ET4386 Estimation and Detection

Detection

Lecture 2: Deterministic Signals (Ch. 4)

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Previous Lecture

- **Important probability density functions (pdfs)**
  - Gaussian pdf
  - Central Chi-squared pdf

- **Optimal binary detection:**
  - Neyman-Pearson Theorem
  - Minimum Probability of Error

Learning Objectives

- LO1: Optimal binary detection
  - Bayes risk
- LO2: Detecting a known signal in noise using the NP criterion.
  - White noise
- LO3: Detecting a known signal in noise using the NP criterion.
  - Colored noise

Previous Lecture: Introduction to Detection Theory

Binary detection: Determine whether a certain signal that is embedded in noise is present or not.

$$\begin{array}{ll} \mathcal{H}_0 & x[n] = w[n] \\ \mathcal{H}_1 & x[n] = s[n] + w[n] \end{array}$$

Note that if the number of hypotheses is more than two, then the problem becomes a multiple hypothesis testing problem. One example is detection of different digits in speech processing.

- $x[n]$ is a single sample measurement
- $s[n]$ is the signal of interest and $w[n]$ the noise, e,g, $w[n] \sim N(0, \sigma^2)$
- $\mathcal{H}_0$ (signal absent) is the Null hypothesis
- $\mathcal{H}_1$ (signal present) is the Alternative hypothesis

Previous Lecture: Neyman-Pearson Theorem

Problem statement

Assume a data set $\mathbf{x} = [x[0], x[1], ..., x[N-1]]^T$ is available. The detection problem is defined as follows

$$\begin{aligned} \mathcal{H}_0 : \quad & T(\mathbf{x}) < \lambda \\ \mathcal{H}_1 : \quad & T(\mathbf{x}) > \lambda \end{aligned}$$

where T is the decision function and λ is the detection threshold. Our goal is to design T so as to maximize P_D subject to $P_{FA} < \alpha$.

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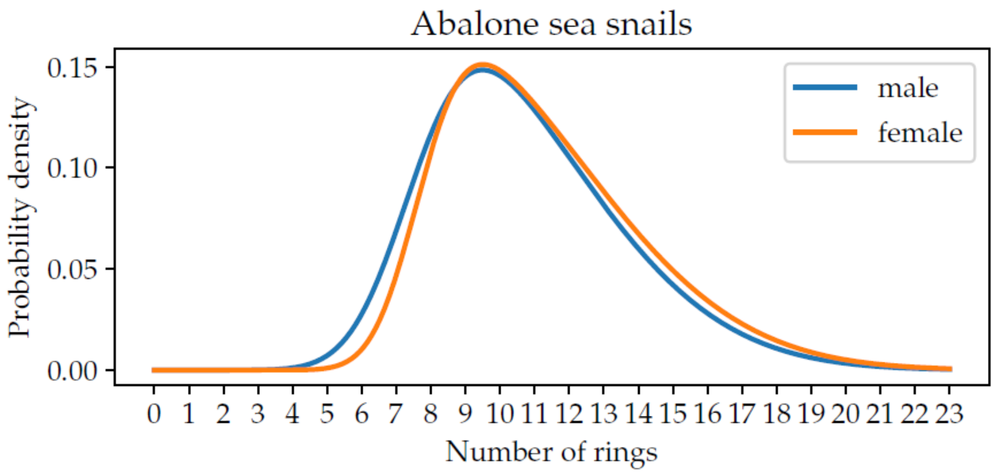
Neyman-Pearson Theorem



Is it a male or female abalone?

To maximize P_D for a given $P_{FA} = \alpha$ decide $\mathcal{H}_1$ if

$$L(\mathbf{x}) = \frac{p(\mathbf{x}; \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} > \lambda$$

where the threshold λ is found from

$$P_{FA} = \int_{\{\mathbf{x}:L(\mathbf{x})>\lambda\}} p(\mathbf{x}; \mathcal{H}_0) d\mathbf{x} = \alpha$$

The function $L(\mathbf{x})$ is called the likelihood ratio and the entire test is called the likelihood ratio test (LRT).

Full derivation in Appendix 3A.

Neyman-Pearson Theorem – Example 1 (1)

Let $p(x;\mathcal{H}_1)$ and $p(x;\mathcal{H}_0)$ be given by the Rayleigh pdfs

$$p(x;\mathcal{H}_1) = \begin{cases} \frac{x}{\sigma_1^2}\exp\left(-\frac{1}{2}\frac{x^2}{\sigma_1^2}\right) & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$

and

$$p(x;\mathcal{H}_0) = \begin{cases} \frac{x}{\sigma_0^2}\exp\left(-\frac{1}{2}\frac{x^2}{\sigma_0^2}\right) & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$

with $\sigma_1^2 > \sigma_0^2$.

Exercise:

- Determine an expression for P_D.

A

$$P_D = \exp\left[\frac{\sigma_0^2 \ln(P_{fa})}{\sigma_1^2}\right]$$

B

$$P_D = \exp\left[\frac{\sigma_1^2 \ln(P_{fa})}{\sigma_0^2}\right]$$

C

$$P_D = P_{fa}\frac{\sigma_1^2}{\sigma_0^2}$$

Neyman-Pearson Theorem – Example 1 (2)

Solution:

$$L(x) = \frac{p(x;\mathcal{H}_1)}{p(x;\mathcal{H}_0)} = \frac{\sigma_0^2}{\sigma_1^2}\exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_1^2} - \frac{x^2}{\sigma_0^2}\right)\right] > \lambda \;\Rightarrow\; T(x) = x^2 > \lambda'$$

$$P_{FA} = P(x^2 > \lambda';\mathcal{H}_0) = P(x > \sqrt{\lambda'};\mathcal{H}_0) = \int_{\sqrt{\lambda'}}^{\infty} \frac{x}{\sigma_0^2}\exp\left[-\frac{1}{2}\frac{x^2}{\sigma_0^2}\right]dx = \exp\left[-\frac{\lambda'}{2\sigma_0^2}\right]$$

with $\lambda' = -2\sigma_0^2 \ln(P_{FA})$.
The detection probability:

$$P_D = \int_{\sqrt{\lambda'}}^{\infty} \frac{x}{\sigma_1^2}\exp\left[-\frac{1}{2}\frac{x^2}{\sigma_1^2}\right]dx = \exp\left[-\frac{\lambda'}{2\sigma_1^2}\right].$$

Neyman-Pearson Theorem – Example 1 (1)

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Exercise:

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C

$$P_D = P_{fa}\frac{\sigma_1^2}{\sigma_0^2}$$

Neyman-Pearson Theorem – Example 2 (1)

Given is a geometrically distribute random variable k, which is the number of failures before the first succes in a series of Bernoulli trials. The pmf is given by

$$f(k;p) = (1-p)^k p.$$

We want to make a binary decision on the distribution of k, which is given by the following two hypotheses:

$$\begin{array}{ll} \mathcal{H}_0 & k \sim f(k;p_0) \\ \mathcal{H}_1 & k \sim f(k;p_1) \end{array}$$

- Find the NP detector $T(k)$.

A $T(k) = 2k$

B $T(k) = log(k)$

C $T(k) = k$

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Neyman-Pearson Theorem – Example 2 (2)

LRT: $\frac{(1-p_1)^k p_1}{(1-p_0)^k p_0} \geq \lambda$

$\frac{(1-p_1)^k}{(1-p_0)^k} \geq \lambda \frac{p_0}{p_1}$

$T(k) = k \geq \frac{\log \lambda \frac{p_0}{p_1}}{\log\left(\frac{1-p_1}{1-p_0}\right)} = \lambda'$

- Determine the detection performance as a function of P_{fa}.

A $P_D = (1 - p_0)^{\frac{\log(P_{fa})}{\log(1-p_1)}}$

B $P_D = \frac{\log(P_{fa})}{\log(1-p_0)}$

C $P_D = (1 - p_1)^{\frac{\log(P_{fa})}{\log(1-p_0)}}$

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- Find the NP detector $T(k)$.

A $T(k) = 2k$

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Neyman-Pearson Theorem – Example 2 (3)

Solution:

$$p_{fa} = P(k \geq \lambda' | H_0) = \sum_{k=\lambda'}^{\infty} (1-p_0)^k p_0 = 1-p_0 \sum_{k=0}^{\lambda'-1} (1-p_0)^k = 1-p_0 \frac{1-(1-p_0)^{\lambda'}}{1-(1-p_0)} = (1-p_0)^{\lambda'}$$

(use $\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$ for $|r| < 1$)

$$\lambda' = \frac{\log(P_{fa})}{\log(1-p_0)}$$

$$P_D = P(k \geq \lambda' | H_1) = \sum_{k=\lambda'}^{\infty} (1-p_1)^k p_1 = (1-p_1)^{\lambda'} = (1-p_1)^{\frac{\log(P_{fa})}{\log(1-p_0)}}$$

Neyman-Pearson Theorem – Example 2 (2)

LRT: $\frac{(1-p_1)^k p_1}{(1-p_0)^k p_0} \geq \lambda$

$\frac{(1-p_1)^k}{(1-p_0)^k} \geq \lambda \frac{p_0}{p_1}$

$T(k) = k \geq \frac{\log \lambda \frac{p_0}{p_1}}{\log\left(\frac{1-p_1}{1-p_0}\right)} = \lambda'$

- Determine the detection performance as a function of P_{fa}.

A $P_D = (1-p_0)^{\frac{\log(P_{fa})}{\log(1-p_1)}}$

B $P_D = \frac{\log(P_{fa})}{\log(1-p_0)}$

C $P_D = (1-p_1)^{\frac{\log(P_{fa})}{\log(1-p_0)}}$

Minimum Probability of Error

Assume the prior probabilities of $\mathcal{H}_0$ and $\mathcal{H}_1$ are known and represented by $P(\mathcal{H}_0)$ and $P(\mathcal{H}_1)$, respectively. The probability of error, P_e, is then defined as

$$P_e = P(\mathcal{H}_1)P(\mathcal{H}_0|\mathcal{H}_1) + P(\mathcal{H}_0)P(\mathcal{H}_1|\mathcal{H}_0) = P(\mathcal{H}_1)P_M + P(\mathcal{H}_0)P_{FA}$$

Our goal is to design a detector that minimizes P_e. It is shown that the following detector is optimal in this case

$$\boxed{\frac{p(\mathbf{x}|\mathcal{H}_1)}{p(\mathbf{x}|\mathcal{H}_0)} > \frac{P(\mathcal{H}_0)}{P(\mathcal{H}_1)} = \lambda}$$

In case $P(\mathcal{H}_0) = P(\mathcal{H}_1)$, the detector is called the maximum likelihood detector.

Learning Objectives

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  - Bayes risk
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Bayes Risk

A generalisation of the minimum P_e criterion is one where costs are assigned to each type of error:

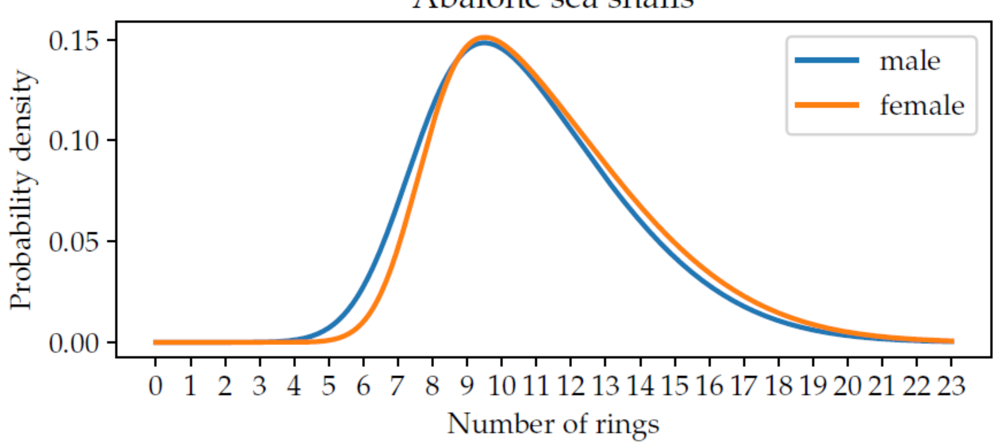
Let C_{ij} be the cost if we decide $\mathcal{H}_i$ while $\mathcal{H}_j$ is true. Minimizing the expected costs we get

$$\mathcal{R} = E[C] = \sum_{i=0}^{1} \sum_{j=0}^{1} C_{ij} P(\mathcal{H}_i | \mathcal{H}_j) P(\mathcal{H}_j)$$

If $C_{10} > C_{00}$ and $C_{01} > C_{11}$ the detector that minimises the Bayes risk is to decide $\mathcal{H}_1$ when

$$\boxed{\frac{p(\mathbf{x}|\mathcal{H}_1)}{p(\mathbf{x}|\mathcal{H}_0)} > \frac{C_{10} - C_{00}}{C_{01} - C_{11}} \frac{P(\mathcal{H}_0)}{P(\mathcal{H}_1)} = \gamma.}$$

Bayes Risk (2)



Is it a male or female abalone?

$$\mathcal{H}_0 \quad : \quad T(\mathbf{x}) < \gamma$$

$$\mathcal{H}_1 \quad : \quad T(\mathbf{x}) > \gamma$$

Using detection theory, rules can be derived on how to chose γ and how to find $T(\mathbf{x})$.

- Neyman-Pearson Theorem: $L(\mathbf{x}) = \frac{p(\mathbf{x};\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)} > \lambda$, where λ found from $P_{FA} = \int_{\{\mathbf{x}:L(\mathbf{x})>\lambda\}} p(\mathbf{x};\mathcal{H}_0)d\mathbf{x} = \alpha$
- Minimum probability of error: $\frac{p(\mathbf{x}|\mathcal{H}_1)}{p(\mathbf{x}|\mathcal{H}_0)} > \frac{P(\mathcal{H}_0)}{P(\mathcal{H}_1)} = \gamma$
- Bayesian detector: $\frac{p(\mathbf{x}|\mathcal{H}_1)}{p(\mathbf{x}|\mathcal{H}_0)} > \frac{C_{10}-C_{00}}{C_{01}-C_{11}} \frac{P(\mathcal{H}_0)}{P(\mathcal{H}_1)} = \gamma.$

Same test statistics
Different threshold

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Learning Objectives

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Deterministic Signals

Binary detection problem:

$$\mathcal{H}_0 \quad x[n] = w[n]$$

$$\mathcal{H}_1 \quad x[n] = s[n] + w[n]$$

Assumptions

- $s[n]$ is deterministic and known.
- $w[n]$ is white Gaussian noise with variance σ^2.

Deterministic Signals

The NP detector decides $\mathcal{H}_1$ if the likelihood ratio exceeds a threshold,

$$L(\mathbf{x}) = \frac{p(\mathbf{x};\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)} > \lambda$$

where $\mathbf{x} = [x[0], x[1], \ldots, x[N-1]]^T$. Since

$$p(\mathbf{x};\mathcal{H}_1) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-s[n])^2\right]$$

$$p(\mathbf{x};\mathcal{H}_0) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right]$$

we have

$$L(\mathbf{x}) = \exp\left[-\frac{1}{2\sigma^2}\left(\sum_{n=0}^{N-1}(x[n]-s[n])^2 - \sum_{n=0}^{N-1}x^2[n]\right)\right] > \lambda.$$

Notice that presence of $s[n]$ implies change in mean of observe signal. Optimal detector will test whether there is a change in the mean of the test statistic.

Deterministic Signals

Taking the logarithm of both sides does not change the inequality, so we have

$$l(\mathbf{x}) = \ln L(\mathbf{x}) = -\frac{1}{2\sigma^2}\left(\sum_{n=0}^{N-1}(x[n]-s[n])^2 - \sum_{n=0}^{N-1}x^2[n]\right) > \ln\lambda$$

We decide $\mathcal{H}_1$ if

$$\frac{1}{\sigma^2}\sum_{n=0}^{N-1}x[n]s[n] - \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}s^2[n] > \ln\lambda$$

Since $s[n]$ is known, we may incorporate the energy term into the threshold to yield

$$T(\mathbf{x}) = \sum_{n=0}^{N-1}x[n]s[n] > \lambda^{'}$$

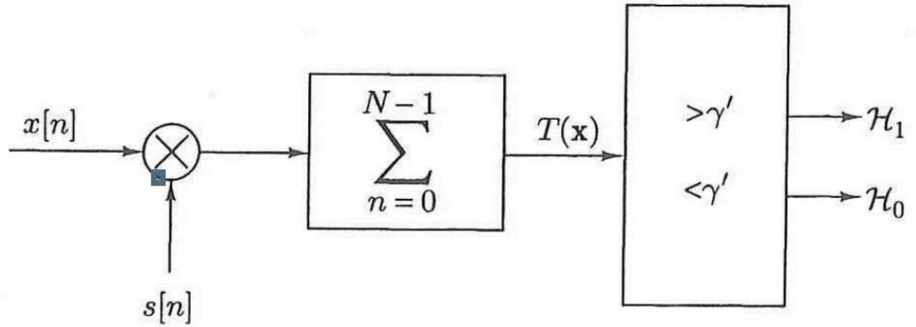
How to interpret this?

where $\lambda^{'} = \sigma^2\ln\lambda + \frac{1}{2}\sum_{n=0}^{N-1}s^2[n]$.

Correlation between observed signal $x[n]$ and $s[n]$.

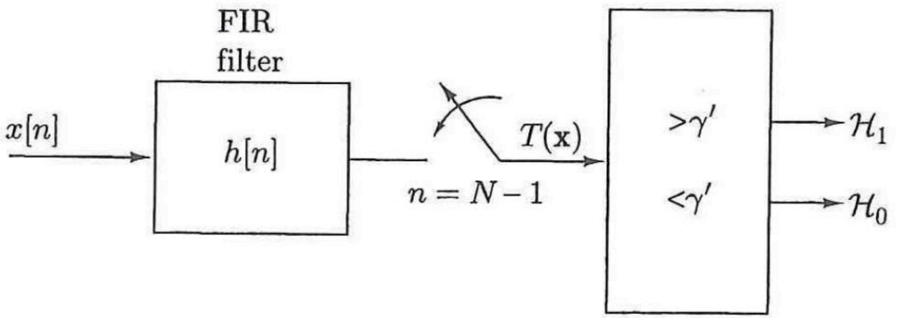
Deterministic Signals - Interpretation

Interpretation 1: The resulting $T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s[n]$ is a correlator. The received data is correlated with a replica of the signal.



(a)

Interpretation 2: The resulting $T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s[n]$ is a matched filter.



$$h[n] = \begin{cases} s[N-1-n] & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

Fig. 4.1 from Kay-II.

Deterministic Signals – Matched Filter Interpretation

- Let $x[n]$ be the input to an FIR filter.
- Impulse response $h[n]$.
- Output $y[n] = \sum_{k=0}^{n} h[n-k]x[k]$
- Select as impulse response $h[n] = s[N-1-n]$ for $n = 0, 1, ..., N-1$

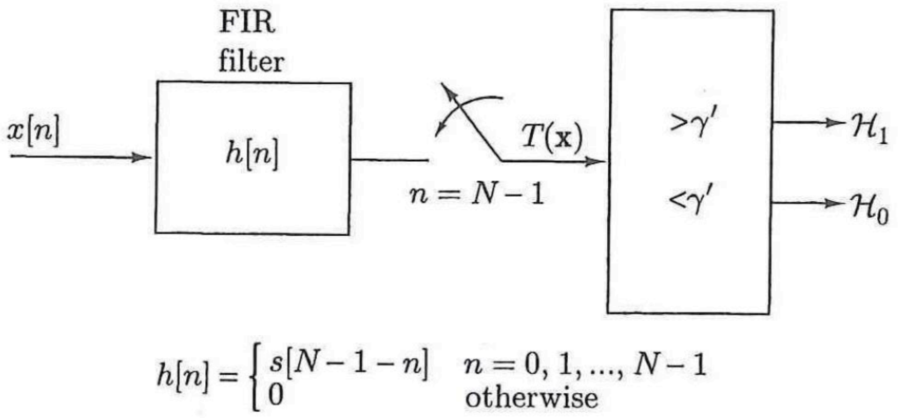


Fig. 4.1 from Kay-II.

Then

$$y[n] = \sum_{k=0}^{n} s[N-1-(n-k)]x[k].$$

Output at $N-1$ is $y[N-1] = \sum_{k=0}^{N-1} s[k]x[k] = T(\mathbf{x})$

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Deterministic Signals – Matched Filter Interpretation

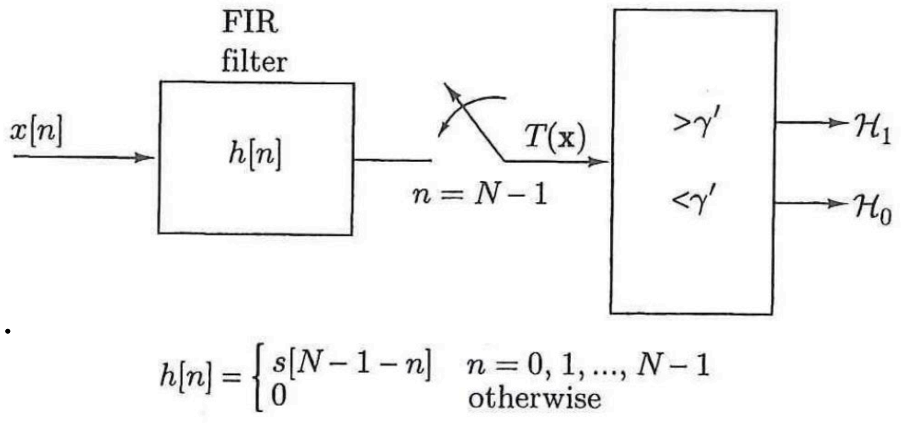

Fig. 4.1 from Kay-II.

Matched filter:

$$y[n] = \sum_{k=0}^{n} s[N-1-(n-k)]x[k].$$

Output at $N-1$ is $y[N-1] = \sum_{k=0}^{N-1} s[k]x[k] = T(\mathbf{x})$

- This implementation of the NP detector is known as the matched filter.
- Matched filter impulse response is obtained by flipping $s[n]$ about $n = 0$ and shifting it to the right with N-1 samples.
- The best detection performance will be at $n = N-1$. Then $y[n] = T(\mathbf{x})$.

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Q1: The match filter is called "matched" since it is tailored to the expected shape of the signal.

True — 0%

False — 0%

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Q1: The match filter is called "matched" since it is tailored to the expected shape of the signal.

True — 83.33% ✓

False — 16.67%

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Matched Filter - SNR Maximizer

Matched filter:

$$y[n] = \sum_{k=0}^{n} s[N-1-(n-k)]x[k].$$

Output at $N-1$ is $y[N-1] = \sum_{k=0}^{N-1} s[k]x[k] = T(\mathbf{x})$

- For deterministic signals, Matched filter is optimal implementation of the NP detector!
- To optimize detection probability P_D, we have seen we should increase the deflection coefficient. Generally this means increasing the SNR.
- The matched filter maximises the SNR at the output of an FIR filter.

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Matched Filter - SNR Maximizer

output SNR:

$$\eta = \frac{E^2(y[N-1];\mathcal{H}_1)}{\mathrm{var}(y[N-1]);\mathcal{H}_1}$$

Let $\mathbf{s} = [s[0], ..., s[N-1]]^T$, $\mathbf{w} = [w[0], ..., w[N-1]]^T$ and $\mathbf{h} = [h[N-1], ...\ h[0]]^T$.

$$\begin{aligned} \eta &= \frac{(\mathbf{h}^T\mathbf{s})^2}{E(\mathbf{h}^T\mathbf{w})^2} = \frac{(\mathbf{h}^T\mathbf{s})^2}{\mathbf{h}^T E(\mathbf{w}\mathbf{w}^\mathbf{T})\mathbf{h}} \\ &= \frac{(\mathbf{h}^T\mathbf{s})^2}{\mathbf{h}^T\sigma^2\mathbf{I}\mathbf{h}} = \frac{1}{\sigma^2}\frac{(\mathbf{h}^T\mathbf{s})^2}{\mathbf{h}^T\mathbf{h}} \end{aligned}$$

Cauchy-Schwarz inequality:$(\mathbf{h}^T\mathbf{s})^2 \leq (\mathbf{h}^T\mathbf{h})(\mathbf{s}^T\mathbf{s})$, with equality if and only if $\mathbf{h} = c\mathbf{s}$.

$$\Rightarrow \eta \leq \frac{1}{\sigma^2}\mathbf{s}^T\mathbf{s} = \frac{\mathcal{E}}{\sigma^2}$$

Taking $c = 1$, maximum SNR is obtained if

$$h[N-1-n] = s[n], \quad n = 0, 1, ..., N-1$$

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Performance of the Matched Filter

What is P_D for a given P_{FA}?

- $\mathcal{H}_1$ is decided when
- $T(\mathbf{x}) = \sum_{k=0}^{N-1} s[k]x[k] = \gamma'$. As $x[n]$ is Gaussian $\Rightarrow T(\mathbf{x})$ is also Gaussian. Therefore,

$$\mathrm{E}(T;\mathcal{H}_0) = \mathrm{E}\left(\sum_{n=0}^{N-1} w[n]s[n]\right) = 0$$

$$\mathrm{E}(T;\mathcal{H}_1) = \mathrm{E}\left(\sum_{n=0}^{N-1} (s[n]+w[n])s[n]\right) = \mathcal{E}$$

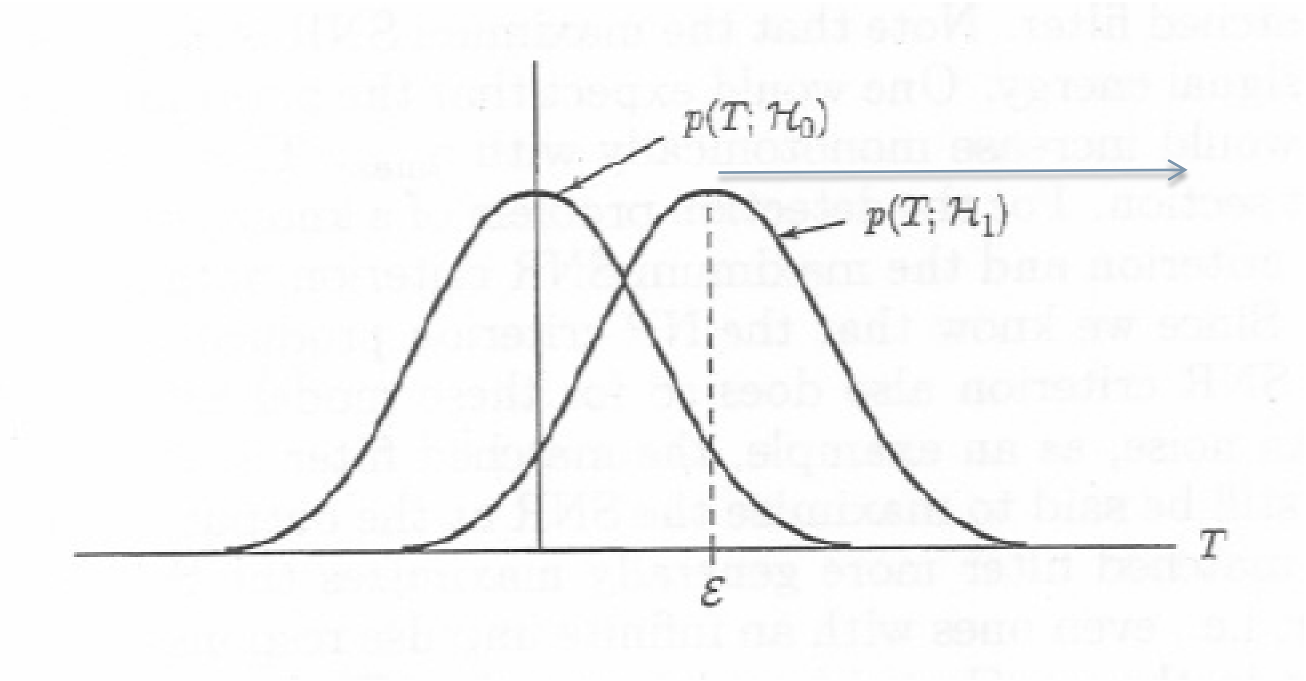
$$\mathrm{var}(T;\mathcal{H}_0) = \mathrm{var}\left(\sum_{n=0}^{N-1} w[n]s[n]\right) = \sigma^2\mathcal{E}$$

$$\mathrm{var}(T;\mathcal{H}_1) = \mathrm{var}\left(\sum_{n=0}^{N-1} (s[n]+w[n])s[n]\right) = \sigma^2\mathcal{E} \quad \text{where } \mathcal{E} \text{ is the signal energy.}$$

Performance of the Matched Filter

$$T \sim \begin{cases} \mathcal{H}_0 : & \mathcal{N}(0, \sigma^2 \mathcal{E}) \\ \mathcal{H}_1 : & \mathcal{N}(\mathcal{E}, \sigma^2 \mathcal{E}) \end{cases}$$

Figure: pdfs of matched filter statistic.(Fig. 4.4 Kay-II)



The larger $\mathcal{E}$, the the further the pdfs move away from each other,

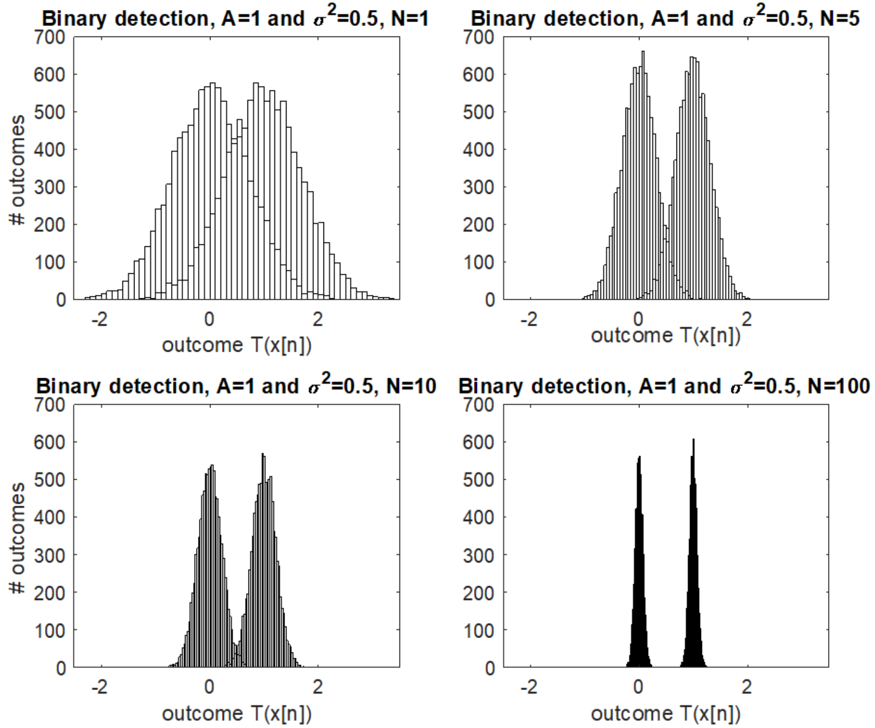
the better the performance will be.

Using the Q-function for the Gaussian pdf

- For $T(x[n]) = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$, evaluate $P(T(x[n]) \geq \gamma)$ when $x \sim \mathcal{N}(\mu, \sigma^2)$.
- $P(T(x[n]) \geq \gamma) = P(\frac{1}{N}\sum_{n=0}^{N-1} x[n] \geq \gamma) = Q\left(\frac{\gamma-\mu}{\sqrt{\sigma^2/N}}\right)$, where

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}t^2\right) dt$$

is the right-tail probability of the Gaussian PDF.



Performance of the Matched Filter

This way, the probability of false alarm and detection are as follows

$$P_{FA} = Pr\left(T > \lambda^{'}; \mathcal{H}_0\right) = Q\left(\frac{\lambda^{'}}{\sqrt{(\sigma^2)\mathcal{E}}}\right)$$
$$P_D = Pr\left(T > \lambda^{'}; \mathcal{H}_1\right) = Q\left(\frac{\lambda^{'} - \mathcal{E}}{\sqrt{(\sigma^2)\mathcal{E}}}\right)$$

Deriving $\lambda^{'} = \sqrt{\sigma^2 \mathcal{E}} Q^{-1}(P_{FA})$ and substituting in P_D, we have $P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{\mathcal{E}}{\sigma^2}}\right)$

where $\mathcal{E}/\sigma^2$ is the energy to noise ratio.

Performance of the Matched Filter

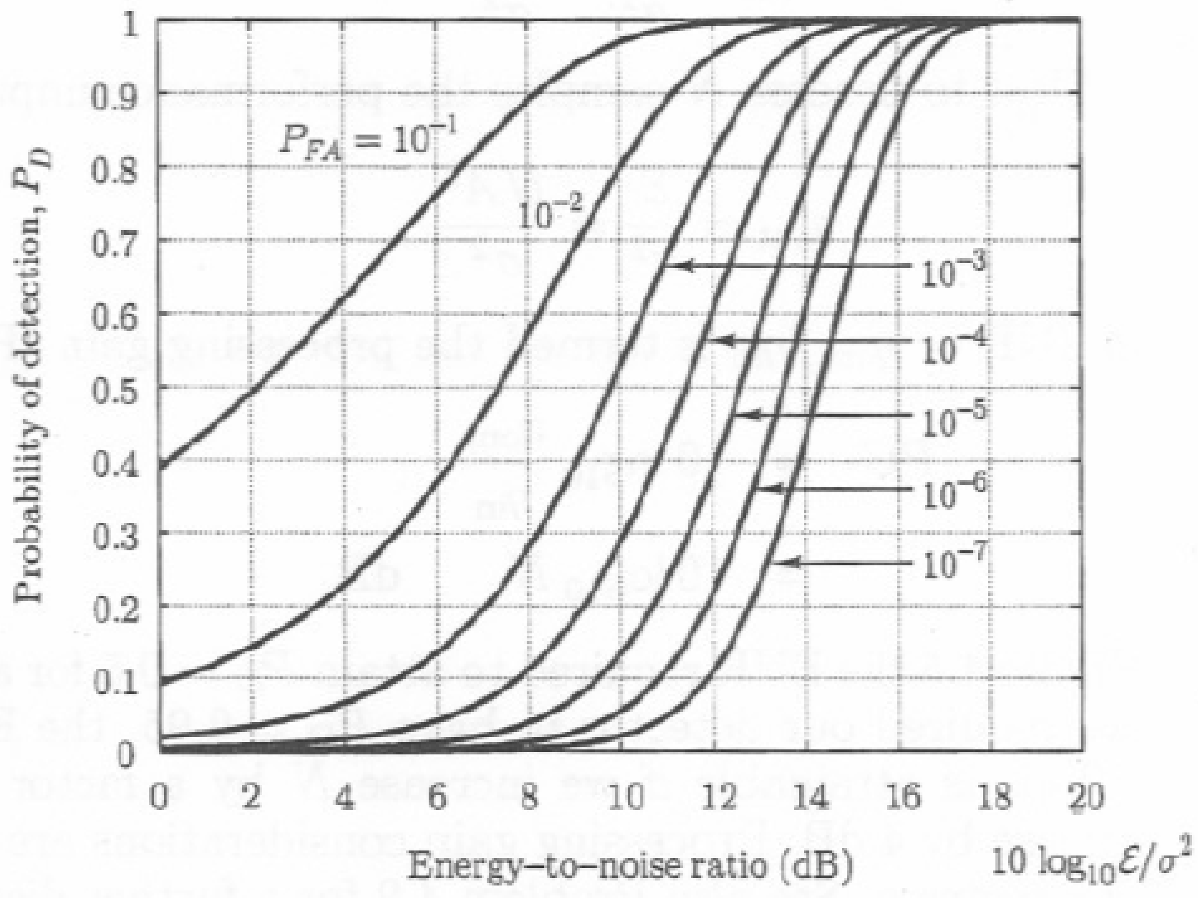


Fig. 4.5 from Kay-II.

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{\mathcal{E}}{\sigma^2}}\right)$$

where $\mathcal{E}/\sigma^2$ is the energy to noise ratio. To increase P_D: Increase P_{FA}, and/or increase SNR $\frac{\mathcal{E}}{\sigma^2}$.

Remember the example from previous lecture: DC in WGN with P_D

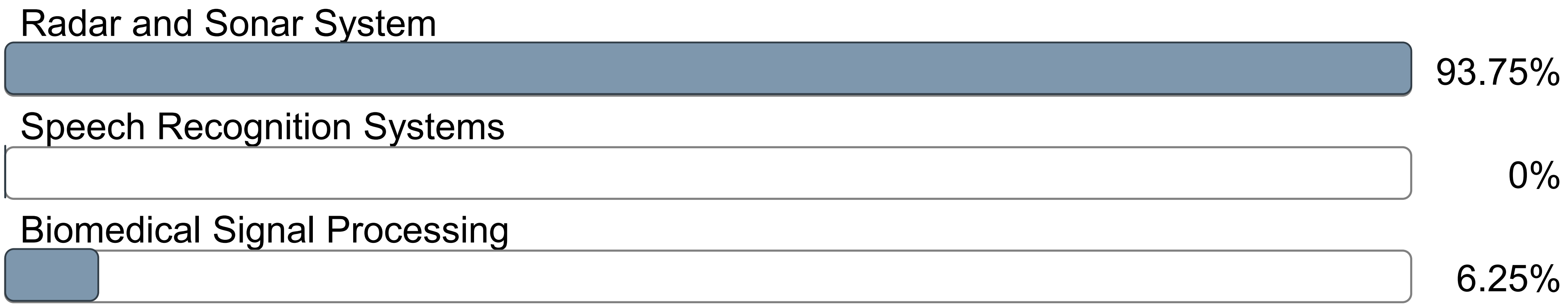
$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{NA^2}{\sigma^2}}\right)$$

In that example $s[n] = A$ and $\mathcal{E} = NA^2$. The shape of the signal does not influence the detection performance for white noise. Only the total energy $\mathcal{E}$. In our example thus N and the amplitude A.

Q2: In which signal processing scenario you think is the matched filter mostly applied?

Radar and Sonar System	0%
Speech Recognition Systems	0%
Biomedical Signal Processing	0%

Q2: In which signal processing scenario you think is the matched filter mostly applied?



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Correlated Noise

What about coloured noise?

$$p(\mathbf{x};\mathcal{H}_1) = \frac{1}{(2\pi)^{\frac{N}{2}}\det^{\frac{1}{2}}(\mathbf{C})}\exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{s})^T\mathbf{C}^{-1}(\mathbf{x}-\mathbf{s})\right]$$

$$p(\mathbf{x};\mathcal{H}_0) = \frac{1}{(2\pi)^{\frac{N}{2}}\det^{\frac{1}{2}}(\mathbf{C})}\exp\left[-\frac{1}{2}\mathbf{x}^T\mathbf{C}^{-1}\mathbf{x}\right].$$

The NP detector decides $\mathcal{H}_1$ if the likelihood ratio exceeds a threshold: $L(\mathbf{x}) = \frac{p(\mathbf{x};\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)} > \lambda$.

$$\ln L(\mathbf{x}) = \mathbf{x}^T\mathbf{C}^{-1}\mathbf{s} - \frac{1}{2}\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s} > \ln\lambda$$

$$\mathbf{x}^T\mathbf{C}^{-1}\mathbf{s} > \ln\lambda + \frac{1}{2}\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s} = \lambda'$$

For WGN ($\mathbf{C} = \sigma^2\mathbf{I}$) we obtain the special case we already know:

$$\frac{\mathbf{x}^T\mathbf{s}}{\sigma^2} > \lambda' \Rightarrow \mathbf{x}^T\mathbf{s} = \sum_{n=0}^{N-1} x[n]s[n] > \sigma^2\lambda'$$

Performance of the Matched Filter – Colored noise

For white Gaussian noise, P_D does not depend on signal shape, only on the energy $\mathbf{s}^T\mathbf{s}$:

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{\mathbf{s}^T\mathbf{s}}{\sigma^2}}\right)$$

What is P_D for a given P_{FA} in colored noise?

Performance of the Matched Filter – Colored noise

$\mathcal{H}_1$ is decided when $T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{s} > \gamma'$.

$\mathbf{x}$ is Gaussian $\Rightarrow T(\mathbf{x})$ is also Gaussian. Therefore,

$$\mathrm{E}(T;\mathcal{H}_0) = \mathrm{E}\left(\mathbf{w^T C^{-1} s}\right) = 0$$

$$\mathrm{E}(T;\mathcal{H}_1) = \mathrm{E}\left((\mathbf{s}+\mathbf{w})^T \mathbf{C}^{-1}\mathbf{s}\right) = \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}$$

$$\mathrm{var}(T;\mathcal{H}_0) = \mathrm{E}\left((\mathbf{w^T C^{-1} s})^\mathbf{2}\right) - \mathrm{E}\left((\mathbf{w^T C^{-1} s})\right)^2 = \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}$$

$$\mathrm{var}(T;\mathcal{H}_1) = \mathrm{E}\left((\mathbf{x^T C^{-1} s})^\mathbf{2}\right) - \mathrm{E}\left((\mathbf{x^T C^{-1} s})\right)^2 = \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}.$$

$$P_{fa} = Q\left(\frac{\gamma'}{\sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}}\right) \rightarrow \gamma' = Q^{-1}\left(P_{fa}\right)\sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}$$

$$P_D = Q\left(\frac{\gamma' - \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}{\sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}}\right) = Q\left(Q^{-1}\left(P_{fa}\right) - \sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}\right),$$

with $\mathrm{s}^T\mathrm{C}^{-1}\mathrm{s}$ the "SNR" of the "whitened"signal.

Colored noise: Optimal Detection Signal

Notice:

For white Gaussian noise, P_D does not depend on signal shape, only on the energy $\mathbf{s}^T\mathbf{s}$:

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{\mathbf{s}^T\mathbf{s}}{\sigma^2}}\right)$$

For colored Gaussian noise, P_D DOES depend on the shape of the $\mathbf{s}$ compared to the statistics of the noise:

$$P_D = Q\left(\frac{\gamma' - \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}{\sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}}\right) = Q\left(Q^{-1}\left(P_{fa}\right) - \sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}\right).$$

What is the optimal $\mathbf{s}$ for the P_D?

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Q3: When dealing with colored noise, what role does the power of the signal play in optimizing the detection process?

A) The power of the signal has no impact on the detection process in the presence of colored noise.	0%
B) Higher signal power is always beneficial for detection in the presence of colored noise.	0%
C) The optimal power of the signal depends on the characteristics of the colored noise.	0%

Q3: When dealing with colored noise, what role does the power of the signal play in optimizing the detection process?

A) The power of the signal has no impact on the detection process in the presence of colored noise. 0%

B) Higher signal power is always beneficial for detection in the presence of colored noise. 26.92%

C) The optimal power of the signal depends on the characteristics of the colored noise. ✓ 73.08%

Colored noise: Optimal Detection Signal

1. Constrain the total energy to be $\mathbf{s}^T\mathbf{s} = E$.

2. Optimize for the shape of $\mathbf{s}$:

$$\max_{\mathbf{s}} \quad \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}$$
$$s.t. \quad \mathbf{s}^T\mathbf{s} = E$$

$$L(\mathbf{s}) = \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s} + \lambda(E - \mathbf{s}^T\mathbf{s})$$

Use $\frac{\partial \mathbf{b}^T\mathbf{x}}{\partial \mathbf{x}} = \mathbf{b}$ and $\frac{\partial \mathbf{x}^T\mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x}$.

$$\frac{\partial L(\mathbf{s})}{\partial \mathbf{s}} = 2\mathbf{C}^{-1}\mathbf{s} - 2\lambda\mathbf{s} = 0 \rightarrow \mathbf{C}^{-1}\mathbf{s} = \lambda\mathbf{s}$$

$\mathbf{s}$ is thus an eigenvector of $\mathbf{C}^{-1}$ with eigenvalue λ.

To maximize $\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}$, we should choose the eigenvector $\mathbf{s}$ that corresponds with the maximum eigenvalue λ of $\mathbf{C}^{-1}$ (or the minimum eigenvalue of $\mathbf{C}$).

Optimal Detection Signal - Example

Let $0 < \rho < 1$ be the correlation coefficient between consecutive noise samples, and let

$$\mathbf{C} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

1. Eigenvalues: $(\lambda - 1 - \rho)(\lambda - 1 + \rho) = 0 \rightarrow \lambda_1 = 1 + \rho$ and $\lambda_2 = 1 - \rho$ with corresponding eigenvectors following from $(\mathbf{C} - \lambda_i \mathbf{I})\,\mathbf{v}_i = \mathbf{0}$, $\mathbf{v_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$.
2. The minimum eigenvalue of $\mathbf{C}$ is thus $\lambda_2 = 1 - \rho$.
3. Therefore, $\mathbf{s} = \sqrt{E} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ with

Keeps the target signal (as $s[0] = -s[1]$) and reduces the noise (as a beamformer).

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{s} = \mathbf{x}^T \mathbf{C}^{-1} \sqrt{E} \mathbf{v}_2 = \sqrt{E} \frac{1}{\lambda_2} \mathbf{x}^T \mathbf{v}_2 = \frac{\sqrt{\frac{E}{2}}}{1 - \rho} (x[0] - x[1]).$$

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Deterministic Signals – Summary

Binary detection problem with $\mathbf{w} \sim N(\mathbf{0}, \mathbf{C})$ and deterministic $\mathbf{s}$:

$$\mathcal{H}_0 \quad x[n] = w[n]$$
$$\mathcal{H}_1 \quad x[n] = s[n] + w[n]$$

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{s}$$

Notice that if C is positive definite, $\mathbf{C}^{-1}$ can be written as $\mathbf{C}^{-1} = \mathbf{D}^T \mathbf{D}$, leading to

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{D}^T \mathbf{D} \mathbf{s}$$

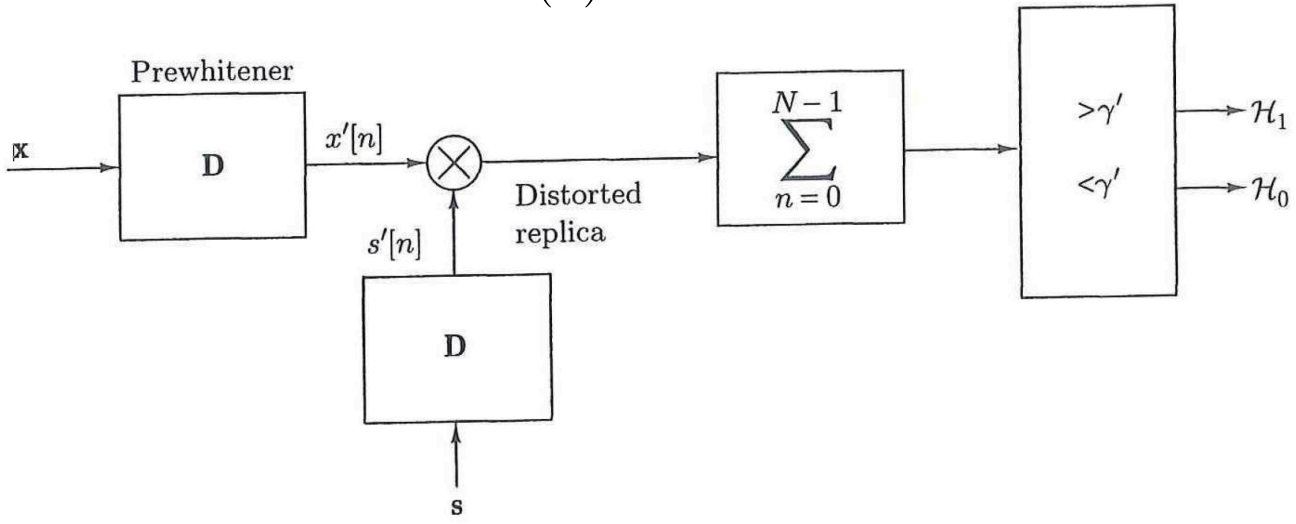

Fig. 4.7 Kay-II.

Problem 1: Binary detection problem with $\mathbf{w} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$ and deterministic signal $s[n] = Ar^n$:

$$\begin{array}{ll} \mathcal{H}_0 & x[n] = w[n] \\ \mathcal{H}_1 & x[n] = Ar^n + w[n] \end{array}$$

Problem 1a: Find the NP detector.

Problem 1b: Determine the detection performance.

Problem 1: Binary detection problem with $\mathbf{w} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$ and deterministic signal $s[n] = Ar^n$:

$$\begin{array}{ll} \mathcal{H}_0 & x[n] = w[n] \\ \mathcal{H}_1 & x[n] = Ar^n + w[n] \end{array}$$

Problem 1a: Let $\mathbf{H} = [1, r, ..., r^{N-1}]^T$.

$$p(\mathbf{x}; \mathcal{H}_1) = \frac{1}{(2\pi)^{\frac{N}{2}} \det^{\frac{1}{2}}(\mathbf{C})} \exp\left[-\frac{1}{2}(\mathbf{x} - A\mathbf{H})^T \mathbf{C}^{-1} (\mathbf{x} - A\mathbf{H})\right]$$

$$p(\mathbf{x}; \mathcal{H}_0) = \frac{1}{(2\pi)^{\frac{N}{2}} \det^{\frac{1}{2}}(\mathbf{C})} \exp\left[-\frac{1}{2}\mathbf{x}^T \mathbf{C}^{-1} \mathbf{x}\right].$$

$L(\mathbf{x}) = \frac{p(\mathbf{x};\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)} > \lambda.$

$$\ln L(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} A\mathbf{H} - \frac{1}{2}\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H} A^2 > \ln \lambda$$

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{H} A > \ln \lambda + \frac{1}{2}\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H} A^2 = \lambda'$$

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{H} A$$

Problem 1b: $T(\mathbf{x})$ is Gaussian distributed under both $\mathcal{H}_1$ and $\mathcal{H}_0$.

$$\mathrm{E}[T; \mathcal{H}_0] = \mathrm{E}[\mathbf{w}^T \mathbf{C}^{-1} \mathbf{H} A] = 0$$

$$\mathrm{E}[T; \mathcal{H}_1] = \mathrm{E}[(A\mathbf{H} + \mathbf{w})^T \mathbf{C}^{-1} \mathbf{H} A] = A^2 \mathbf{H}^T \mathbf{C}^{-1} \mathbf{H} = \frac{A^2}{\sigma^2} \sum_{n=0}^{N-1} r^{2n}$$

$$\mathrm{var}[T; \mathcal{H}_0] = \mathrm{E}[(\mathbf{w}^T \mathbf{C}^{-1} \mathbf{H} A)^2] = A^2 \mathbf{H}^T \mathbf{C}^{-1} \mathbf{H} = \frac{A^2}{\sigma^2} \sum_{n=0}^{N-1} r^{2n}$$

$$\mathrm{var}[T; \mathcal{H}_1] = \mathrm{E}[((A\mathbf{H} + \mathbf{w})^T \mathbf{C}^{-1} \mathbf{H} A - \mathrm{E}[(A\mathbf{H} + \mathbf{w})^T \mathbf{C}^{-1} \mathbf{H} A])^2$$

$$= \mathrm{E}\left[\left(((A\mathbf{H} + \mathbf{w}) - \mathrm{E}\left[(A\mathbf{H} + \mathbf{w})\right])^T \mathbf{C}^{-1} \mathbf{H} A\right)^2\right] = \mathrm{E}\left[\left(\mathbf{w}^T \mathbf{C}^{-1} \mathbf{H} A\right)^2\right] = var[T; \mathcal{H}_0] = \frac{A^2}{\sigma^2} \sum_{n=0}^{N-1} r^{2n}$$

$$P_{fa} = Q\left(\frac{\lambda'}{\sqrt{\frac{A^2}{\sigma^2} \sum_{n=0}^{N-1} r^{2n}}}\right) \rightarrow \lambda' = Q^{-1}\left(P_{fa}\right) \sqrt{\frac{A^2}{\sigma^2} \sum_{n=0}^{N-1} r^{2n}}$$

$$P_D = Q\left(\frac{\lambda' - \frac{A^2}{\sigma^2} \sum_{n=0}^{N-1} r^{2n}}{\sqrt{\frac{A^2}{\sigma^2} \sum_{n=0}^{N-1} r^{2n}}}\right) = Q\left(Q^{-1}\left(P_{fa}\right) - \sqrt{\frac{A^2}{\sigma^2} \sum_{n=0}^{N-1} r^{2n}}\right),$$

Learning Objectives

- LO1: Optimal binary detection
  - Bayes risk
- LO2: Detecting a known signal in noise using the NP criterion.
  - White noise
- LO3: Detecting a known signal in noise using the NP criterion.
  - Colored noise

Reading Tasks

- Chapter 4 – 4.4
- Exercise 1-3 from course website

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Next Lecture

- Random signals
  - Random process, modelling, scenario…
- NP detector for:
  - Zero mean Gaussian random process with known covariance
  - Generalized Gaussian detection (arbitrary covariance matrix)

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