Estimation and Detection

Random Signals (Ch. 5)

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Recap

- **Detection theory**
  - Neyman-Pearson Theorem (NP)
  - Minimum Probability of Error
  - Bayes Risk

- **Detecting a known deterministic signal in noise using the NP criterion**
  - White noise
  - Colored noise

Learning objectives

LO1: Analyze **optimal detectors** for white Gaussian signals

LO2: Analyze **optimal detectors** for Gaussian signals with arbitrary covariance matrix buried in white noise and show how NP leads to **the estimator-correlator**

LO3: Analyze **optimal detectors** for Gaussian signals with arbitrary covariance matrix buried in colored noise and show how NP leads to **the estimator-dewhitener**

LO4: Analyze **optimal general Gaussian detection**

Q1: What criterion should be employed to maximize the probability of detection subject to a constant probability of false alarm?

Neyman-Pearson	0%
Bayesian risk	0%

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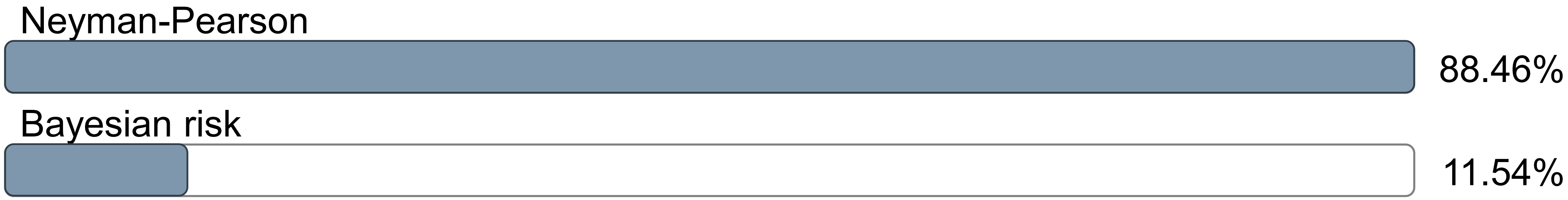
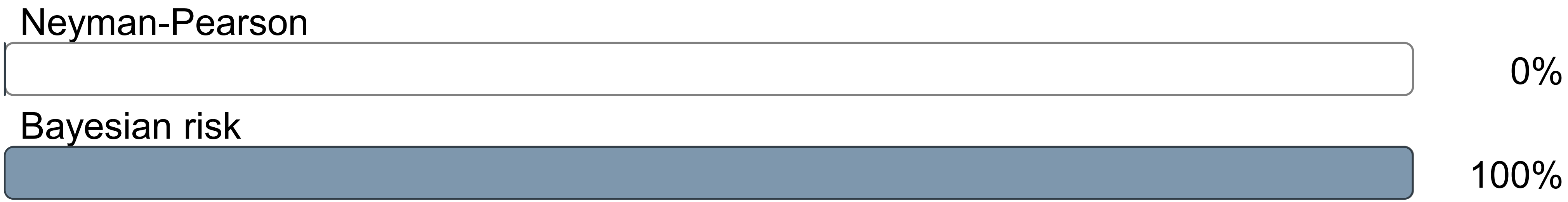


Q2: What criterion should be used if we want to minimize the average cost?

Neyman-Pearson	0%
Bayesian risk	0%

Q2: What criterion should be used if we want to minimize the average cost?



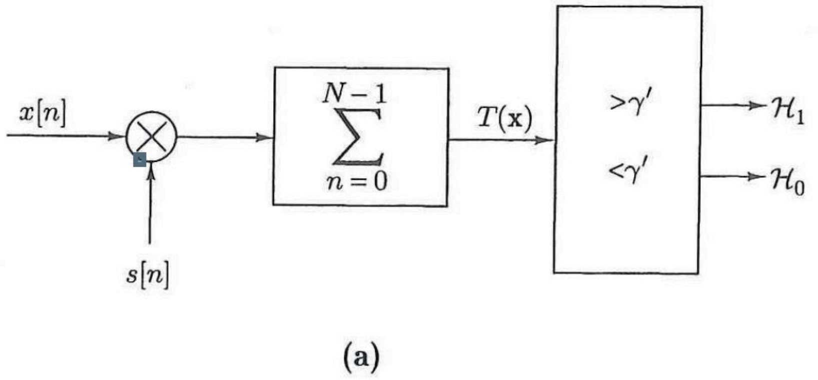
Deterministic Signals (WGN) - Interpretation

Binary detection problem with $\mathbf{w} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$ and deterministic $\mathbf{s}$:

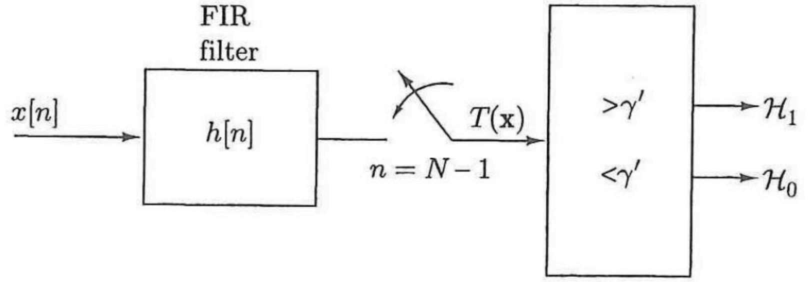
White noise

$$\mathcal{H}_0 \quad x[n] = w[n]$$
$$\mathcal{H}_1 \quad x[n] = s[n] + w[n]$$

Interpretation 1: The resulting $T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s[n]$ is a correlator. The received data is correlated with a replica of the signal.



Interpretation 2: The resulting $T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s[n]$ is a matched filter.



$$h[n] = \begin{cases} s[N-1-n] & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

Fig. 4.1 from Kay-II.

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Deterministic Signals – Summary

Binary detection problem with $\mathbf{w} \sim N(\mathbf{0}, \mathbf{C})$ and deterministic $\mathbf{s}$:

Colored noise

$$\mathcal{H}_0 \quad x[n] = w[n]$$

$$\mathcal{H}_1 \quad x[n] = s[n] + w[n]$$

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{s}$$

Notice that if C is positive definite, $\mathbf{C}^{-1}$ can be written as $\mathbf{C}^{-1} = \mathbf{D}^T \mathbf{D}$, leading to

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{D}^T \mathbf{D} \mathbf{s}$$

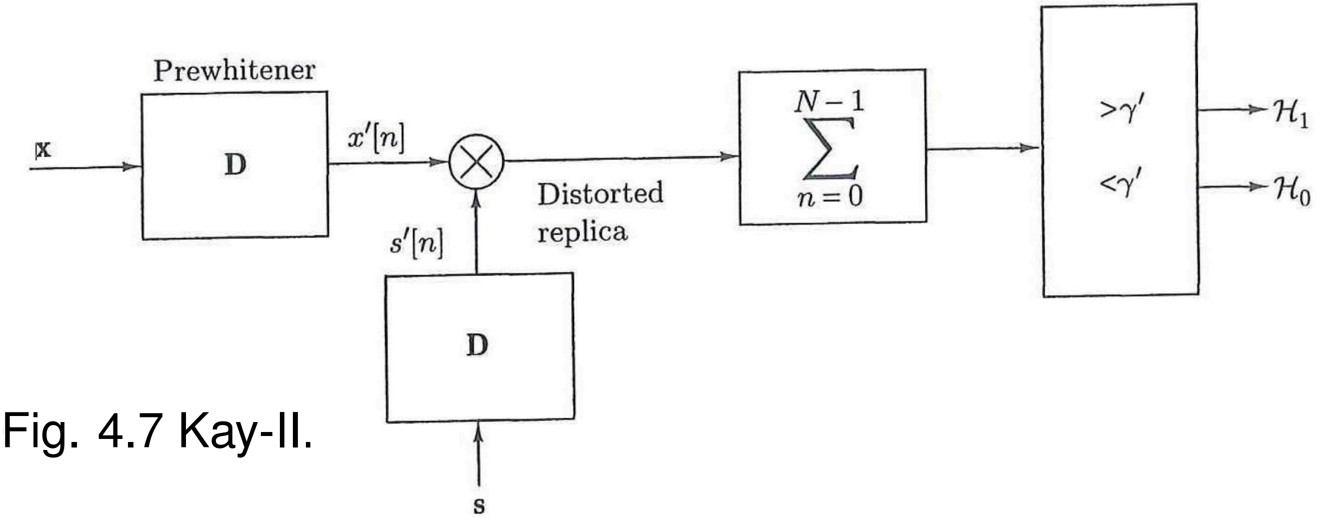

Fig. 4.7 Kay-II.

Quiz

3. Write down the Neyman-Pearson detector when the Gaussian noise is not white ($\mathrm{w} \sim \mathcal{N}(0, \mathbf{C})$, where $\mathbf{C}$ is the covariance matrix):

Write your answer.

$T(\mathbf{x}) = \mathbf{x}^T\mathbf{C}^{-1}\mathbf{s}$ See Equation (4.16). The detector of (4.16) is referred to as a genemralized replica-correlator or matched filter.

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Colored noise: Optimal Detection Signal

Notice:

For white Gaussian noise, P_D does not depend on signal shape, only on the energy $\mathrm{s}^T\mathrm{s}$:

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{\mathbf{s}^T\mathbf{s}}{\sigma^2}}\right)$$

For colored Gaussian noise, P_D DOES depend on the shape of the s compared to the statistics of the noise:

$$P_D = Q\left(\frac{\gamma' - \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}{\sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}}\right) = Q\left(Q^{-1}\left(P_{fa}\right) - \sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}\right).$$

What is the optimal s for the P_D?

Colored noise: Optimal Detection Signal

1. Constrain the total energy to be $\mathbf{s}^T\mathbf{s} = E$.

2. Optimize for the shape of $\mathbf{s}$:

$$\max_{\mathbf{s}} \quad \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}$$
$$s.t. \quad \mathbf{s}^T\mathbf{s} = E$$

$$L(\mathbf{s}) = \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s} + \lambda(E - \mathbf{s}^T\mathbf{s})$$

Use $\frac{\partial \mathbf{b}^T\mathbf{x}}{\partial \mathbf{x}} = \mathbf{b}$ and $\frac{\partial \mathbf{x}^T\mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x}$.

$$\frac{\partial L(\mathbf{s})}{\partial \mathbf{s}} = 2\mathbf{C}^{-1}\mathbf{s} - 2\lambda\mathbf{s} = 0 \;\rightarrow \mathbf{C}^{-1}\mathbf{s} = \lambda\mathbf{s}$$

$\mathbf{s}$ is thus an eigenvector of $\mathbf{C}^{-1}$ with eigenvalue λ.

To maximize $\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}$, we should choose the eigenvector $\mathbf{s}$ that corresponds with the maximum eigenvalue λ of $\mathbf{C}^{-1}$ (or the minimum eigenvalue of $\mathbf{C}$).

Exercise 1

Problem 1: Binary detection problem with $\mathbf{w} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$ and deterministic signal $s[n] = Ar^n$:

$$\begin{array}{ll} \mathcal{H}_0 & x[n] = w[n] \\ \mathcal{H}_1 & x[n] = Ar^n + w[n] \end{array}$$

Problem 1a: Find the NP detector.

Problem 1b: Determine the detection performance.

Problem 1c: What happens as $N \to \infty$ for $0 \leq r \leq 1$, $r = 1$ and $r \geq 1$?

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Exercise 1

Problem 1: Binary detection problem with $\mathbf{w} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$ and deterministic signal $s[n] = Ar^n$:

$$\begin{array}{ll} \mathcal{H}_0 & x[n] = w[n] \\ \mathcal{H}_1 & x[n] = Ar^n + w[n] \end{array}$$

Problem 1a: Find the NP detector.

Problem 1a: Let $\mathbf{H} = [1, r, ..., r^{N-1}]^T$.

$$\begin{aligned} p(\mathbf{x}; \mathcal{H}_1) &= \frac{1}{(2\pi)^{\frac{N}{2}} \det^{\frac{1}{2}}(\mathbf{C})} \exp\left[-\frac{1}{2}(\mathbf{x} - A\mathbf{H})^T \mathbf{C}^{-1} (\mathbf{x} - A\mathbf{H})\right] \\ p(\mathbf{x}; \mathcal{H}_0) &= \frac{1}{(2\pi)^{\frac{N}{2}} \det^{\frac{1}{2}}(\mathbf{C})} \exp\left[-\frac{1}{2}\mathbf{x}^T \mathbf{C}^{-1} \mathbf{x}\right]. \end{aligned}$$

$$L(\mathbf{x}) = \frac{p(\mathbf{x};\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)} > \lambda.$$

$$\ln L(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} A\mathbf{H} - \frac{1}{2}\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H} A^2 > \ln \lambda$$

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{H} A > \ln \lambda + \frac{1}{2}\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H} A^2 = \lambda'$$

Exercise 1

Problem 1b: Determine the detection performance.

Problem 1b: $T(\mathbf{x})$ is Gaussian distributed under both $\mathcal{H}_1$ and $\mathcal{H}_0$.

$$[T;\mathcal{H}_0] = [\mathbf{w}^T\mathbf{C}^{-1}\mathbf{H}A] = 0$$

$$[T;\mathcal{H}_1] = [(A\mathbf{H}+\mathbf{w})^T\mathbf{C}^{-1}\mathbf{H}A] = A^2\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H} = \frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}$$

$$\text{var}[T;\mathcal{H}_0] = [(\mathbf{w}^T\mathbf{C}^{-1}\mathbf{H}A)^2] = A^2\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H} = \frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}$$

$$\text{var}[T;\mathcal{H}_1] = [((A\mathbf{H}+\mathbf{w})^T\mathbf{C}^{-1}\mathbf{H}A - [(A\mathbf{H}+\mathbf{w})^T\mathbf{C}^{-1}\mathbf{H}A])^2$$

$$= \left[\left(((A\mathbf{H}+\mathbf{w}) - [(A\mathbf{H}+\mathbf{w})])^T\,\mathbf{C}^{-1}\mathbf{H}A\right)^2\right] = \left[(\mathbf{w}^T\mathbf{C}^{-1}\mathbf{H}A)^2\right] = var[T;\mathcal{H}_0] = \frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}$$

$$P_{fa} = Q\left(\frac{\lambda'}{\sqrt{\frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}}}\right) \rightarrow \lambda' = Q^{-1}(P_{fa})\sqrt{\frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}}$$

$$P_D = Q\left(\frac{\lambda' - \frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}}{\sqrt{\frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}}}\right) = Q\left(Q^{-1}(P_{fa}) - \sqrt{\frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}}\right),$$

Exercise 1

Problem 1c: What happens as $N \to \infty$ for $0 \le r \le 1$, $r = 1$ and $r \ge 1$?

$$P_{fa} = Q\left(\frac{\lambda'}{\sqrt{\frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}}}\right) \to \lambda' = Q^{-1}\left(P_{fa}\right)\sqrt{\frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}}$$

$$P_D = Q\left(\frac{\lambda' - \frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}}{\sqrt{\frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}}}\right) = Q\left(Q^{-1}\left(P_{fa}\right) - \sqrt{\frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}}\right),$$

Problem 1c: For $0 \le r \le 1$, $\sum_{n=0}^{N-1} r^{2n} = \frac{1-r^{2N}}{1-r^2}$ and $P_D = Q\left(Q^{-1}\left(P_{fa}\right) - \sqrt{\frac{A^2}{\sigma^2}\frac{1-r^{2N}}{1-r^2}}\right)$

When $N \to \infty$ for $0 \le r \le 1$, P_D will become $P_D = Q\left(Q^{-1}\left(P_{fa}\right) - \sqrt{\frac{A^2}{\sigma^2}\frac{1}{1-r^2}}\right)$ which will be smaller than 1 for $P_{fa} < 1$.

For $r = 1$ P_D will become $P_D = Q\left(Q^{-1}\left(P_{fa}\right) - \sqrt{\frac{NA^2}{\sigma^2}}\right)$ and for $N \to \infty$ P_D will approach 1. For $r \ge 1$ P_D will also approach 1 as $\lim_{N\to\infty}\sum_{n=0}^{N-1} r^{2n}$ will then $\to \infty$.

Exercise 2a

White noise

To optimize the detection probability of a signal in WGN, different signals are investigated. These are

$$s_1[n] = A$$

and

$$s_2[n] = A(-1)^n,$$

both for $n = 0, ..., N-1$.

- Which signal will have the best detection performance?

A $s_1[n]$

B $s_2[n]$

C Equal detection performance.

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Exercise 2a

White noise

To optimize the detection probability of a signal in WGN, different signals are investigated. These are

$$s_1[n] = A$$

and

$$s_2[n] = A(-1)^n,$$

both for $n = 0, ..., N-1$.

- Which signal will have the best detection performance?

Problem 2a: As the noise is white and Gaussian, the shape of the signal does not influence the detection performancce, but the power does. As both signals have an equal power, the detection performance will be equal.

Exercise 2b

Now consider the case where the noise has correlation matrix $\mathbf{C} = \sigma^2\mathbf{I} + P\mathbf{1}\mathbf{1}^T$ and signals

$$s_1[n] = A$$

and

$$s_2[n] = A(-1)^n,$$

both for $n = 0, ..., N-1$.

- Which signal will have the best detection performance?

A $s_1[n]$

B $s_2[n]$

C Equal detection performance.

Exercise 2b

Colored noise

Problem 2b: Using the matrix inversion lemma we can calculate $\mathbf{C}^{-1}$, that is, $\mathbf{C}^{-1} = \frac{1}{\sigma^2}\boldsymbol{I} - \frac{\frac{1}{\sigma^4}\mathbf{1}\mathbf{1}^T}{1+\frac{N}{\sigma^2}}$. We can use this result to calculate the P_D:

$$P_D = Q\left(Q^{-1}(P_{fa}) - \sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}\right).$$

For $s_1[n]$ we then get $P_D = Q\left(Q^{-1}(P_{fa}) - \sqrt{\frac{A^2}{\frac{\sigma^2}{N}+1}}\right) = Q\left(Q^{-1}(P_{fa}) - \sqrt{\frac{A^2\frac{N}{\sigma^2}}{\frac{N}{\sigma^2}+1}}\right)$.

For $s_2[n]$ and (even) N we get $P_D = Q\left(Q^{-1}(P_{fa}) - \sqrt{A^2\frac{N}{\sigma^2}}\right)$. The P_D for even N and $s_2[n]$ will thus always be larger.
One can also argue that $s[n]$ should ideally equal the eigenvector of $\mathbf{C}$ that corresponds to the minimum eigenvalue. The largest eigenvalue is $\mathbf{1}$. This corresponds with $s_1[n]$. Signal $s_2[n]$ is at least orthogonal to this eigenvector and corresponds to the minimum eigenvalue. $s_2[n]$ will thus have the best detection performance.

Overview of Chapter 5

Topic	Section
Estimator-Correlator	Chapter 5.3
Linear Model	Chapter 5.4
Estimator-Correlator for Large Data Records	Chapter 5.5
General Gaussian Detection	Chapter 5.6
Signal Processing Examples	Chapter 5.7

Learning objectives

LO1: Analyze **optimal detectors** for white Gaussian signals

LO2: Analyze **optimal detectors** for Gaussian signals with arbitrary covariance matrix buried in white noise and show how NP leads to **the estimator-correlator**

LO3: Analyze **optimal detectors** for Gaussian signals with arbitrary covariance matrix buried in colored noise and show how NP leads to **the estimator-dewhitener**

LO4: Analyze **optimal general Gaussian detection**

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Q4: Statement: we can detect all signals in the presence of noise by detecting the change in the mean of a test statistic.

True	0%
False	0%

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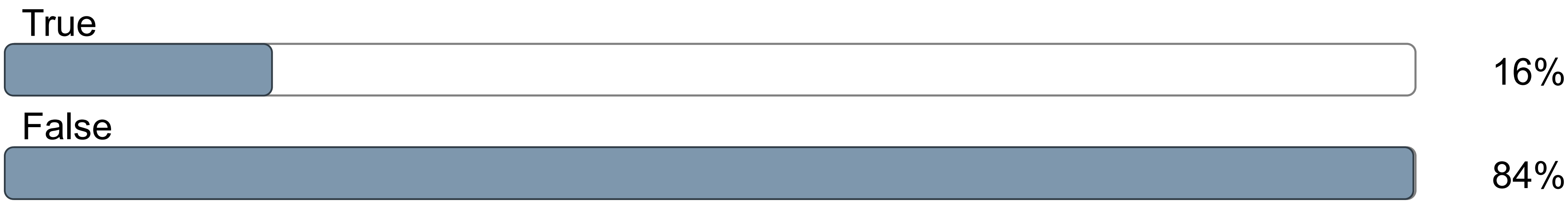
Q4: Statement: we can detect all signals in the presence of noise by detecting the change in the mean of a test statistic.



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Quiz

4. Statement: we can detect all signals in the presence of noise by detecting the change in the mean of a test statistic.

a. True
b. False

No. For deterministic signals, we can detect them by the change in the mean of a test statistic. However, this is not universally true. The effectiveness of this approach depends on the characteristics of the signals and the nature of the noise.
1. In scenarios where the noise is not Gaussian, relying solely on changes in the mean may lead to suboptimal performance.
2. In some cases, a signal is more appropriately modeled as a random process. It may not result in a significant change in the mean of a test statistic.

Random Signals

Assumptions:

- Target signal $s[n]$: Random, zero-mean Gaussian random process with known covariance.
- Noise $w[n]$: Random, zero-mean Gaussian random process with known covariance. and independent from $s[n]$.

Binary detection problem:

$$\begin{array}{ll} \mathcal{H}_0 & x[n] = w[n] \\ \mathcal{H}_1 & x[n] = s[n] + w[n] \end{array}$$

The NP detector: Decides $\mathcal{H}_1$ if

$$L(\mathbf{x}) = \frac{p(\mathbf{x}; \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} > \lambda$$

Random Signals - Example

$$\mathcal{H}_0 : \mathbf{x} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$$

$$\mathcal{H}_1 : \mathbf{x} \sim \mathcal{N}(\mathbf{0}, (\sigma_s^2 + \sigma^2) \mathbf{I})$$

Thus, we have $L(\mathbf{x}) = \dfrac{\frac{1}{\left(2\pi(\sigma_s^2+\sigma^2)\right)^{\frac{N}{2}}} \exp\left[-\frac{1}{2(\sigma_s^2+\sigma^2)} \sum_{n=0}^{N-1} x^2[n]\right]}{\frac{1}{\left(2\pi\sigma^2\right)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} x^2[n]\right]}$.

Calculating the Log-Likelihood Ratio (LLR), we have

$$\begin{aligned} l(\mathbf{x}) &= \frac{N}{2} \ln\left(\frac{\sigma^2}{\sigma_s^2 + \sigma^2}\right) - \frac{1}{2}\left(\frac{1}{\sigma_s^2 + \sigma^2} - \frac{1}{\sigma^2}\right) \sum_{n=0}^{N-1} x^2[n] \\ &= \frac{N}{2} \ln\left(\frac{\sigma^2}{\sigma_s^2 + \sigma^2}\right) + \frac{1}{2} \frac{\sigma_s^2}{\sigma^2(\sigma_s^2 + \sigma^2)} \sum_{n=0}^{N-1} x^2[n] \end{aligned}$$

Notice: Scalar Wiener filter!

Random Signals - Example

Hence, we decide $\mathcal{H}_1$ if

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} x^2[n] > \lambda^{'}$$

The NP detector computes the energy in the received data and compares it to a threshold.

$T(\mathbf{x})$ under $\mathcal{H}_0$ and $\mathcal{H}_1$ is distributed as follows

$$\mathcal{H}_0 : \quad \frac{T(\mathbf{x})}{\sigma^2} \sim \chi_N^2$$

$$\mathcal{H}_1 : \quad \frac{T(\mathbf{x})}{\sigma_s^2 + \sigma^2} \sim \chi_N^2$$

S is χ_N^2 distributed if $S = \sum_i^N x_i^2$ and $x_i \sim N(0,1)$.

Random Signals - Example

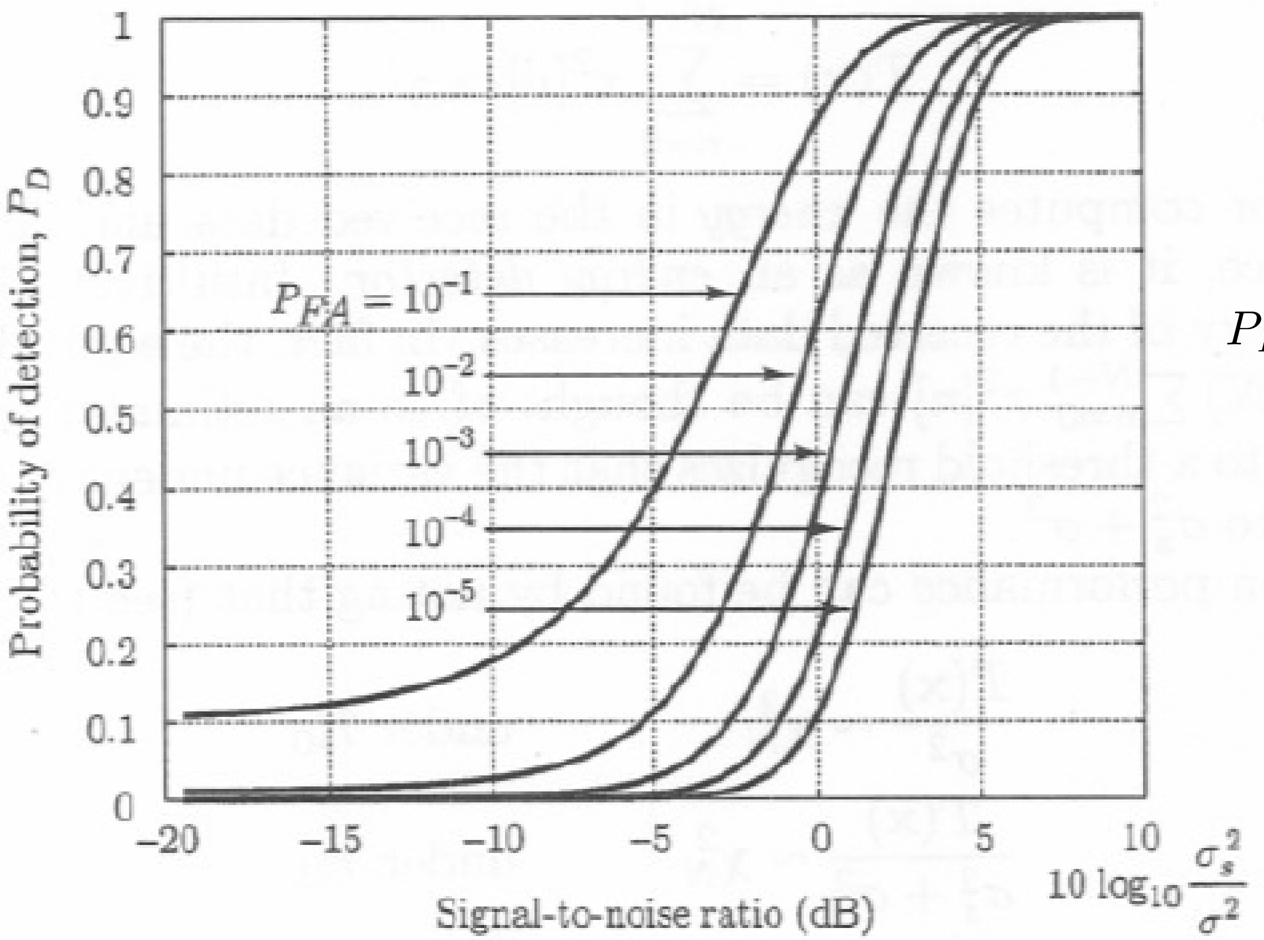
Therefore, we have

$$
\begin{aligned}
P_{FA} &= Pr\{T(\mathbf{x}) > \lambda^{'}; \mathcal{H}_0\} \\
&= Pr\left\{\frac{T(\mathbf{x})}{\sigma^2} > \frac{\lambda^{'}}{\sigma^2}; \mathcal{H}_0\right\} \\
&= Q_{\chi^2_N}\left(\frac{\lambda^{'}}{\sigma^2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
P_D &= Pr\{T(\mathbf{x}) > \lambda^{'}; \mathcal{H}_1\} \\
&= Pr\left\{\frac{T(\mathbf{x})}{\sigma_s^2 + \sigma^2} > \frac{\lambda^{'}}{\sigma_s^2 + \sigma^2}; \mathcal{H}_1\right\} \\
&= Q_{\chi^2_N}\left(\frac{\lambda^{'}}{\sigma_s^2 + \sigma^2}\right)
\end{aligned}
$$

Random Signals - Example



Energy Detector Performance for $N = 25$

(Fig. 5.1 Kay-II)

$$P_D = Q_{\chi_N^2}\left(\frac{\lambda'}{\sigma_s^2 + \sigma^2}\right) = Q_{\chi_N^2}\left(\frac{\frac{\lambda'}{\sigma^2}}{\frac{\sigma_s^2}{\sigma^2} + 1}\right).$$

P_D thus increases with SNR σ_s^2/σ^2

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LO4: Analyze **optimal general Gaussian detection**

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Random Signals – Generalization 1

$$\mathcal{H}_0 : \mathbf{x} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$$

$$\mathcal{H}_1 : \mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_s + \sigma^2\mathbf{I})$$

Thus, we have $L(\mathbf{x}) = \dfrac{\frac{1}{\left(2\pi\right)^{\frac{N}{2}} det^{\frac{1}{2}}\left(\mathbf{C}_s+\sigma^2\mathbf{I}\right)} \exp\left[-\frac{1}{2}\mathbf{x}^T\left(\mathbf{C}_s+\sigma^2\mathbf{I}\right)^{-1}\mathbf{x}\right]}{\frac{1}{\left(2\pi\sigma^2\right)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\mathbf{x}^T\mathbf{x}\right]}.$

Calculating the Log-Likelihood Ratio (LLR), we have

$$T(\mathbf{x}) = \sigma^2\mathbf{x}^T\left[\frac{1}{\sigma^2}\mathbf{I} - \left(\mathbf{C}_s + \sigma^2\mathbf{I}\right)^{-1}\right]\mathbf{x} > 2\gamma^{'}\sigma^2$$

Random Signals – Generalization 1

$$T(\mathbf{x}) = \sigma^2 \mathbf{x}^T \left[\frac{1}{\sigma^2}\mathbf{I} - \left(\mathbf{C}_s + \sigma^2 \mathbf{I}\right)^{-1} \right] \mathbf{x} > 2\gamma' \sigma^2$$

Using the matrix inversion lemma

$$(\mathbf{A} + \mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{DA}^{-1}\mathbf{B} + \mathbf{C}^{-1})^{-1}\mathbf{DA}^{-1}$$

we can write

$$\left(\mathbf{C}_s + \sigma^2 \mathbf{I}\right)^{-1}$$

as

$$\frac{1}{\sigma^2}\mathbf{I} - \frac{1}{\sigma^4}\left(\mathbf{C}_s^{-1} + \frac{1}{\sigma^2}\mathbf{I}\right)^{-1}$$

such that

$$T(\mathbf{x}) = \mathbf{x}^T \left[\frac{1}{\sigma^2}\left(\mathbf{C}_s^{-1} + \frac{1}{\sigma^2}\mathbf{I}\right)^{-1} \right] \mathbf{x} > 2\gamma' \sigma^2$$

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Random Signals – Generalization 1

$$T(\mathbf{x}) = \mathbf{x}^T \left[\frac{1}{\sigma^2} \left(\mathbf{C}_s^{-1} + \frac{1}{\sigma^2}\mathbf{I} \right)^{-1} \right] \mathbf{x} > 2\gamma' \sigma^2$$

$\hat{\mathbf{s}}$

Now set $\hat{s}$ equal to

$$\hat{\mathbf{s}} = \left[\frac{1}{\sigma^2} \left(\mathbf{C}_s^{-1} + \frac{1}{\sigma^2}\mathbf{I} \right)^{-1} \right] \mathbf{x}$$

Recognize this as the LMMSE filter!!

$(\boldsymbol{\theta} = \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}\mathbf{x})$

which can be rewritten as

$$\hat{\mathbf{s}} = \frac{1}{\sigma^2} \left[\frac{1}{\sigma^2} \left(\mathbf{C}_s + \sigma^2\mathbf{I} \right) \mathbf{C}_s^{-1} \right]^{-1} \mathbf{x} = \mathbf{C}_s(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\mathbf{x}$$

Hence, we decide for $\mathcal{H}_1$ if

$$T(\mathbf{x}) = \mathbf{x}^T\hat{\mathbf{s}} > \gamma''$$

Quiz

5. Find the **NP detector** for the problem of a **random signal** $s[n]$ with mean zero and covariance matrix $\mathbf{C}_s = \text{diag}\left(\sigma^2_{s_0}, \sigma^2_{s_1}, \ldots, \sigma^2_{s_{N-1}}\right)$ embedded in **WGN** with variance σ^2. Assume that the data samples observed are $x[n]$ for $n = 0, 1, \ldots, N-1$. Do not explicitly evaluate the threshold, see Equation (5.5).

Write your answer.

$$\hat{\mathbf{s}} = \mathbf{C_s}\left(\mathbf{C_s} + \sigma^2\mathbf{I}\right)^{-1}\mathbf{x}$$

$$= \begin{bmatrix} \frac{\sigma^2_{S_0}}{\sigma^2_{S_0}+\sigma^2} & \ddots & 0 \\ 0 & & \frac{\sigma^2_{S_{N-1}}}{\sigma^2_{S_{N-1}}+\sigma^2} \end{bmatrix}\mathbf{x}$$

$$T(\mathbf{x}) = \mathbf{x}^\top\hat{\mathbf{s}} > \gamma''$$

$$T(\mathbf{x}) = \mathbf{x}^\top\mathbf{C_S}\left(\mathbf{C_S} + \sigma^2\mathbf{I}\right)^{-1}\mathbf{x} > \gamma''$$

$$= \sum_{n=0}^{N-1}\frac{\sigma^2_{s_n}}{\sigma_{s^2_n} + \sigma^2}x^2[n]$$

Random Signals – Estimator Correlator

Hence, we decide for $\mathcal{H}_1$ if

$$T(\mathbf{x}) = \mathbf{x}^T \hat{\mathbf{s}} > \gamma''$$

with

$$\hat{\mathbf{s}} = \frac{1}{\sigma^2}\left[\frac{1}{\sigma^2}\left(\mathbf{C}_s + \sigma^2\mathbf{I}\right)\mathbf{C}_s^{-1}\right]^{-1}\mathbf{x} = \mathbf{C}_s(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\mathbf{x}$$

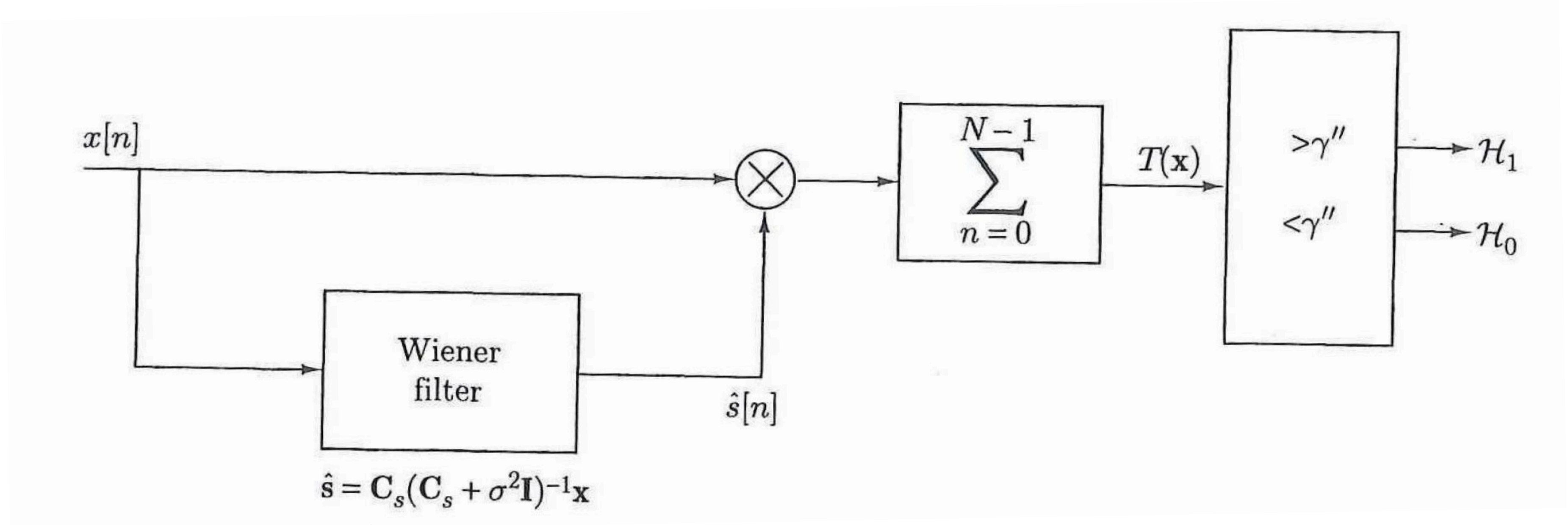

Fig. 5.2 Kay-II.

Random (Correlated) Signal – Example (1)

Let $0 < \rho < 1$ be the correlation coefficient between consecutive signal samples $s[n]$, and let

$$\mathbf{C_s} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

- Eigenvalues: $(\lambda - 1 - \rho)(\lambda - 1 + \rho) = 0 \rightarrow \lambda_1 = 1 + \rho$ and $\lambda_2 = 1 - \rho$ with corresponding eigenvectors following from $(\mathbf{C_s} - \lambda_i \mathbf{I})\, \mathbf{v}_i = \mathbf{0}$, $\mathbf{v_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$.

Now use the fact that $\mathbf{C}_s = \mathbf{V} \mathbf{\Lambda}_s \mathbf{V}^T$ (= eigenvalue decomposition)

$$\hat{\mathbf{s}} = \frac{1}{\sigma^2} \left[\frac{1}{\sigma^2} \left(\mathbf{C}_s + \sigma^2 \mathbf{I} \right) \mathbf{C}_s^{-1} \right]^{-1} \mathbf{x} = \left[\left(\mathbf{C}_s + \sigma^2 \mathbf{I} \right) \mathbf{C}_s^{-1} \right]^{-1} \mathbf{x}$$

$$= \left[\left(\mathbf{V} \mathbf{\Lambda}_s \mathbf{V}^T + \sigma^2 \mathbf{I} \right) \left(\mathbf{V} \mathbf{\Lambda}_s \mathbf{V}^T \right)^{-1} \right]^{-1} \mathbf{x}$$

$$= \mathbf{V} \left(\mathbf{I} + \boldsymbol{\sigma^2} \mathbf{\Lambda_s}^{-\mathbf{1}} \right)^{-1} \mathbf{V}^T \mathbf{x}$$

Random (Correlated) Signal – Example (2)

So the total test statistic becomes:

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \left(\mathbf{I} + \boldsymbol{\sigma^2} \boldsymbol{\Lambda}_\mathbf{s}{}^{-\mathbf{1}}\right)^{-1} \mathbf{V}^T \mathbf{x}$$

If we set $\mathbf{y} = \mathbf{V}^T \mathbf{x}$, then $\mathbf{y}$ is a decorrelated version of $\mathbf{x}$ (Hence $E[\mathbf{y}\mathbf{y}^T] = \lambda_\mathbf{s} + \sigma^2 \mathbf{I}$ is diagonal) and we get

$$T(\mathbf{x}) = \mathbf{y}^T \left(\mathbf{I} + \boldsymbol{\sigma^2} \boldsymbol{\Lambda}_\mathbf{s}{}^{-\mathbf{1}}\right)^{-1} \mathbf{y} = \sum_{n=0}^{N-1} y^2[n] \frac{\lambda_{s[n]}}{\lambda_{s[n]} + \sigma^2}$$

Hence, in the transformed space we have obtained a weighted energy detector.

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Random (Correlated) Signal – Example (3)

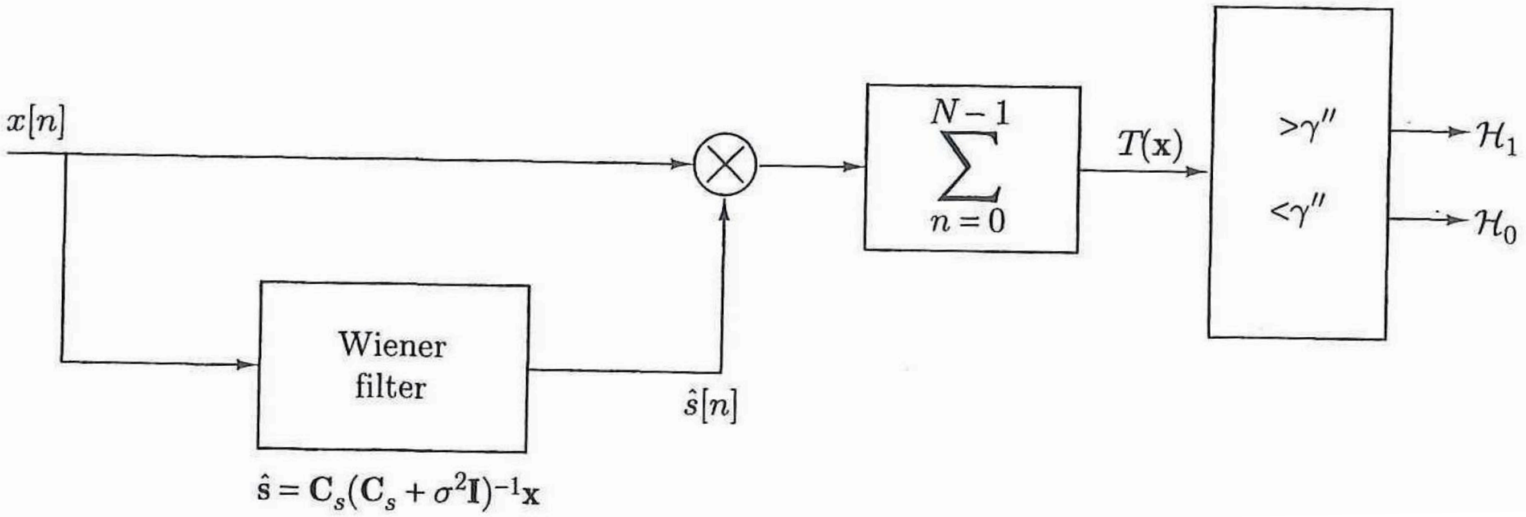


Fig. 5.2 Kay-II.

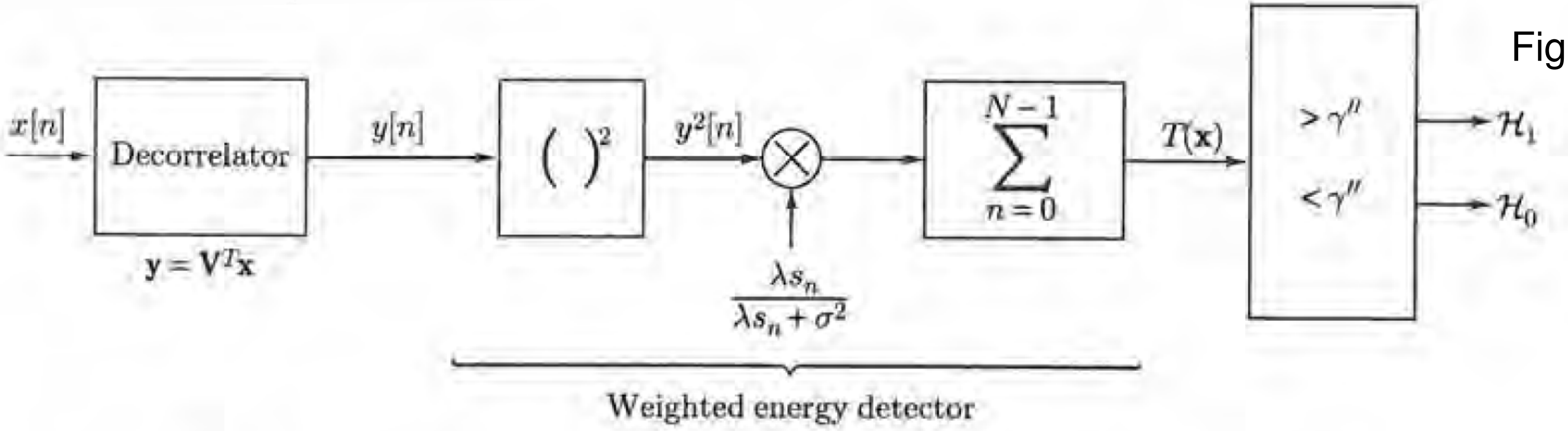


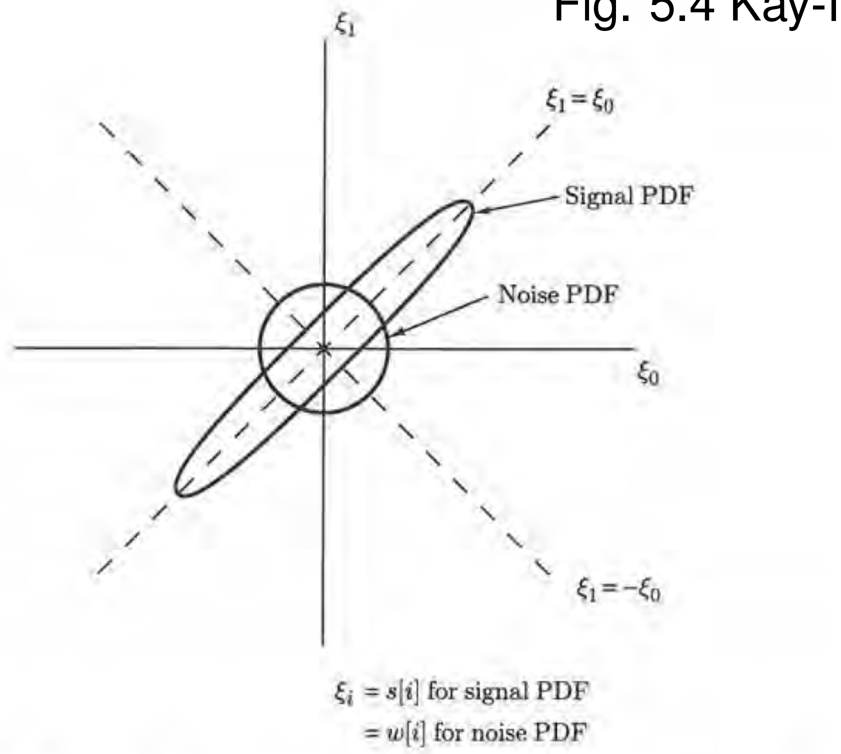
Fig. 5.3 Kay-II.

Random (Correlated) Signal – Example (4)

- Eigenvalues: $(\lambda - 1 - \rho)(\lambda - 1 + \rho) = 0 \rightarrow \lambda_1 = 1 + \rho$ and $\lambda_2 = 1 - \rho$ with corresponding eigenvectors following from $(\mathbf{C_s} - \lambda_i \mathbf{I})\,\mathbf{v}_i = \mathbf{0}$, $\mathbf{v_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$.
- $\mathbf{y} = \mathbf{V}^T \mathbf{x}$
- $T(\mathbf{x}) = \mathbf{y}^T \left(\mathbf{I} + \sigma^2 \mathbf{\Lambda_s}^{-1}\right)^{-1} \mathbf{y} = \sum_{n=0}^{N-1} y^2[n] \frac{\lambda_{s[n]}}{\lambda_{s[n]} + \sigma^2}$

Weighted sum of signal energies
Give more weightage to signal components with large power

Fig. 5.4 Kay-II



Learning objectives

LO1: Analyze **optimal detectors** for white Gaussian signals

LO2: Analyze **optimal detectors** for Gaussian signals with arbitrary covariance matrix buried in white noise and show how NP leads to **the estimator-correlator**

LO3: Analyze **optimal detectors** for Gaussian signals with arbitrary covariance matrix buried in colored noise and show how NP leads to **the estimator-dewhitener**

LO4: Analyze **optimal general Gaussian detection**

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Random Signals – Generalization 2

$$\mathcal{H}_0 : \mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_w)$$

$$\mathcal{H}_1 : \mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_s + \mathbf{C}_w)$$

Thus, we have $L(\mathbf{x}) = \dfrac{\frac{1}{\left(2\pi\right)^{\frac{N}{2}} det^{\frac{1}{2}}(\mathbf{C}_s+\mathbf{C}_w)} \exp\left[-\frac{1}{2}\mathbf{x}^T\left(\mathbf{C}_s+\mathbf{C}_w\right)^{-1}\mathbf{x}\right]}{\frac{1}{\left(2\pi\right)^{\frac{N}{2}} det^{\frac{1}{2}}(\mathbf{C}_w)} \exp\left[-\frac{1}{2}\mathbf{x}^T\mathbf{C}_w^{-1}\mathbf{x}\right]}.$

Calculating the Log-Likelihood Ratio (LLR), we have

$$T(\mathbf{x}) = \mathbf{x}^T\left[\mathbf{C}_w^{-1} - \left(\mathbf{C}_s+\mathbf{C}_w\right)^{-1}\right]\mathbf{x} > 2\gamma^{'}$$

From the matrix inversion lemma it follows that: $\mathbf{C}_w^{-1} - \left(\mathbf{C}_s+\mathbf{C}_w\right)^{-1} = \mathbf{C}_w^{-1}\mathbf{C}_s\left(\mathbf{C}_s+\mathbf{C}_w\right)^{-1}$

so we get

$$T(\mathbf{x}) = \mathbf{x}^T\left[\mathbf{C}_w^{-1}\mathbf{C}_s\left(\mathbf{C}_s+\mathbf{C}_w\right)^{-1}\right]\mathbf{x} > 2\gamma^{'}$$

Random Signals – Generalization 2

$$\mathcal{H}_0 : \mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_w)$$

$$\mathcal{H}_1 : \mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_s + \mathbf{C}_w)$$

Thus, we have $L(\mathbf{x}) = \dfrac{\frac{1}{(2\pi)^{\frac{N}{2}} det^{\frac{1}{2}}(\mathbf{C}_s+\mathbf{C}_w)} \exp\left[-\frac{1}{2}\mathbf{x}^T(\mathbf{C}_s+\mathbf{C}_w)^{-1}\mathbf{x}\right]}{\frac{1}{(2\pi)^{\frac{N}{2}} det^{\frac{1}{2}}(\mathbf{C}_w)} \exp\left[-\frac{1}{2}\mathbf{x}^T\mathbf{C}_w^{-1}\mathbf{x}\right]}$.

Calculating the Log-Likelihood Ratio (LLR), we have

$$T(\mathbf{x}) = \mathbf{x}^T\left[\mathbf{C}_w^{-1} - (\mathbf{C}_s + \mathbf{C}_w)^{-1}\right]\mathbf{x} > 2\gamma^{'}$$

From the matrix inversion lemma it follows that: $\mathbf{C}_w^{-1} - (\mathbf{C}_s + \mathbf{C}_w)^{-1} = \mathbf{C}_w^{-1}\mathbf{C}_s(\mathbf{C}_s + \mathbf{C}_w)^{-1}$

$$(\mathbf{A}+\mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{DA}^{-1}\mathbf{B}+\mathbf{C}^{-1})^{-1}\mathbf{DA}^{-1}$$

with **B = D = I, A = $\mathbf{C}_w$** and **C = $\mathbf{C}_s$**

Random Signals – Generalization 2

$$T(\mathbf{x}) = \mathbf{x}^T \left[\mathbf{C}_w^{-1} \mathbf{C}_s \left(\mathbf{C}_s + \mathbf{C}_w \right)^{-1} \right] \mathbf{x} > 2\gamma'$$

Notice that this can be written as

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}_w^{-1} \hat{\mathbf{s}}$$

with $\hat{\mathbf{s}} = \mathbf{C}_s \left(\mathbf{C}_s + \mathbf{C}_w \right)^{-1} \mathbf{x}$

Writing $\mathbf{C}_w^{-1} = \mathbf{D}^T \mathbf{D}$ this can also be written in terms of a whitening of $\mathbf{x}$ and $\hat{\mathbf{s}}$:

$$T(\mathbf{x}) = \left(\mathbf{D}\mathbf{x} \right)^T \mathbf{D}\hat{\mathbf{s}}$$

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Deterministic Signals – Summary

Binary detection problem with $\mathbf{w} \sim N(\mathbf{0}, \mathbf{C})$ and deterministic $\mathbf{s}$:

$$\mathcal{H}_0 \quad x[n] = w[n]$$
$$\mathcal{H}_1 \quad x[n] = s[n] + w[n]$$

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{s}$$

Notice that if C is positive definite, $\mathbf{C}^{-1}$ can be written as $\mathbf{C}^{-1} = \mathbf{D}^T\mathbf{D}$, leading to

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{D}^T \mathbf{D} \mathbf{s}$$

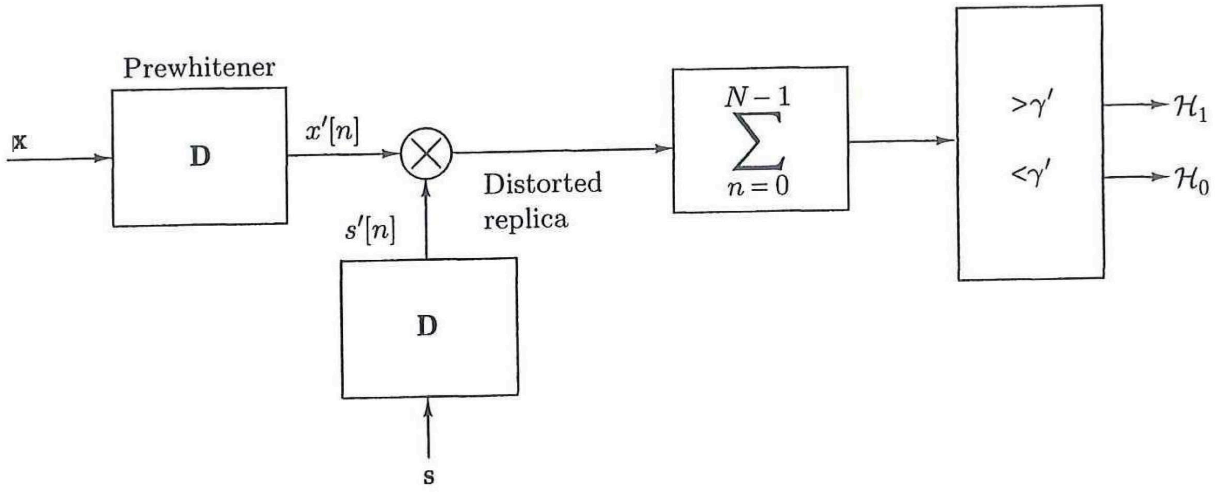
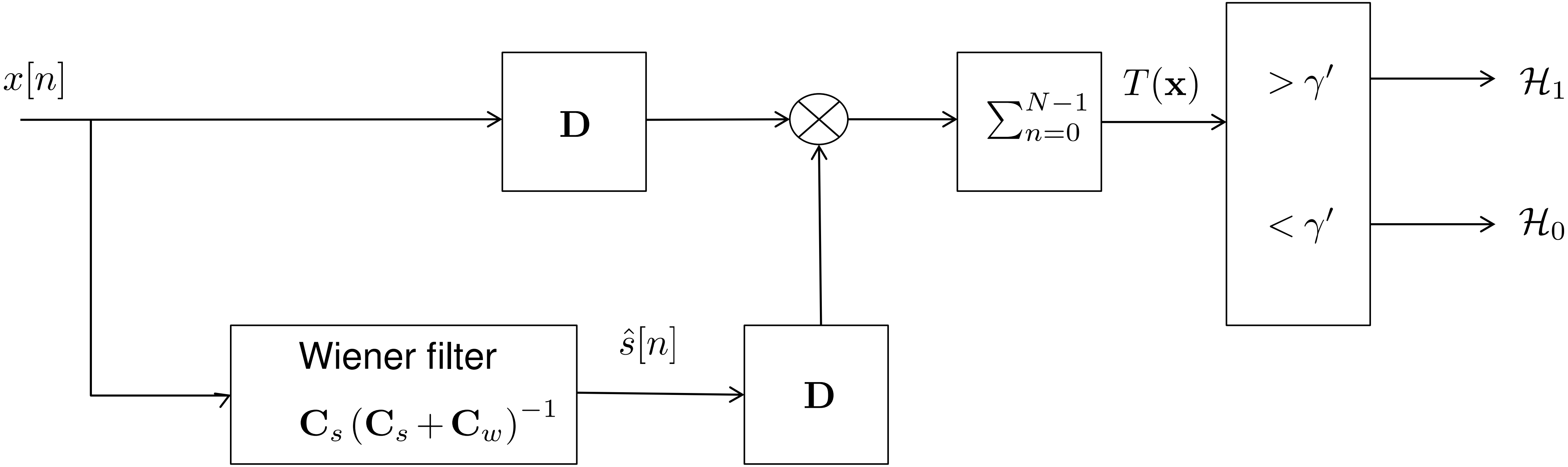


Fig. 4.7 Kay-II.

Summary (2)



$T(\mathbf{x}) = (\mathbf{D}\mathbf{x})^T \mathbf{D}\hat{\mathbf{s}}$ with $\hat{\mathbf{s}} = \mathbf{C}_s(\mathbf{C}_s+\mathbf{C}_w)^{-1}\mathbf{x}$

Learning objectives

LO1: Analyze **optimal detectors** for white Gaussian signals

LO2: Analyze **optimal detectors** for Gaussian signals with arbitrary covariance matrix buried in white noise and show how NP leads to **the estimator-correlator**

LO3: Analyze **optimal detectors** for Gaussian signals with arbitrary covariance matrix buried in colored noise and show how NP leads to **the estimator-dewhitener**

LO4: Analyze **optimal general Gaussian detection**

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General Gaussian Detection: both deterministic & random

$$\mathcal{H}_0 : \mathbf{x} = \mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_w)$$

$$\mathcal{H}_1 : \mathbf{x} = \mathbf{s} + \mathbf{w} \sim \mathcal{N}(\boldsymbol{\mu}_s, \mathbf{C}_s + \mathbf{C}_w)$$

Thus, we have $L(\mathbf{x}) = \dfrac{\frac{1}{\left(2\pi\right)^{\frac{N}{2}} det^{\frac{1}{2}}(\mathbf{C}_s+\mathbf{C}_w)} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_s)^T(\mathbf{C}_s+\mathbf{C}_w)^{-1}(\mathbf{x}-\boldsymbol{\mu}_s)\right]}{\frac{1}{\left(2\pi\right)^{\frac{N}{2}} det^{\frac{1}{2}}(\mathbf{C}_w)} \exp\left[-\frac{1}{2}\mathbf{x}^T\mathbf{C}_w^{-1}\mathbf{x}\right]}$.

Calculating the Log-Likelihood Ratio (LLR), we get

$$T(\mathbf{x}) = \mathbf{x}^T\left[\mathbf{C}_w^{-1} - (\mathbf{C}_s+\mathbf{C}_w)^{-1}\right]\mathbf{x} + 2\mathbf{x}^T(\mathbf{C}_s+\mathbf{C}_w)^{-1}\boldsymbol{\mu}_s - \boldsymbol{\mu}_s^T(\mathbf{C}_s+\mathbf{C}_w)^{-1}\boldsymbol{\mu}_s$$

Using matrix inversion lemma, leaving out the data independent terms and scaling we get:

$$T'(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\left[\mathbf{C}_w^{-1}\mathbf{C}_s(\mathbf{C}_s+\mathbf{C}_w)^{-1}\right]\mathbf{x} + \mathbf{x}^T(\mathbf{C}_s+\mathbf{C}_w)^{-1}\boldsymbol{\mu}_s$$

General Gaussian Detection: both deterministic & random

$$T'(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \left[\mathbf{C}_w^{-1}\mathbf{C}_s\left(\mathbf{C}_s + \mathbf{C}_w\right)^{-1}\right]\mathbf{x} + \mathbf{x}^T\left(\mathbf{C}_s + \mathbf{C}_w\right)^{-1}\boldsymbol{\mu}_s$$

Test statistic $T'(\mathbf{x})$ contains term quadratic in $\mathbf{x}$ (due the randomness modelled using a covariance matrix) and a term linear in $\mathbf{x}$ due to the deterministic part (non-zero mean).

special case 1: $\mathbf{C}_s = 0$, that is, a deterministic signal with $\mathbf{s} = \mu_s$. Then,

$$T'(\mathbf{x}) = \mathbf{x}^T\mathbf{C}_w^{-1}\boldsymbol{\mu}_s$$

special case 2: $\boldsymbol{\mu}_s = \mathbf{0}$, that is, a random signal with $\mathbf{s} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_s)$. Then,

$$T'(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\left[\mathbf{C}_w^{-1}\mathbf{C}_s\left(\mathbf{C}_s + \mathbf{C}_w\right)^{-1}\right]\mathbf{x} = \frac{1}{2}\mathbf{x}^T\mathbf{C}_w^{-1}\hat{\mathbf{s}}$$

with $\hat{\mathbf{s}} = \mathbf{C}_s\left(\mathbf{C}_s + \mathbf{C}_w\right)^{-1}\mathbf{x}$.

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Random Signals – Exercise

Problem 4: Find the NP detector for the problem of a random Gaussian signal $s[n]$ for $n = 0, ..., N-1$ in white Gaussian noise. The covariance matrix $\mathbf{C}_s$ is given by $\mathbf{C}_s = \text{diag}(\sigma_{s_0}^2, \sigma_{s_1}^2, ..., \sigma_{s_{N-1}}^2)$ and $\mathbf{s} \sim N(\mathbf{0}, \mathbf{C}_s)$.

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Random Signals – Exercise

Problem 4: $\mathbf{s} \sim N(\mathbf{0}, \mathbf{C}_s)$ and $\mathbf{w} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$

$$T(\mathbf{x}) = \mathbf{x}^T\mathbf{C}_s\left(\mathbf{C}_s + \sigma^2\mathbf{I}\right)^{-1}\mathbf{x} = \mathbf{x}^T\mathbf{\Lambda}\mathbf{x}$$

with $\mathbf{\Lambda} = \mathrm{diag}\left(\frac{\sigma_{s_0}^2}{\sigma_{s_0}^2+\sigma^2}, \frac{\sigma_{s_1}^2}{\sigma_{s_1}^2+\sigma^2}, \ldots, \frac{\sigma_{s_{N-1}}^2}{\sigma_{s_{N-1}}^2+\sigma^2}\right)$

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} x^2[n]\frac{\sigma_{s_n}^2}{\sigma_{s_n}^2 + \sigma^2}$$

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Random Signals – Exercise

Problem 5: We want to detect a random DC level A embedded in WGN with variance σ^2.

The two hypotheses are given by

$$\begin{aligned} \mathcal{H}_0 \quad & x[n] = w[n] \\ \mathcal{H}_1 \quad & x[n] = A + w[n] \end{aligned}$$

for $n = 0, ..., N-1$ and $A \sim N(0, \sigma_A^2)$.

Problem 5a: Find the MMSE estimator of the signal $\mathbf{s}$.

Problem 5b: Find the NP detector $T(\mathbf{x})$.

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Random Signals – Exercise

Problem 5a: We need to calculate $\hat{\mathbf{s}} = E[\mathbf{s}|\mathbf{x}]$. However, A and $\mathbf{w}$ are Gaussian (and thus also jointly Gaussian) distributed. In addition, the model is linear:

$$\mathbf{x} = \mathbf{1}A + \mathbf{w} = \mathbf{s} + \mathbf{w}.$$

In this case the MMSE estimator is given by $\hat{\mathbf{s}} = E[A|\mathbf{x}]\mathbf{1} = \left(\mathbf{C}_A^{-1} + \mathbf{H}^T\mathbf{C}_\mathbf{w}^{-1}\mathbf{H}^{-1}\right)^{-1}\mathbf{H}^T\mathbf{C}_\mathbf{w}^{-1}\mathbf{x} = \left(\frac{1}{N\sigma_A^2} + \frac{1}{\sigma^2}\right)^{-1}\frac{\bar{x}}{\sigma^2} = \frac{\sigma_A^2\bar{x}}{\sigma_A^2 + \frac{\sigma^2}{N}}\mathbf{1}$

Problem 5b: NP: $T(\mathbf{x}) = \mathbf{x}^T\hat{\mathbf{s}}$

Random Signals – Exercise

Problem 6: We have the following binary detection problem

$$\begin{aligned} \mathcal{H}_0 \quad & x[n] = w[n] \\ \mathcal{H}_1 \quad & x[n] = Ar^n + w[n] \end{aligned}$$

with $0 < r < 1$ and $A \sim N(0, \sigma_A^2)$ and $w[n]$ white Gaussian noise with variance σ^2. A and $w[n]$ are independent.

Problem 6a: Find the test statistic $T(\mathbf{x})$.

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Random Signals – Exercise

Problem 6:

Problem 6a: $\mathbf{s} = A\mathbf{H}$, $\mathbf{H} = [1, r^1, ..., r^{N-1}]^T$ with $\mathbf{s} \sim N(0, \sigma_A^2 \mathbf{H}\mathbf{H}^T)$.

$$\hat{\mathbf{s}} = \mathbf{C}_s \left(\mathbf{C}_s + \sigma^2 \mathbf{I}\right)^{-1} \mathbf{x}$$

Using the matrix inversion lemma it follows that

$$\hat{\mathbf{s}} = \mathbf{C}_s \left(\mathbf{C}_s + \sigma^2 \mathbf{I}\right)^{-1} \mathbf{x} = \frac{\sigma_A^2}{\sigma^2} \mathbf{H}\mathbf{H}^T \left(1 - \frac{\mathbf{H}^T\mathbf{H}\sigma_A^2}{\sigma^2 + \mathbf{H}^T\mathbf{H}\sigma_A^2}\right)\mathbf{x} = \frac{\sigma_A^2 \mathbf{H}\mathbf{H}^T}{\sigma^2 + \mathbf{H}^T\mathbf{H}\sigma_A^2}\mathbf{x}$$

We then get

$$T(\mathbf{x}) = \mathbf{x}^T\hat{\mathbf{s}} = \frac{\sigma_A^2 \mathbf{x}^T \mathbf{H}\mathbf{H}^T \mathbf{x}}{\sigma^2 + \mathbf{H}^T\mathbf{H}\sigma_A^2} = \frac{\left(\sum_{n=0}^{N-1} r^n x[n]\right)^2}{\left(\frac{\sigma^2}{\sigma_A^2} + \sum_{n=0}^{N-1} r^{2n}\right)}$$

or (using 14.7 vol - I):

$$\hat{A} = \left(\mathbf{C}_A^{-1} + \mathbf{H}^T\mathbf{C}_w^{-1}\mathbf{H}\right)^{-1} \mathbf{H}^T\mathbf{C}_w^{-1}\mathbf{x} = \left(\frac{1}{\sigma_A^2} + \frac{\sum_{n=0}^{N-1} r^{2n}}{\sigma^2}\right)^{-1} \frac{\sum_{n=0}^{N-1} r^n x[n]}{\sigma^2}$$

$$= \left(\frac{\sigma^2}{\sigma_A^2} + \sum_{n=0}^{N-1} r^{2n}\right)^{-1} \sum_{n=0}^{N-1} r^n x[n]$$

$$\hat{\mathbf{s}} = \hat{A}\mathbf{H}$$

$$T(\mathbf{x}) = \mathbf{x}^T\hat{\mathbf{s}} = \mathbf{x}^T\mathbf{H}\hat{A} = \frac{\left(\sum_{n=0}^{N-1} r^n x[n]\right)^2}{\left(\frac{\sigma^2}{\sigma_A^2} + \sum_{n=0}^{N-1} r^{2n}\right)}$$

Random Signals – Exercise

Problem 8:

Problem 8a: Find the NP detector for the case where $\mathbf{x}$ has dimension $N \times 1$ and where

$$\mathcal{H}_0 \quad x[n] = w[n]$$

$$\mathcal{H}_1 \quad x[n] = s[n] + w[n]$$

with $\mathbf{w} \sim N(\mathbf{0}, \mathbf{C}_w)$ and $\mathbf{s} \sim N(\mathbf{0}, \mathbf{C}_s) = N(\mathbf{0}, \mathbf{C}_w \eta)$ with $\eta > 0$.

Problem 8b: Determine P_{fa} and P_D for general N as well as for $N = 2$. Hint: if $\mathbf{x} \sim N(\mathbf{0}, \mathbf{C})$, then $\mathbf{x}^T \mathbf{C}^{-1} \mathbf{x} \sim \chi^2_N$.

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Random Signals – Exercise

Problem 8:

Problem 8a: We have $w \sim N(\mathbf{0}, \mathbf{C}_w)$ and $s \sim N(\mathbf{0}, \mathbf{C}_s) = N(\mathbf{0}, \mathbf{C}_w \eta)$. So, $T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}_w^{-1} \mathbf{C}_s (\mathbf{C}_s + \mathbf{C}_w)^{-1} \mathbf{x} = \frac{\mathbf{x}^T \mathbf{C}_w^{-1} \mathbf{x} \eta}{1+\eta} \geq \gamma$ and $T'(\mathbf{x}) = \mathbf{x}^T \mathbf{C}_w^{-1} \mathbf{x} \geq \gamma'$. We know that $\mathbf{x}^T \mathbf{C}^{-1} \mathbf{x} \sim \chi_N^2$ (whitening of $\mathbf{x}$)

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Random Signals – Exercise

Problem 8b:

$$\begin{array}{ll} \mathcal{H}_0 & \mathbf{x} \sim N(0, \mathbf{C}_w) \\ \mathcal{H}_1 & \mathbf{x} \sim N(0, (1+\eta)\mathbf{C}_w) \end{array}$$

so,

$$\begin{array}{ll} \mathcal{H}_0 & T(\mathbf{x}) \sim \chi^2_N \\ \mathcal{H}_1 & \frac{T(\mathbf{x})}{1+\eta} \sim \chi^2_N \end{array}$$

$P_{fa} = P(T(\mathbf{x}) \geq \gamma'; H_0) = Q_{\chi^2_N}(\gamma') \Rightarrow \gamma' = Q^{-1}_{\chi^2_N}(P_{fa})$

$P_D = P(T(\mathbf{x}) \geq \gamma'; H_1) = P(\frac{T(\mathbf{x})}{1+\eta} \geq \frac{\gamma'}{1+\eta}; H_1) = Q_{\chi^2_N}(\frac{\gamma'}{1+\eta})$.

Notice that for $N = 2$, χ^2_2−distributed RVs becomes exponentially distributed.

$Q_{\chi^2_N}(\gamma') = \int_{\gamma'}^{\infty} \frac{1}{2} e^{-\frac{x}{2}} dx = e^{-\frac{\gamma'}{2}} = P_{fa} \Rightarrow \gamma' = -2\log P_{fa}$. $P_D = Q_{\chi^2_N}(\frac{-2\log P_{fa}}{1+\eta}) =$

$\int_{\frac{-2\log P_{fa}}{1+\eta}}^{\infty} \frac{1}{2} e^{-\frac{x}{2}} dx = e^{\frac{\log P_{fa}}{1+\eta}} = P_{fa}^{\frac{1}{1+\eta}}$

Random Signals – Exercise

Problem 9: Let

$$\mathcal{H}_0 \quad x[n] = w[n]$$
$$\mathcal{H}_1 \quad x[n] = s[n] + w[n]$$

with $w[n] \sim N(0, \sigma^2)$ and $s[n] \sim N(A, \sigma_s^2)$. Give the NP detector for the case that the IID samples $n = 0, ..., N-1$ are observed.

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Random Signals – Exercise

Problem 9: We can use the expression for general Gaussian detection. That is

$$\begin{aligned}
T'(\mathbf{x}) = & \frac{1}{2}\mathbf{x}^T\left[\mathbf{C}_w^{-1}\mathbf{C}_s\left(\mathbf{C}_s+\mathbf{C}_w\right)^{-1}\right]\mathbf{x}+\mathbf{x}^T\left(\mathbf{C}_s+\mathbf{C}_w\right)^{-1}\boldsymbol{\mu}_s \\
= & \frac{1}{2}\mathbf{x}^T\frac{\sigma_s^2}{\sigma^2}\left(\sigma_s^2+\sigma^2\right)^{-1}\mathbf{x}+\mathbf{x}^T\left(\sigma_s^2+\sigma^2\right)^{-1}A\mathbf{1} \\
= & \frac{1}{2}\frac{\sigma_s^2}{\sigma^2}\left(\sigma_s^2+\sigma^2\right)^{-1}\mathbf{x}^T\mathbf{x}+\frac{A}{\sigma_s^2+\sigma^2}\mathbf{x}^T\mathbf{1} \\
= & \frac{N}{2}\frac{\sigma_s^2}{\sigma^2}\frac{1}{\sigma_s^2+\sigma^2}\underbrace{\frac{1}{N}\sum_{n=0}^{N-1}x^2[n]}_{\text{estimate of variance}}+\frac{NA}{\sigma_s^2+\sigma^2}\underbrace{\frac{1}{N}\sum_{n}^{N-1}x[n]}_{\text{estimate of mean}}
\end{aligned}$$

From this we can clearly see the contribution in the detector based on the deterministic component (mean) of the data and the random component (variance) of the data.

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LO2: Analyze **optimal detectors** for Gaussian signals with arbitrary covariance matrix buried in white noise and show how NP leads to **the estimator-correlator**

LO3: Analyze **optimal detectors** for Gaussian signals with arbitrary covariance matrix buried in colored noise and show how NP leads to **the estimator-dewhitener**

LO4: Analyze **optimal general Gaussian detection**

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Reading Tasks

- Chapter 5
- Exercise 4,5,6,8,9 from course website

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Next class

- Composite Hypothesis Testing
- Locally Most Powerful Detectors
- Bayesian approach
- Generalized likelihood ratio