Time [Sec]	N, Number of samples						
	50	75	100	125	150	175	200
AMLE	0.865	0.995	1.129	1.248	1.325	1.686	1.704
SEES	0.0120	0.0121	0.0144	0.0163	0.0186	0.0219	0.030

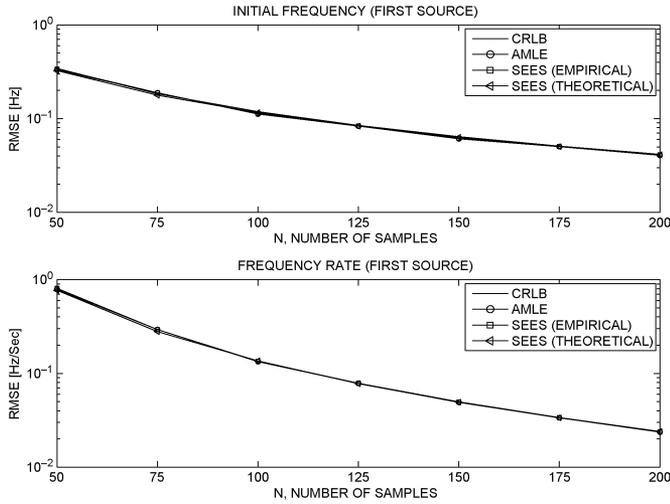


Fig. 2. RMSEs of the estimated initial frequency and the frequency rate with their associated CRLB versus the number of samples.

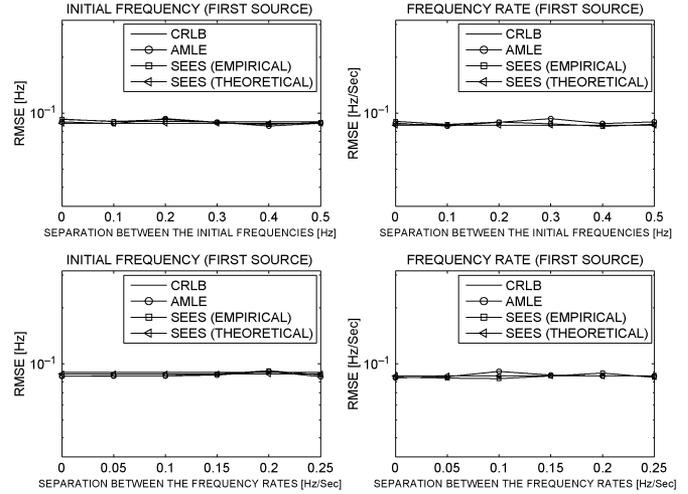


Fig. 4. RMSEs of the estimated initial frequency and the frequency rate with their associated CRLB versus the separation between the initial frequencies and versus the separation between the frequency rates.

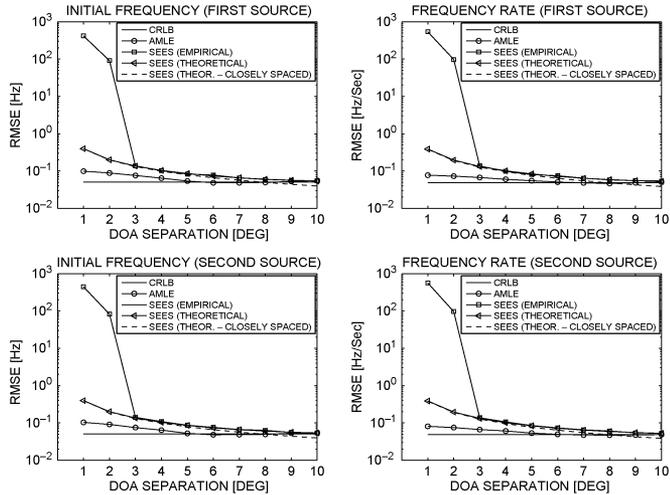


Fig. 3. RMSEs of the estimated initial frequency and the frequency rate with their associated CRLB versus the DOA separation.

clear that the processing time of the AMLE is much larger than that of the proposed method.

3) *RMSE Versus the DOA Separation:* We compare the RMSE of the estimated initial frequency ( $b_1$ ) and the estimated frequency rate ( $2b_2$ ) versus the separation in the DOA of the two signals. We set the DOA of the first signal at 0 [deg]. The DOA of the second signal is varied from 1 to 10 [deg] with a step of 1 [deg]. The SNR is 5 [dB]. We also added the approximated closed form expressions in (58) and (59). The RMSE results are presented in Fig. 3. As can be seen, the proposed SEES estimator is inferior to the AMLE at small DOA separation, while both have similar performance at large

separation. Also the expressions in (58) and (59) predict well the RMSE performance.

4) *RMSE Versus the Coefficients' Separation:* We compare the RMSE of the estimated initial frequency ( $b_1$ ) and the estimated frequency rate ( $2b_2$ ) versus the separation between the initial frequencies and versus the separation between the frequency rates of the signals. The DOAs of the signals are  $\theta_1 = -10$  [deg] and  $\theta_2 = 20$  [deg]. The SNR is 5 [dB]. We varied  $b_{1,1}$  (with  $b_{2,1} = 2\pi\xi f_s 0.6$ ,  $b_{1,2} = -2\pi\xi f_s 0.15$ , and  $b_{2,2} = -2\pi\xi f_s 0.3$ ) from  $2\pi\xi f_s 0.1$  to  $2\pi\xi f_s 0.6$  with a step of  $2\pi\xi f_s 0.1$ . The RMSE results of the first source versus the separation between the initial frequencies are presented in Fig. 4 (two upper plots). We also varied  $b_{1,2}$  (with  $b_{2,2} = -2\pi\xi f_s 0.15$ ,  $b_{1,1} = 2\pi\xi f_s 0.8$ , and  $b_{2,1} = 2\pi\xi f_s 0.2$ ) from  $-2\pi\xi f_s 0.15$  to  $-2\pi\xi f_s 0.4$  with a step of  $2\pi\xi f_s 0.05$ . The RMSE results of the first source versus the separation between the frequency rates are presented in Fig. 4 (two lower plots). The results for the second signal are similar and are therefore not presented. As can be observed, the AMLE and the SEES estimator are not sensitive to the separation between the initial frequencies and to the separation between the frequency rates of the signals.

5) *RMSE for High Order PPSs:* We compare the RMSE of the estimated polynomial coefficients for quadratic FM signal (that is,  $\{b_p\}_{p=1}^3$ ) and for fourth-order PPS (that is,  $\{b_p\}_{p=1}^4$ ) versus the SNR. The continuous time quadratic FM signals are given by [10, p. 344, Eq. (41)]  $s_1(t) = e^{j(2\pi\xi f_s(t-4t^2/T+8t^3/3T^2))}$ ,  $s_2(t) = e^{j(2\pi\xi f_s(-t+4t^2/T-8t^3/3T^2))}$ , and the fourth-order signals [10, p. 344, Eq. (43)]  $s_1(t) = e^{j(2\pi\xi f_s(t-10t^2/T+18t^3/T^2-9t^4/T^3))}$ ,

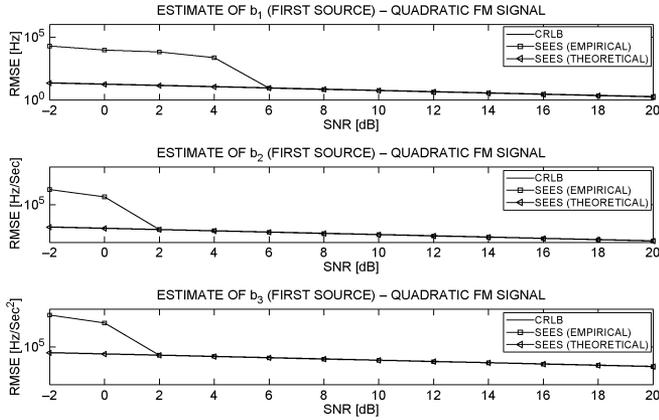


Fig. 5. RMSEs of the estimated polynomial coefficients with their associated CRLB versus the SNR for quadratic FM signals.

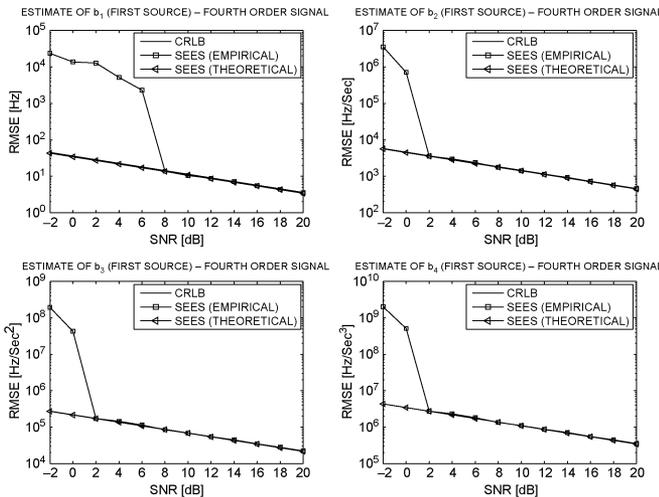


Fig. 6. RMSEs of the estimated polynomial coefficients with their associated CRLB versus the SNR for fourth-order PPSs.

$s_2(t) = e^{j(2\pi\xi f_s(-t+10t^2/T-18t^3/T^2+9t^4/T^3))}$ . In both cases  $\xi = 0.4$ , and  $T = N/f_s$ , where  $N = 256$ , and  $f_s = 1/T_s = 8192$  [Hz]. The DOAs of the signals are  $\theta_1 = -10$  [deg] and  $\theta_2 = 20$  [deg]. We varied the SNR from  $-2$  to  $20$  [dB] with a step of  $2$  [dB]. The complexity of the AMLE is very high in these cases, and since it is known that this estimator agrees with the CRLB for any value of SNR, we only consider the RMSE performance of the SEES estimator. The RMSE results are presented in Fig. 5 (for quadratic FM signals) and in Fig. 6 (for fourth-order PPSs). In both plots we only presented the results for the first signal, since the results of the second signal are similar. As can be observed, the SEES estimator has similar performance at high SNR as predicted by the CRLB.

## VIII. CONCLUSION

In this paper we presented a new technique for estimating polynomial phase signals observed by a sensor array with an unknown array manifold. The approximated MLE requires a multidimensional search in the polynomial coefficients space. We propose an estimation approach which first separates the signals using a blind source separation method, and then estimates

the coefficients of the polynomial phase of each signal using a LS method. The complexity of the algorithm increases linearly with respect to the polynomial order and to the number of samples. Simulation results demonstrated that the proposed estimator achieves the CRLB at moderate or high SNR.

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