Estimation and Detection

Composite hypothesis testing (Ch. 6)

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Recap

- **Detection theory**
  - Neyman-Pearson Theorem (NP)
  - Minimum Probability of Error
  - Bayes Risk
- **Detecting a known deterministic signal in noise using the NP criterion**
  - White noise
  - Colored noise
- **Detecting a random signal in noise using the NP criterion**
  - Random signal in white noise
  - Random signal in colored noise

Learning objectives

LO1: Understand scenarios where **hypothesis testing** needs to be applied to **PDFs with unknown parameters**

LO2: Understand and apply the **uniformly most powerful** test

LO3: Apply the Bayesian approach in situations where a **prior Probability Density Function (PDF)** for detection problems with unknown parameters

LO4: Apply the **generalized likelihood ratio test** for detection problems with unknown parameters

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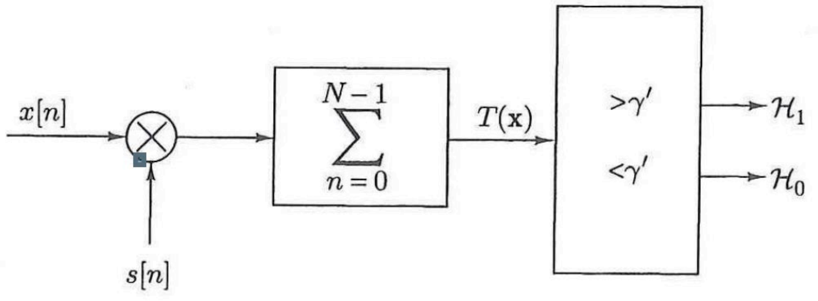
Deterministic signals in white Gaussian noise

Binary detection problem with $\mathbf{w} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$ and deterministic $\mathbf{s}$:

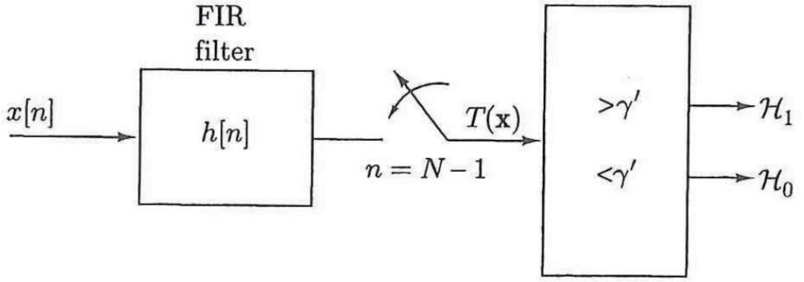
$$\begin{aligned} &\mathcal{H}_0 \quad x[n] = w[n] \\ &\mathcal{H}_1 \quad x[n] = s[n] + w[n] \end{aligned}$$

Interpretation 1: The resulting $T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s[n]$ is a correlator. The received data is correlated with a replica of the signal.



(a)

Interpretation 2: The resulting $T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s[n]$ is a matched filter.



$$h[n] = \begin{cases} s[N-1-n] & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

Fig. 4.1 from Kay-II.

Deterministic signals in colored Gaussian noise

Colored noise

Binary detection problem with $\mathbf{w} \sim N(\mathbf{0}, \mathbf{C})$ and deterministic $\mathbf{s}$:

$$\begin{aligned} \mathcal{H}_0 \quad & x[n] = w[n] \\ \mathcal{H}_1 \quad & x[n] = s[n] + w[n] \end{aligned}$$

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{s}$$

Notice that if C is positive definite, $\mathbf{C}^{-1}$ can be written as $\mathbf{C}^{-1} = \mathbf{D}^T \mathbf{D}$, leading to

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{D}^T \mathbf{D} \mathbf{s}$$

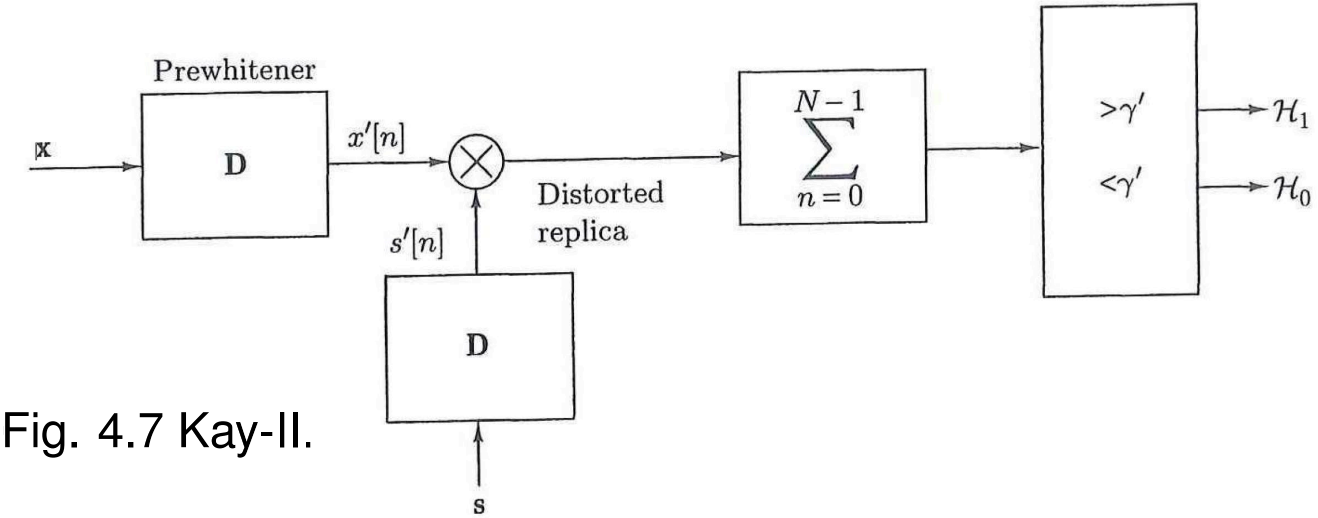


Fig. 4.7 Kay-II.

Random (correlated) signal in WGN

Hence, we decide for $\mathcal{H}_1$ if

$$T(\mathbf{x}) = \mathbf{x}^T \hat{\mathbf{s}} > \gamma''$$

with

$$\hat{\mathbf{s}} = \frac{1}{\sigma^2}\left[\frac{1}{\sigma^2}\left(\mathbf{C}_s + \sigma^2\mathbf{I}\right)\mathbf{C}_s^{-1}\right]^{-1}\mathbf{x} = \mathbf{C}_s(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\mathbf{x}$$

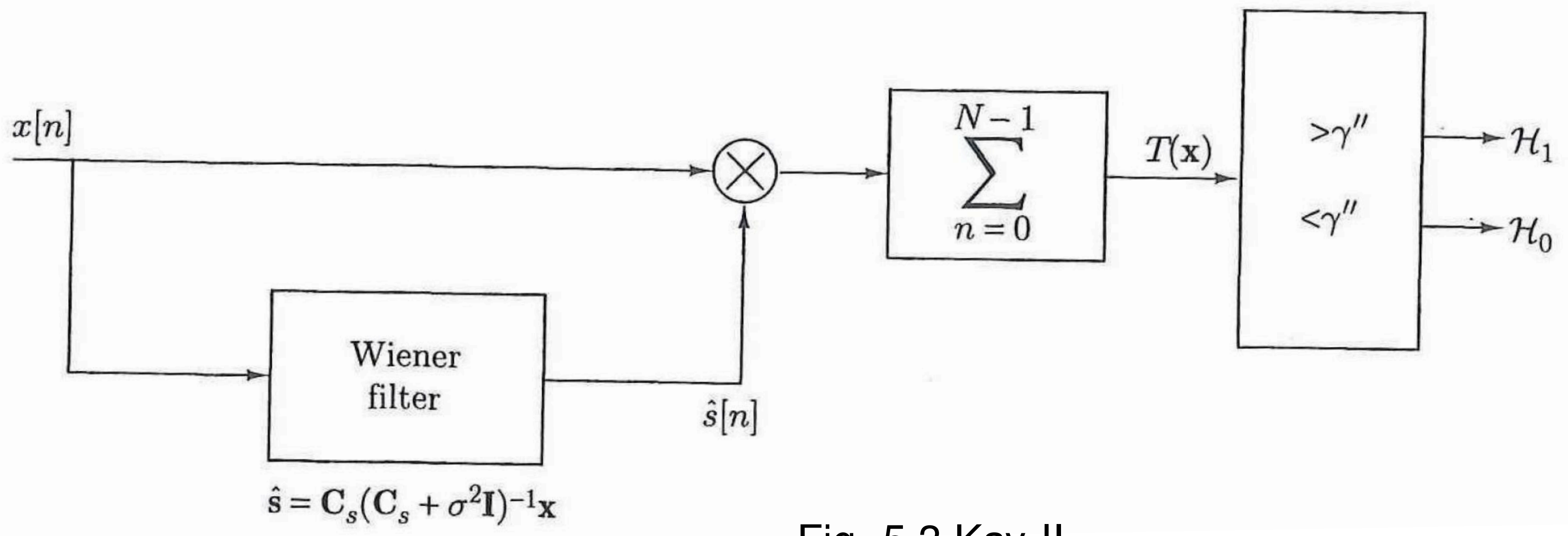


Fig. 5.3 Kay-II.

Random (correlated) signal in WGN

Hence, we decide for $\mathcal{H}_1$ if

$$T(\mathbf{x}) = \mathbf{x}^T \hat{\mathbf{s}} > \gamma''$$

with

$$\hat{\mathbf{s}} = \frac{1}{\sigma^2} \left[\frac{1}{\sigma^2} \left(\mathbf{C}_s + \sigma^2 \mathbf{I} \right) \mathbf{C}_s^{-1} \right]^{-1} \mathbf{x} = \mathbf{C}_s (\mathbf{C}_s + \sigma^2 \mathbf{I})^{-1} \mathbf{x}$$

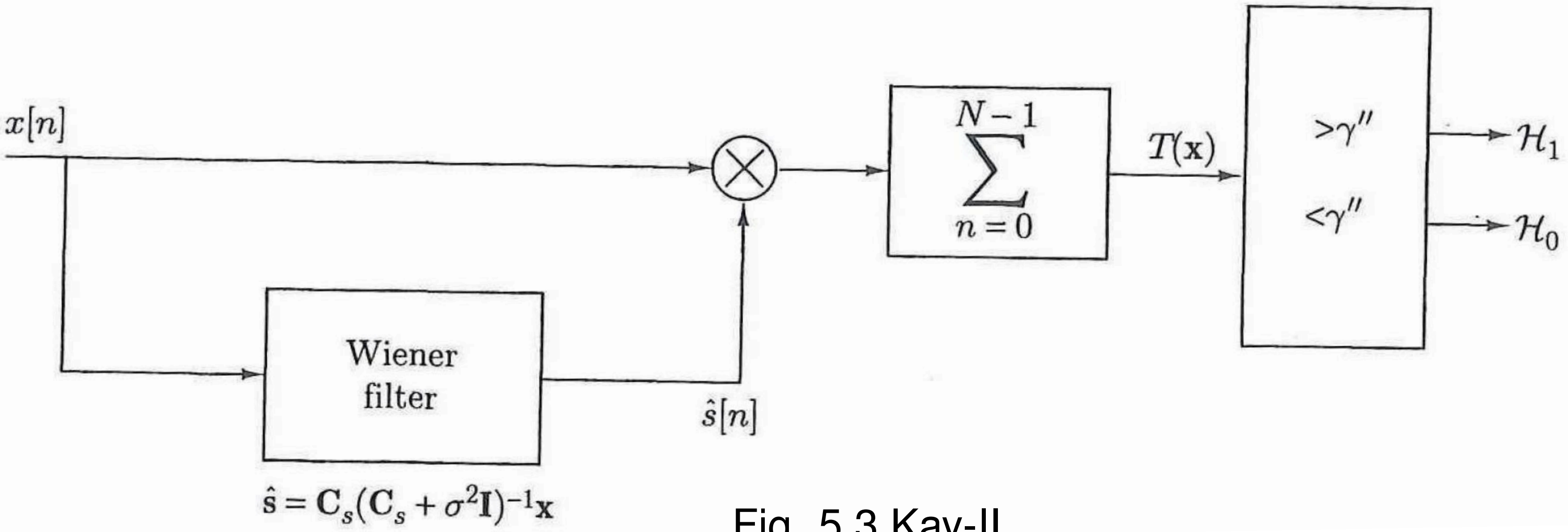
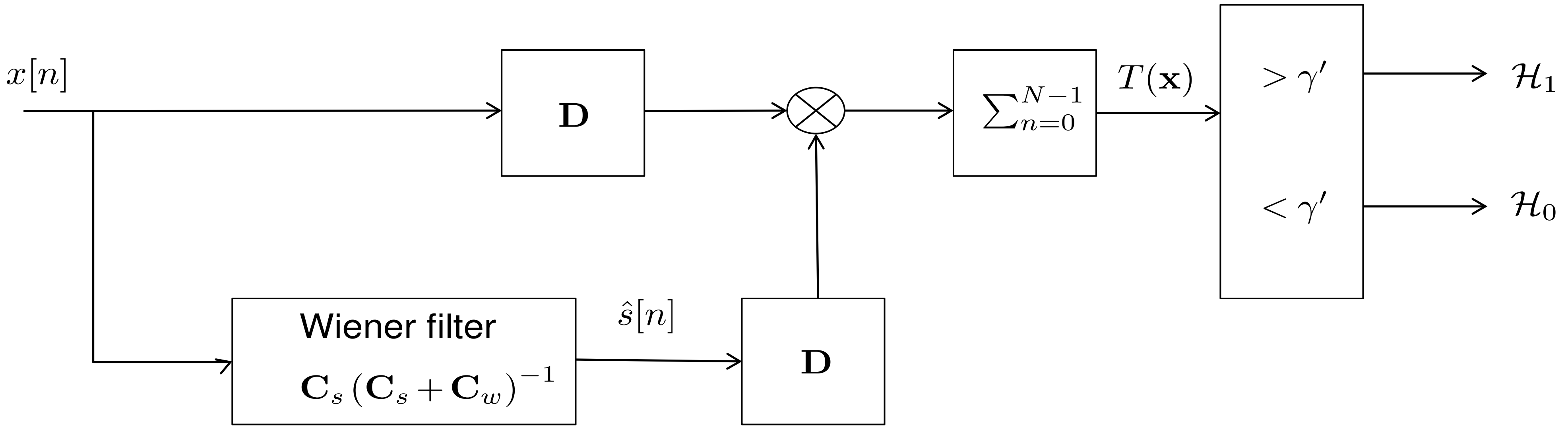


Fig. 5.3 Kay-II.

Random (Correlated) Signal in colored Gaussian noise



$$T(\mathbf{x}) = (\mathbf{D}\mathbf{x})^T \mathbf{D}\hat{\mathbf{s}} \text{ with } \hat{\mathbf{s}} = \mathbf{C}_s\left(\mathbf{C}_s + \mathbf{C}_w\right)^{-1}\mathbf{x}$$

Generalized Gaussian detection

$$\mathcal{H}_0 : \mathbf{x} = \mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_w)$$

$$\mathcal{H}_1 : \mathbf{x} = \mathbf{s} + \mathbf{w} \sim \mathcal{N}(\boldsymbol{\mu}_s, \mathbf{C}_s + \mathbf{C}_w)$$

Thus, we have $L(\mathbf{x}) = \dfrac{\frac{1}{\left(2\pi\right)^{\frac{N}{2}} det^{\frac{1}{2}}\left(\mathbf{C}_s + \mathbf{C}_w\right)} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_s)^T\left(\mathbf{C}_s+\mathbf{C}_w\right)^{-1}(\mathbf{x}-\boldsymbol{\mu}_s)\right]}{\frac{1}{\left(2\pi\right)^{\frac{N}{2}} det^{\frac{1}{2}}\left(\mathbf{C}_w\right)} \exp\left[-\frac{1}{2}\mathbf{x}^T\mathbf{C}_w^{-1}\mathbf{x}\right]}$.

Calculating the Log-Likelihood Ratio (LLR), we get

$$T(\mathbf{x}) = \mathbf{x}^T\left[\mathbf{C}_w^{-1} - \left(\mathbf{C}_s+\mathbf{C}_w\right)^{-1}\right]\mathbf{x} + 2\mathbf{x}^T\left(\mathbf{C}_s+\mathbf{C}_w\right)^{-1}\boldsymbol{\mu}_s - \boldsymbol{\mu}_s^T\left(\mathbf{C}_s+\mathbf{C}_w\right)^{-1}\boldsymbol{\mu}_s$$

Using matrix inversion lemma, leaving out the data independent terms and scaling we get:

$$T'(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\left[\mathbf{C}_w^{-1}\mathbf{C}_s\left(\mathbf{C}_s+\mathbf{C}_w\right)^{-1}\right]\mathbf{x} + \mathbf{x}^T\left(\mathbf{C}_s+\mathbf{C}_w\right)^{-1}\boldsymbol{\mu}_s$$

Learning objectives

LO1: Understand scenarios where **hypothesis testing** needs to be applied to **PDFs with unknown parameters**

LO2: Understand and apply the **uniformly most powerful** test

LO3: Apply the Bayesian approach in situations where a **prior Probability Density Function (PDF)** for detection problems with unknown parameters

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Overview of Chapter 6

Topic	Index
Summary of Composite Hypothesis Testing	Chapter 6.2
Composite Hypothesis Testing	Chapter 6.3
Bayesian approach	Chapter 6.4.1
GLRT	Chapter 6.4.2
Performances	Chapter 6.5
Locally Most Powerful Detectors	Chapter 6.7

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Q1: Is it more realistic that situations where the probability density functions (PDFs) under H0 and H1 are not completely known, and why?

Yes	0%
No	0%

##/##

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Results slide

Q1: Is it more realistic that situations where the probability density functions (PDFs) under H0 and H1 are not completely known, and why?

Yes	0%
No	0%

RESULTS SLIDE

Composite hypothesis testing

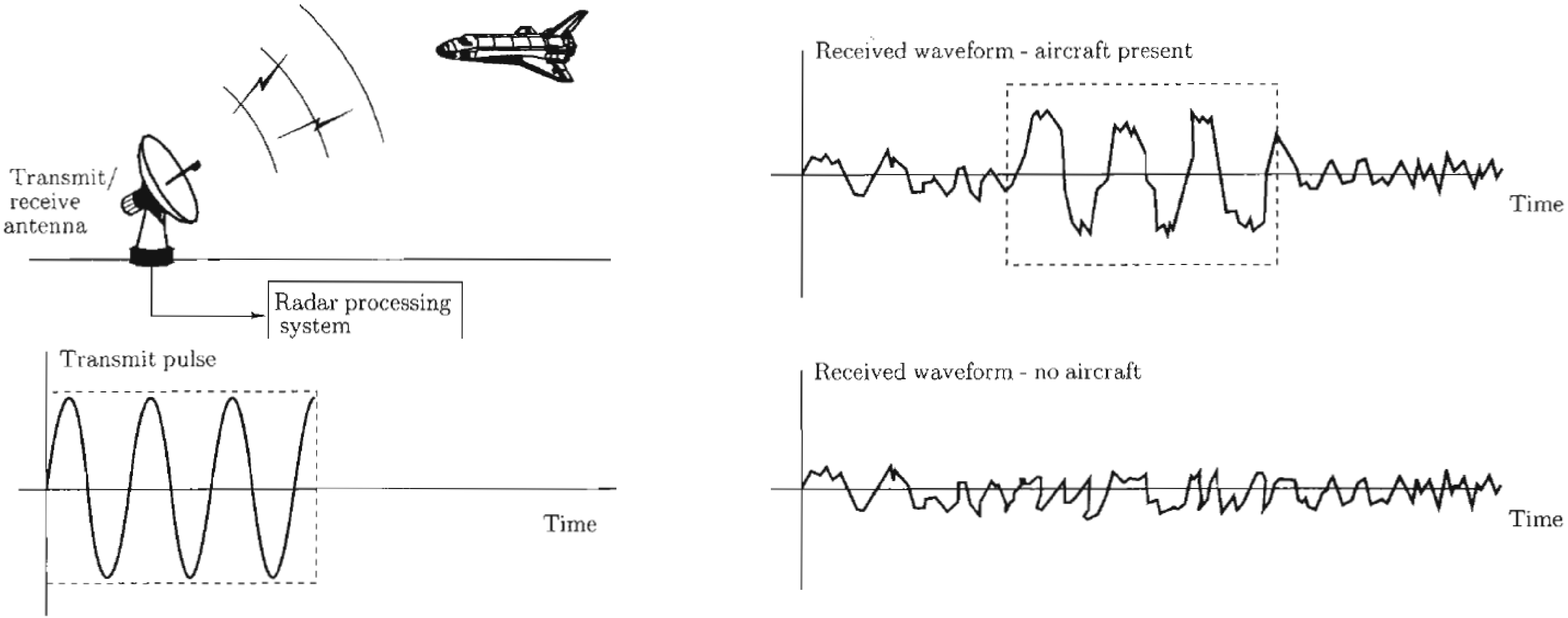
Motivation

- Neyman-Pearson detectors require perfect knowledge of the PDFs
  - What if this information is unknown ?
  - Are there detectors for such scenarios ?
  - Example applications: Radar, Sonar

Composite hypothesis testing

- Approach 1:
  - Design the NP detector, assuming the parameters are known.
  - Manipulate the Test so that it is not dependent on the parameters.
- Approach 2:
  - Bayesian approach: Consider unknown parameters as realizations of random variables and assign a prior pdf.
- Approach 3:
  - Generalized likelihood ratio: Estimate unknown parameters using MLEs.

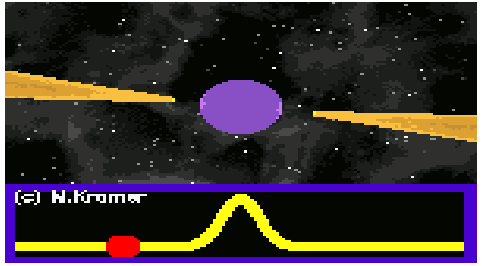
Example – Radar



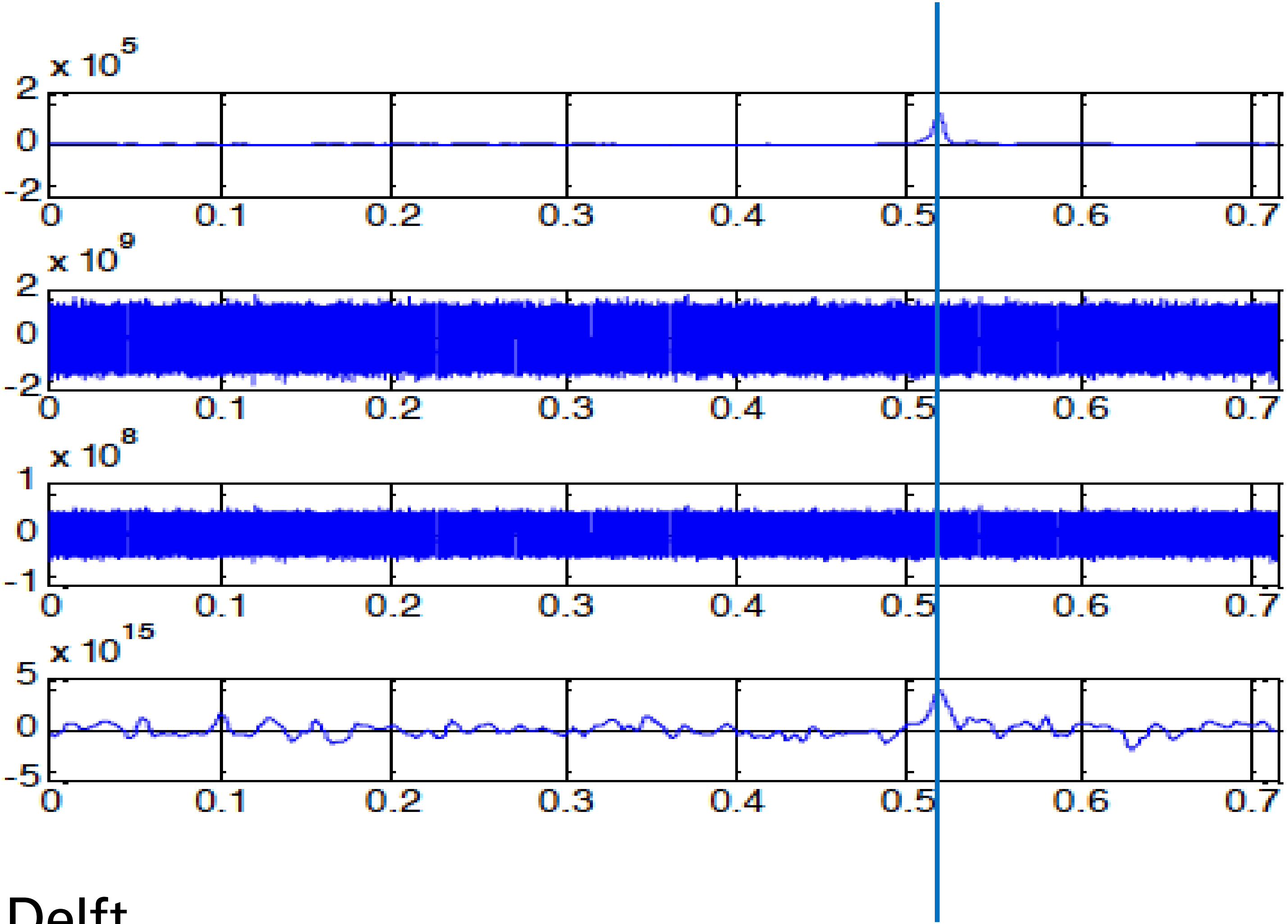
From Kay, part II.

- Detecting the presence of an aircraft using Radar.
- Transmit a known signal, analyse received waveform.

Example – Radio Pulsar Navigation



Kramer (University of Manchester)



pulsar B0329+54

observation (SNR = -90 dB)

epoch folding ($K = 10^3$)

matched filtering

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Composite hypothesis testing

Remember this example

$$\begin{aligned}\mathcal{H}_0 &: \quad x[n] = w[n] \; n = 0, 1, \ldots, N-1 \\ \mathcal{H}_1 &: \quad x[n] = A + w[n] \; n = 0, 1, \ldots, N-1\end{aligned}$$

where $A > 0$ and $w[n]$ is WGN with variance σ^2. NP detector decides $\mathcal{H}_1$ if

$$L(\mathbf{x}) = \frac{p(\mathbf{x}; \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} > \lambda$$

Composite hypothesis testing

We then have

$$\frac{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right]}{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right]} > \lambda$$

Taking the logarithm of both sides and simplification results in

$$\frac{1}{N}\sum_{n=0}^{N-1}x[n] > \frac{\sigma^2}{NA}\ln\lambda + \frac{A}{2} = \lambda^{'}$$

Where the threshold is found by

$$P_{FA} = Pr(T(\mathbf{x}) > \lambda^{'};\mathcal{H}_0) = Q\left(\frac{\lambda^{'}}{\sqrt{\sigma^2/N}}\right) \quad \Rightarrow \lambda^{'} = \sqrt{\frac{\sigma^2}{N}}Q^{-1}(P_{FA})$$

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Composite hypothesis testing (positive *A*)

What if the value of A is unknown?

One-sided test

$$\mathcal{H}_0 \quad : \quad x[n] = w[n] \; n = 0, 1, \ldots, N-1 \qquad \textbf{(A = 0)}$$

$$\mathcal{H}_1 \quad : \quad x[n] = A + w[n] \; n = 0, 1, \ldots, N-1 \qquad \textbf{(A > 0)}$$

where A is unknown, but we know $\underline{A > 0}$. Further we know that $w[n]$ is WGN with variance σ^2. NP detector decides $\mathcal{H}_1$ if

$$L(\mathbf{x}) = \frac{p(\mathbf{x}; A, \mathcal{H}_1)}{p(\mathbf{x}; A, \mathcal{H}_0)} > \lambda$$

Composite hypothesis testing (positive *A*)

What if the value of A is unknown?

One-sided test

$$\mathcal{H}_0 \quad : \quad x[n] = w[n] \; n = 0, 1, \ldots, N-1 \qquad \textbf{(A = 0)}$$

$$\mathcal{H}_1 \quad : \quad x[n] = A + w[n] \; n = 0, 1, \ldots, N-1 \qquad \textbf{(A > 0)}$$

A ***one-sided test*** results from an alternative hypothesis which specifies a direction, i.e., when the alternative hypothesis states that the parameter is in fact either **bigger or smaller** than the value specified in the null hypothesis.

A ***two-sided test*** results from an alternative hypothesis which does not specify a direction. i.e. when the alternative hypothesis states that the **null hypothesis is wrong**.

Composite Hypothesis Testing (positive *A*)

We now have

$$\frac{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right]}{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right]} > \lambda$$

Taking the logarithm of both sides and simplification results in

$$\frac{1}{N}\sum_{n=0}^{N-1}x[n] > \frac{\sigma^2}{NA}\ln\lambda + \frac{A}{2} = \lambda'$$

In this case, Both the test statistic $T(\mathbf{x})$ and threshold γ' do not depend on A.

Even though A is unknown, we can thus implement this detector.

(Although P_D does depend on A)

Where the threshold is found by

$$P_{FA} = Pr(T(\mathbf{x}) > \lambda'; \mathcal{H}_0) = Q\left(\frac{\lambda'}{\sqrt{\sigma^2/N}}\right) \quad \Rightarrow \lambda' = \sqrt{\frac{\sigma^2}{N}}Q^{-1}(P_{FA})$$

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{NA^2}{\sigma^2}}\right)$$

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Can we calculate P_D? What can we say about P_D?

Composite Hypothesis Testing – UMP (positive *A*)

The test

$$\frac{1}{N}\sum_{n=0}^{N-1} x[n] > \frac{\sigma^2}{NA}\ln\lambda + \frac{A}{2} = \sqrt{\frac{\sigma^2}{N}}Q^{-1}(P_{FA}),$$

leads to the highest P_D (remember NP maximises P_D) for any value A. (as long as $A > 0$). Such a test is called a Uniformly Most Powerful (UMP) test. Any other test will have a poorer performance.

However...often an UMP does not exist.

DC level in WGN unknown A (positive or negative)

Let us first calculate the clairvoyant detector for the case $-\infty < A < \infty$

The sign of *A* is **known**

$$A > 0: \quad \frac{1}{N}\sum_{n=0}^{N-1} x[n] > \gamma_+^{'}$$

$$A < 0: \quad \frac{1}{N}\sum_{n=0}^{N-1} x[n] < \gamma_-^{'}$$

For $A > 0$

$$P_{FA} = Pr\{\bar{x} > \gamma_+^{'}; \mathcal{H}_0\} = Q\left(\frac{\gamma_+^{'}}{\sqrt{\sigma^2/N}}\right)$$

For $A < 0$

$$P_{FA} = Pr\{\bar{x} < \gamma_-^{'}; \mathcal{H}_0\} = 1 - Q\left(\frac{\gamma_-^{'}}{\sqrt{\sigma^2/N}}\right) = Q\left(\frac{-\gamma_-^{'}}{\sqrt{\sigma^2/N}}\right)$$

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Example – DC level in WGN unknown A

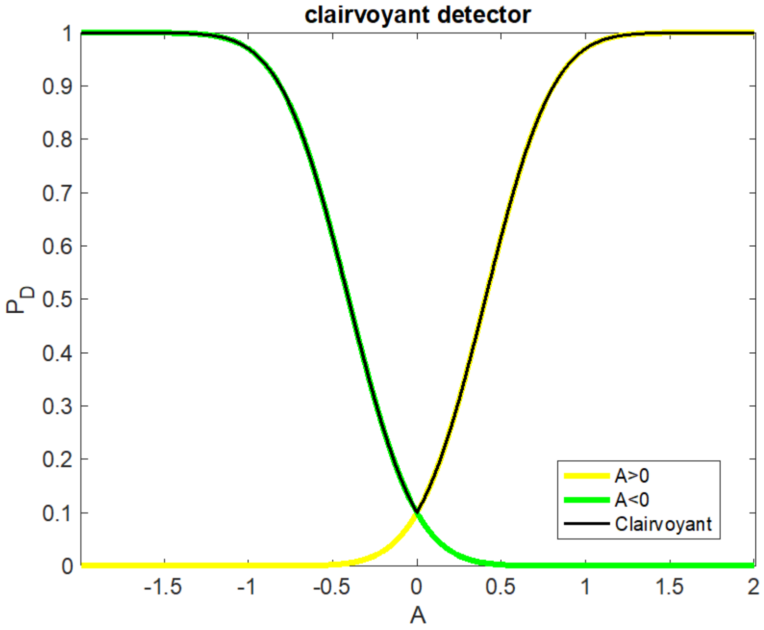
$$= \frac{|A|}{\sqrt{\sigma^2/N}}$$

The detection performance of the clairvoyant detector:

for $\quad A > 0 \quad P_D = Pr\{\bar{x} > \gamma_+^{'}; \mathcal{H}_1\} = Q\left(\frac{\gamma_+^{'} - A}{\sqrt{\sigma^2/N}}\right) = Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{NA^2}{\sigma^2}}\right)$

for $\quad A < 0 \quad P_D = 1 - Q\left(\frac{\gamma_-^{'} - A}{\sqrt{\sigma^2/N}}\right) = Q\left(\frac{-\gamma_-^{'} + A}{\sqrt{\sigma^2/N}}\right) = Q\left(Q^{-1}(P_{FA}) + \frac{A}{\sqrt{\sigma^2/N}}\right)$

$$= Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{NA^2}{\sigma^2}}\right)$$



Example – DC level in WGN unknown A

Instead of the clairvoyant detector, let's look at the realisable detector:

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} x[n] \right| > \gamma'',$$

where A is now unknown and $\boxed{-\infty < A < \infty.}$

Two-sided test

$$\mathcal{H}_0 \quad : \quad x[n] = w[n] \; n = 0, 1, \ldots, N-1$$

(A = 0)

$$\mathcal{H}_1 \quad : \quad x[n] = A + w[n] \; n = 0, 1, \ldots, N-1$$

($A \neq 0$)

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Example – DC level in WGN unknown A

Instead of the clairvoyant detector, let's look at the realisable detector:

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} x[n]\right| > \gamma^{''},$$

where A is now unknown and $-\infty < A < \infty$.

$$P_{FA} = Pr\{|\bar{x}| > \gamma^{''}; \mathcal{H}_0\} =?$$

- A: $P_{FA} = 2Q\left(\frac{\gamma^{''}}{\sqrt{\sigma^2/N}}\right)$
- B: $P_{FA} = Q\left(\frac{\gamma^{''}}{\sqrt{\sigma^2/N}}\right)$
- C: $P_{FA} = 2Q\left(-\frac{\gamma^{''}}{\sqrt{\sigma^2/N}}\right)$

Example – DC level in WGN unknown A

$$\boxed{P_{FA} = Pr\{|\bar{x}| > \gamma^{''}; \mathcal{H}_0\} = 2Pr\{\bar{x} > \gamma^{''}; \mathcal{H}_0\} = 2Q\left(\frac{\gamma^{''}}{\sqrt{\sigma^2/N}}\right)}$$

$$\gamma^{''} = \sqrt{\sigma^2/N}Q^{-1}\left(P_{FA}/2\right)$$

$$P_D = Pr\{|\bar{x}| > \gamma^{''}; \mathcal{H}_1\} = ?$$

- A: $P_D = Q\left(Q^{-1}(P_{FA}/2) + \frac{A}{\sqrt{\sigma^2/N}}\right)$
- B: $P_D = Q\left(Q^{-1}(P_{FA}/2) - \sqrt{\frac{NA^2}{\sigma^2}}\right)$
- C: $P_D = Q\left(Q^{-1}(P_{FA}/2) - \sqrt{\frac{NA^2}{\sigma^2}}\right) + Q\left(Q^{-1}(P_{FA}/2) + \sqrt{\frac{NA^2}{\sigma^2}}\right)$

Example – DC level in WGN unknown A

Instead of the clairvoyant detector, let's look at the realisable detector:

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} x[n]\right| > \gamma'',$$

where A is now unknown and $-\infty < A < \infty$.

$$\boxed{P_{FA} = Pr\{|\bar{x}| > \gamma''; \mathcal{H}_0\} = 2Pr\{\bar{x} > \gamma''; \mathcal{H}_0\} = 2Q\left(\frac{\gamma''}{\sqrt{\sigma^2/N}}\right)}$$

$$\gamma'' = \sqrt{\sigma^2/N}Q^{-1}(P_{FA}/2)$$

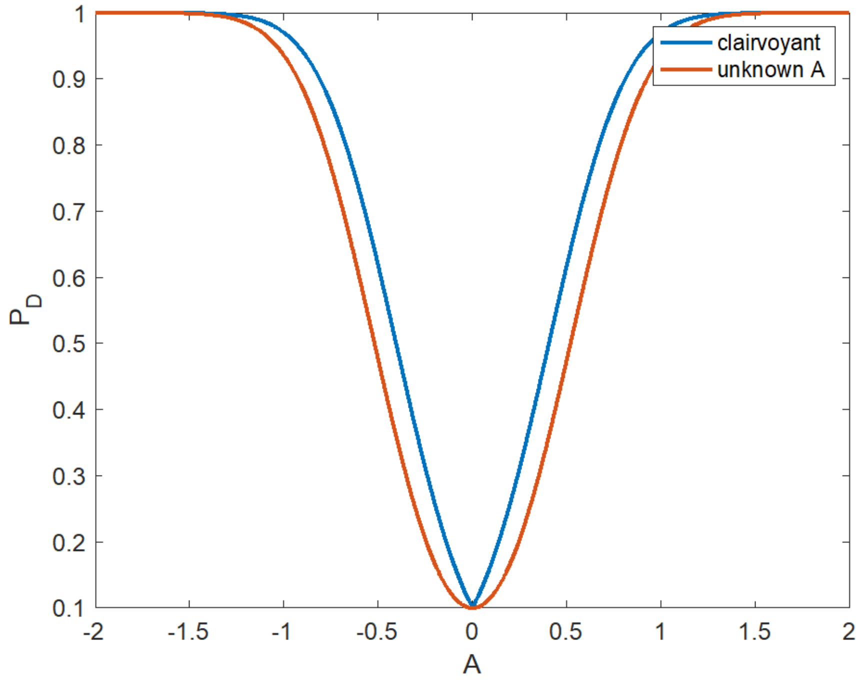
$$\boxed{P_D = Pr\{|\bar{x}| > \gamma''; \mathcal{H}_1\} = Q\left(Q^{-1}(P_{FA}/2) - \sqrt{\frac{NA^2}{\sigma^2}}\right) + Q\left(Q^{-1}(P_{FA}/2) + \sqrt{\frac{NA^2}{\sigma^2}}\right)}$$

$\bar{x} > \gamma''$

$\bar{x} < -\gamma''$

Example – DC level in WGN unknown A

$$P_D = Pr\{|\bar{x}| > \gamma''; \mathcal{H}_1\} = Q\left(Q^{-1}(P_{FA}/2) - \sqrt{\frac{NA^2}{\sigma^2}}\right) + Q\left(Q^{-1}(P_{FA}/2) + \sqrt{\frac{NA^2}{\sigma^2}}\right)$$



The performance of this realisable detector is thus not optimal, but close to the optimal clairvoyant detector.

$$N = 10,\ \sigma^2 = 1, P_{fa} = 0.1$$

Statement: For a UMP test to exist the parameter test must be one-sided.

True

0%

False

0%

A *one-sided test* results from an alternative hypothesis which specifies a direction, i.e., when the alternative hypothesis states that the parameter is in fact either **bigger or smaller** than the value specified in the null hypothesis.

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0/0

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Showing Results

Statement: For a UMP test to exist the parameter test must be one-sided.

True	0%
False	0%

Statement: All one-sided testing problems have UMP tests.

True	0%
False	0%

Statement: All one-sided testing problems have UMP tests.

True	0%
False	0%

Statement: Two-sided testing problems never produce UMP tests.

True

0%

False

0%

0/0

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Preparing Results

Statement: Two-sided testing problems never produce UMP tests.

True	0%
False	0%

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LO1: Understand scenarios where **hypothesis testing** needs to be applied to **PDFs with unknown parameters**

LO2: Understand and apply the **uniformly most powerful** test

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Approaches for composite hypothesis testing

Two approaches:

- Bayesian approach: Consider unknown parameters as realizations of random variables and assign a prior pdf.
- Generalized likelihood ratio: Estimate unknown parameters using MLEs.

Bayesian approach:

Assign priors to unknown parameters ϕ_0 and ϕ_1 under hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively:

$$p(\mathbf{x};\mathcal{H}_0) = \int p(\mathbf{x}|\phi_0;\mathcal{H}_0)p(\phi_0)d\phi_0$$

$$p(\mathbf{x};\mathcal{H}_1) = \int p(\mathbf{x}|\phi_1;\mathcal{H}_1)p(\phi_1)d\phi_1$$

- Need to choose prior pdf.
- Integration can be difficult.

NP detector: $\boxed{\frac{p(\mathbf{x};\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)} = \frac{\int p(\mathbf{x}|\phi_1;\mathcal{H}_1)p(\phi_1)d\phi_1}{\int p(\mathbf{x}|\phi_0;\mathcal{H}_0)p(\phi_0)d\phi_0} > \gamma}$

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Generalized Likelihood Ratio Test

GLRT:

- Replace unknown parameters by their MLEs.
- GLRT:

$$\boxed{L_G(\mathbf{x}) = \frac{p(\mathbf{x};\hat{\phi}_1,\mathcal{H}_1)}{p(\mathbf{x};\hat{\phi}_0,\mathcal{H}_0)} > \gamma}$$

with $p(\mathbf{x};\hat{\phi}_i,\mathcal{H}_i)$ given by

$$\boxed{p(\mathbf{x};\hat{\phi}_i,\mathcal{H}_i) = \max_{\phi_i} p(\mathbf{x};\phi_i,\mathcal{H}_i)}$$

Example: DC in WGN with Unknown Amplitude - GRLT

Remember this example

$$\begin{aligned}\mathcal{H}_0 &: \quad x[n] = w[n] \; n = 0, 1, \ldots, N-1 \\ \mathcal{H}_1 &: \quad x[n] = A + w[n] \; n = 0, 1, \ldots, N-1\end{aligned}$$

where $-\infty < A < \infty$ and $w[n]$ is WGN with variance σ^2. NP detector decides $\mathcal{H}_1$ if the GLRT:

$$\boxed{L_G(\mathbf{x}) = \frac{p(\mathbf{x}; \hat{\phi}_1, \mathcal{H}_1)}{p(\mathbf{x}; \hat{\phi}_0, \mathcal{H}_0)} > \gamma}$$

with $p(\mathbf{x}; \hat{\phi}_i, \mathcal{H}_i)$ given by

$$\boxed{p(\mathbf{x}; \hat{\phi}_i, \mathcal{H}_i) = \max_{\phi_i} p(\mathbf{x}; \phi_i, \mathcal{H}_i)}$$

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Example: DC in WGN with Unknown Amplitude - GRLT

MLE of A:

$$p(\mathbf{x};\hat{A},\mathcal{H}_1) = \max_A p(\mathbf{x};A,\mathcal{H}_1)$$

$$= \max_A \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right].$$

This will lead to $\hat{A} = \frac{1}{N}\sum_{n=0}^{N} x[n] = \bar{x}$.

Thus, the GLRT

$$L_G(\mathbf{x}) = \frac{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-\bar{x})^2\right]}{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right]} > \gamma$$

Taking the logarithm of both sides we have

$$\ln L_G(\mathbf{x}) = -\frac{1}{2\sigma^2}(-2N\bar{x}^2 + N\bar{x}^2) = \frac{N\bar{x}^2}{2\sigma^2}$$

Example: DC in WGN with Unknown Amplitude - GRLT

$$\ln L_G(\mathbf{x}) = -\frac{1}{2\sigma^2}(-2N\bar{x}^2 + N\bar{x}^2) = \frac{N\bar{x}^2}{2\sigma^2}$$

We decide thus for $\mathcal{H}_1$ if

$$\boxed{|\bar{x}| > \gamma''.}$$

This detector is identical to realisable detector we looked at before. Remember:

$$P_{FA} = Pr\{|\bar{x}| > \gamma''; \mathcal{H}_0\} = 2Pr\{\bar{x} > \gamma''; \mathcal{H}_0\} = 2Q\left(\frac{\gamma''}{\sqrt{\sigma^2/N}}\right)$$

$$\gamma'' = \sqrt{\sigma^2/N}Q^{-1}\left(P_{FA}/2\right)$$

$$\boxed{P_D = Pr\{|\bar{x}| > \gamma''; \mathcal{H}_1\} = Q\left(Q^{-1}(P_{FA}/2) - \sqrt{\frac{NA^2}{\sigma^2}}\right) + Q\left(Q^{-1}(P_{FA}/2) + \sqrt{\frac{NA^2}{\sigma^2}}\right).}$$

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Example: Unknown Amplitude and variance - GRLT

Unknown amplitude and variance:

$$\begin{aligned} \mathcal{H}_0 &: \quad x[n] = w[n] \;\; n = 0, 1, \ldots, N-1 \\ \mathcal{H}_1 &: \quad x[n] = A + w[n] \;\; n = 0, 1, \ldots, N-1 \end{aligned}$$

where $-\infty < A < \infty$ and $w[n]$ is WGN with unknown variance σ^2. NP detector decides $\mathcal{H}_1$ if the GLRT:

$$\boxed{L_G(\mathbf{x}) = \frac{p(\mathbf{x}; \hat{A}_1, \hat{\sigma}_1^2, \mathcal{H}_1)}{p(\mathbf{x}; \hat{\sigma}_0^2, \mathcal{H}_0)} > \gamma.}$$

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Example: Unknown Amplitude and variance - GRLT

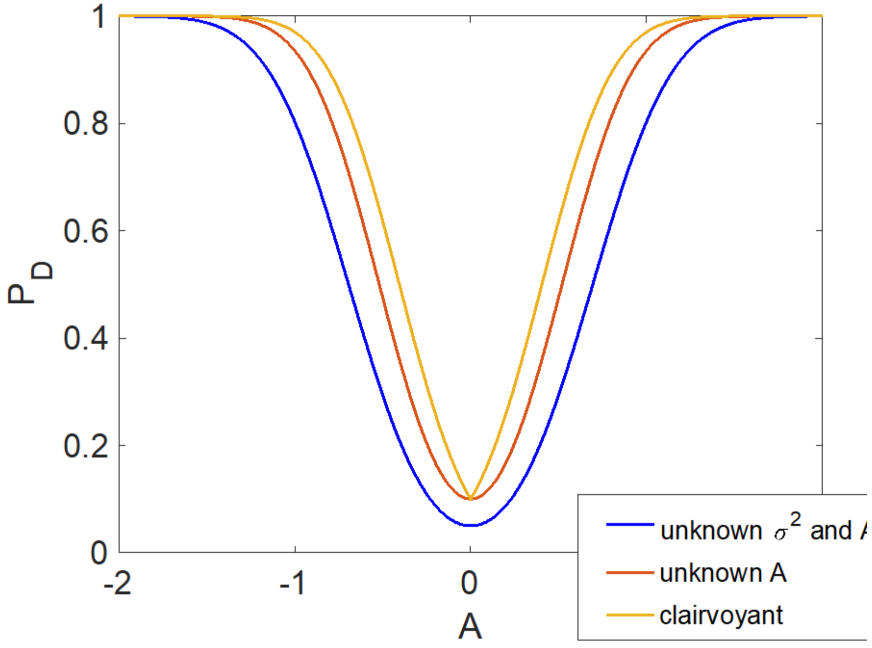
With

- $\hat{A}_{1,MLE} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$
- $\hat{\sigma}^2_{1,MLE} = \frac{1}{N}\sum_{n=0}^{N-1}(x[n] - \hat{A}_{1,MLE})^2$
- $\hat{\sigma}^2_{0,MLE} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]^2$

the GLRT becomes

$$L_G(\mathbf{x}) = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right)^{N/2} \quad \Rightarrow \quad T(\mathbf{x}) = \frac{\hat{A}_1^2}{\hat{\sigma}_1^2}$$

with $P_{fa} = Q_{F_{1,N-1}}(\gamma')$ and $P_D = Q_{F'_{1,N-1(\lambda)}}(\gamma')$.



$N = 10,\ \sigma^2 = 1, P_{fa} = 0.1$

F-distributed test statistic (ratio of Chi-distributed variables)

0/0

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Question slide

5. **Assumptions**: The data $\mathbf{x}$ has the PDF $p(\mathbf{x}; \boldsymbol{\theta}_0, \mathcal{H}_0)$ or $p(\mathbf{x} \mid \boldsymbol{\theta}_0; \mathcal{H}_0)$ under $\mathcal{H}_0$ and $p(\mathbf{x}; \boldsymbol{\theta}_1, \mathcal{H}_1)$ or $p(\mathrm{x} \mid \boldsymbol{\theta}_1; \mathcal{H}_1)$ under $\mathcal{H}_1$. The forms of the PDFs as well as the dimensionalities of the unknown parameter vectors θ_0 and θ_1 may be different under each hypothesis.
   Question: Which general test is this?
   Decide $\mathcal{H}_1$ if

$$L_1(\mathrm{x}) = \frac{p\left(\mathrm{x}; \hat{\theta}_1, \mathcal{H}_1\right)}{p\left(\mathbf{x}; \hat{\theta}_0, \mathcal{H}_0\right)} > \gamma,$$

   where $\hat{\theta}_i$ is the MLE of θ_i (maximizes $p(\mathbf{x}; \theta_i, \mathcal{H}_i)$).

Rao Test 0%

Generalized likelihood ratio test 0%

Bayesian approach 0%

Locally most powerful test 0%

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5. **Assumptions**: The data $\mathbf{x}$ has the PDF $p(\mathbf{x}; \boldsymbol{\theta}_0, \mathcal{H}_0)$ or $p(\mathbf{x} \mid \boldsymbol{\theta}_0; \mathcal{H}_0)$ under $\mathcal{H}_0$ and $p(\mathbf{x}; \boldsymbol{\theta}_1, \mathcal{H}_1)$ or $p(\mathrm{x} \mid \boldsymbol{\theta}_1; \mathcal{H}_1)$ under $\mathcal{H}_1$. The forms of the PDFs as well as the dimensionalities of the unknown parameter vectors θ_0 and θ_1 may be different under each hypothesis.
Question: Which general test is this?
Decide $\mathcal{H}_1$ if

$$L_1(\mathrm{x}) = \frac{p\left(\mathrm{x}; \hat{\theta}_1, \mathcal{H}_1\right)}{p\left(\mathbf{x}; \hat{\theta}_0, \mathcal{H}_0\right)} > \gamma,$$

where $\hat{\theta}_i$ is the MLE of θ_i (maximizes $p(\mathbf{x}; \theta_i, \mathcal{H}_i)$).

Rao Test 0%

Generalized likelihood ratio test 0%

Bayesian approach 0%

Locally most powerful test 0%

6. **Assumptions**: Same as the previous question.
Question: Which general test is this?
Decide $\mathcal{H}_1$ if

$$\frac{p\left(\mathbf{x};\mathcal{H}_1\right)}{p\left(\mathbf{x};\mathcal{H}_0\right)}=\frac{\int p\left(\mathbf{x}\mid\boldsymbol{\theta}_1;\mathcal{H}_1\right)p\left(\boldsymbol{\theta}_1\right)d\boldsymbol{\theta}_1}{\int p\left(\mathrm{x}\mid\boldsymbol{\theta}_0;\mathcal{H}_0\right)p\left(\boldsymbol{\theta}_0\right)d\boldsymbol{\theta}_0}>\gamma,$$

where $p\left(\mathbf{x};\mathcal{H}_i\right)$ is the unconditional data PDF, $p\left(\mathbf{x}\mid\boldsymbol{\theta}_i;\mathcal{H}_i\right)$ is the conditional data PDF and $p\left(\boldsymbol{\theta}_i\right)$ is the prior PDF.

Rao Test — 0%

Generalized likelihood ratio test — 0%

Bayesian approach — 0%

Locally most powerful test — 0%

6. **Assumptions**: Same as the previous question.
Question: Which general test is this?
Decide $\mathcal{H}_1$ if

$$\frac{p(\mathbf{x};\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)} = \frac{\int p(\mathbf{x} \mid \boldsymbol{\theta}_1;\mathcal{H}_1)\, p(\boldsymbol{\theta}_1)\, d\boldsymbol{\theta}_1}{\int p(\mathrm{x} \mid \boldsymbol{\theta}_0;\mathcal{H}_0)\, p(\boldsymbol{\theta}_0)\, d\boldsymbol{\theta}_0} > \gamma,$$

where $p(\mathbf{x};\mathcal{H}_i)$ is the unconditional data PDF, $p(\mathbf{x} \mid \boldsymbol{\theta}_i;\mathcal{H}_i)$ is the conditional data PDF and $p(\boldsymbol{\theta}_i)$ is the prior PDF.

Rao Test 0%

Generalized likelihood ratio test 0%

Bayesian approach 0%

Locally most powerful test 0%

Learning objectives

LO1: Understand scenarios where **hypothesis testing** needs to be applied to **PDFs with unknown parameters**

LO2: Understand and apply the **uniformly most powerful** test

LO3: Apply the Bayesian approach in situations where a **prior Probability Density Function (PDF)** for detection problems with unknown parameters

LO4: Apply the **generalized likelihood ratio test** for detection problems with unknown parameters

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All the Best