ET4386 (2020-2021)

SOLUTIONS TO ESTIMATION THEORY EXCERCISES

① Consider the random variables $\{x_1, x_2, \dots x_N\}$. If $x_n \sim U[0, \theta]$, $\forall\ 0 \leq n < N-1$, then

$$\text{PDF} = p(x_n;\theta) \begin{cases} 1/\theta & 0 < x_n < \theta \\ 0 & \text{otherwise} \end{cases}$$

$$\text{MEAN} = \mathbb{E}(x_n) = \theta/2$$

$$\text{VARIANCE} = \text{var}(x_n) = \theta^2/12$$

a) Let $\hat{\theta} = 2\sum_{n=0}^{N-1} x_n$ be an estimator, then

$$\mathbb{E}[\hat{\theta} - \theta] = 2\sum_{n=0}^{N-1} \mathbb{E}[x_n] - \theta$$

$$= 2\sum_{n=0}^{N-1} \theta/2 - \theta = 0$$

$\therefore$ $\hat{\theta}$ is an unbiased estimator.

b) Note that the regularity conditions hold iff.

$$\int \frac{\partial p(\underline{x};\theta)}{\partial \theta} dx = \frac{\partial}{\partial \theta}\int p(\underline{x};\theta)\, dx$$

which holds true only if the support of the PDF does not depend on θ, which is not the case here.

(2) $x[n] = A\gamma^n + w[n] \qquad n = 0, 1, \dots N-1$

where $w[n] \sim \mathcal{N}(0, \sigma^2) \quad , \gamma > 0$

a) Let $\underline{x} = [x[0], x[1], \dots x[N-1]]^T$, then

$$p(\underline{x}; A) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[\frac{-1}{2\sigma^2} \sum_n (x[n] - A\gamma^n)^2\right]$$

$$\ln p(\underline{x}; A) = \ln \frac{1}{(2\pi\sigma^2)^{N/2}} - \frac{1}{2\sigma^2} \sum_n (x[n] - A\gamma^n)^2$$

$$s(\underline{x}; A) = \frac{\partial}{\partial A} \ln p(x; A)$$

$$= -\frac{\partial}{\partial \theta} \frac{1}{2\sigma^2} \sum_n (x[n] - A\gamma^n)^2$$

$$= \frac{1}{\sigma^2} \sum_n (x[n] - A\gamma^n)\gamma^n$$

$$\frac{\partial^2}{\partial A^2} \ln p(x; A) = -\frac{1}{\sigma^2} \sum_n \gamma^{2n}$$

$$I(A) = -\mathbb{E}\left[\frac{\partial^2 \ln p(x; A)}{\partial A^2}\right] = \frac{1}{\sigma^2} \sum_n \gamma^{2n}$$

$$\therefore \quad \text{CRLB} \Rightarrow \text{var}(\hat{A}) \geq \frac{\sigma^2}{\sum_n \gamma^{2n}}$$

b) Observe that the score can be rewritten

$$
\begin{aligned}
s(\underline{x};A) &= \frac{\partial}{\partial A} \ln p(x;A) \\
&= \frac{1}{\sigma^2} \sum_n \left(x[n] - A r^n\right) r^n \\
&= \frac{1}{\sigma^2} \left[\sum_n x[n] r^n - A \sum_n r^{2n}\right] \\
&= \underbrace{\frac{\sum_n r^{2n}}{\sigma^2}}_{\frac{1}{I(A)}} \left[\underbrace{\frac{\sum_n x[n] r^n}{\sum_n r^{2n}}}_{\hat{A} = g(\underline{x})} - A \right]
\end{aligned}
$$

Hence, $\hat{A}$ is an efficient estimator, with $\text{var}(\hat{A}) = \frac{1}{I(A)} = \frac{1}{\sigma^2} \sum_n r^{2n}$

(3) $x[n] = A + w[n]$, $n = 0, 1, 2, \ldots N-1$

where $w[n] \sim \mathcal{N}\left(0, \frac{A}{2}\right)$, $A > 0$

Note:

$$p(\underline{x}; A) = \frac{1}{(\pi A)^{N/2}} \exp\left[\frac{-1}{A} \sum_n (x[n] - A)^2\right]$$

$$\ln p(\underline{x}; A) = \ln \frac{1}{(\pi A)^{N/2}} - \frac{1}{A} \sum_n (x[n] - A)^2$$

$$s(\underline{x}; A) = \frac{\partial}{\partial A} \ln p(\underline{x}; A)$$

$$= \frac{-N}{2A} + \sum_n \frac{x^2[n]}{A^2} - N$$

$\Rightarrow$ REGULARITY CONDITIONS.

$$\mathbb{E}\left[\frac{\partial \ln}{\partial \theta} p(x; \theta)\right] = \mathbb{E}\left[\frac{-N}{2A} + \sum_n \frac{x^2[n]}{A^2} - N\right]$$

$$= \frac{-N}{2A} - N + \mathbb{E} \sum_n \frac{x^2[n]}{A^2} = 0$$

a) CRLB:

$$\frac{\partial^2}{\partial A^2} p(\underline{x};A) = \frac{\partial}{\partial A}\left[-\frac{N}{2A} + \sum_n \frac{x^2[n]}{A^2} - N\right]$$

$$= \frac{N}{2A^2} - 2\sum_n \frac{x^2[n]}{A^3}$$

$$I(A) = -\mathbb{E}\,\frac{N}{2A^2} + 2\,\frac{\mathbb{E}[x^2[n]]}{A^3}$$

Using $\mathbb{E}[x^2[n]] = N[\text{var}(x[n]] + \mathbb{E}^2[x(n)]]$,

$$I(A) = \frac{N}{2A^2}\{1 + 4A]$$

Since Regularity conditions are met.

$$\text{var}(A) \geq \frac{1}{I(A)} = \frac{2A^2}{N(1+4A)}$$

b) MLE

Solve: $\frac{\partial}{\partial A} \ln p(\underline{x}; A) = 0$

$$\Rightarrow -\frac{N}{2\hat{A}} + \frac{\sum_n x^2[n]}{\hat{A}^2} - N = 0$$

Multiply by $-\frac{\hat{A}^2}{N}$

$$\Rightarrow \frac{\hat{A}}{2} - \frac{1}{N}\sum_n x^2[n] + \hat{A}^2 = 0$$

$$\Rightarrow \hat{A}^2 + \frac{\hat{A}}{2} - \frac{\sum_n x^2[n]}{N} = 0$$

Solve the roots of the above equation to obtain $\hat{A}_{MLE}$.

c) BLUE :

$\underline{x}$ $(N \times 1)$ → observation vector

$\mathbb{E}[\underline{x}] = A$ → Mean

$C = 0.5 A I_N$ → Covarience.

$h = 1_N^T$ vector of ones

$$\hat{A}_{BLUE} = \frac{h^T C^{-1} x}{h^T C^{-1} h} = \frac{1_N^T (0.5 A I_N)^{-1} x}{1_N^T (0.5 A I_N)^{-1} 1_N}$$

$$= \frac{1_N^T x}{N} = \frac{1}{N} \sum_n x = \overline{x}$$

$$\mathrm{var}(\hat{A}_{BLUE}) = \frac{1}{h^T C^{-1} h} = \frac{1}{(0.5A)^{-1} 1_N^T 1_N} = \frac{A}{2N}$$

d) LS :

Let $\underline{x} = A 1_N + \underline{w}$ where $\underline{w} \sim \mathcal{N}\left(0, \frac{A}{2} I\right)$

then LS minimizes.

$$\min_A \; (\underline{x} - A 1_N)^T (\underline{x} - A 1_N)$$

Differentiating w.r.t A and setting to 0,

$$\hat{A}_{LS} = (1_N^T 1_N)^{-1} 1_N^T \underline{x} = \frac{1_N^T \underline{x}}{N}$$

④ $\underline{x} \sim \mathcal{N}(A, \sigma^2 I_N)$

$$p(\underline{x}; A, \sigma^2) = \frac{1}{(2\pi)^{N/2} \sigma^{N/2}} \exp\left[-\frac{1}{2\sigma^2} (\underline{x} - A\underline{1}_N)^T (\underline{x} - A\underline{1}_N) \right]$$

$$\ln p(\underline{x}; A, \sigma^2) = \ln \frac{1}{(2\pi)^{N/2} \sigma^{N/2}} - \frac{1}{2\sigma^2} (\underline{x} - A\underline{1}_N)^T (\underline{x} - A\underline{1}_N)$$

ⓐ $\frac{\partial}{\partial A} \ln p(\underline{x}; A, \sigma^2) = \frac{1}{\sigma^2} (\underline{x} - A) \underline{1}_N^T$

or $\hat{A} = \frac{\underline{x} \underline{1}_N^T}{N} = \overline{x}$

ⓑ $\frac{\partial}{\partial \sigma^2} \ln p(\underline{x}; A, \sigma^2) = -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4} (\underline{x} - A\underline{1}_N)^T (\underline{x} - A\underline{1}_N)$

or $\hat{\sigma}^2 = \frac{(\underline{x} - \overline{x})^T (\underline{x} - \overline{x})}{N}$

FROM ⓐ and ⓑ, using invariance

$$\alpha = \frac{\hat{A}^2}{\hat{\sigma}^2} = \frac{N \overline{x}^2}{(\underline{x} - \overline{x})^T (\underline{x} - \overline{x})}$$

(5) We can write the data model as

$$\underline{x} = \mu \mathbb{1}_N + \underline{w}$$

where $\mathbb{E}(\underline{w}) = 0$ and $\mathbb{E}(\underline{w}\underline{w}^T) = C_w$

(a) BLUE:

$$\hat{\mu}_{BLUE} = \frac{\mathbb{1}_N^T C_w \underline{x}}{\mathbb{1}_N^T C_w \mathbb{1}_N} = \frac{\mathbb{1}_N \underline{x}}{N} = \overline{x}$$

for both Laplacian & Gaussian PDFs

(b) LS:

$$\hat{\mu}_{LS} = \left(\mathbb{1}_N^T \mathbb{1}_N\right)^{-1} \mathbb{1}_N^T \underline{x} = \overline{x}$$

(6) PROVE :

$$\|\hat{S}\|^2 + \|x - \hat{s}\|^2 = \|x\|^2$$

where

$\hat{S}$ = Signal estimate

x = observation vector

$\|\cdot\|$ indicates the Euclidean norm.

Recollect

(a) $\hat{S} = H\hat{\theta} = H[(H^TH)^{-1}H^Tx] = P_H x$

where P_H is the Projection matrix.

(b) Along similar lines,

$x - \hat{s} = x - P_H x = (I - P_H)x = P_H^{\perp} x$

where $P_H^{\perp}$ is the orthogonal complement

Combining (a) + (b), we have

$$\|P_H x\|^2 + \|P_H^{\perp} x\|$$

$$= x^T P_H^T P_H x + x^T P_H^{\perp T} P_H^{\perp} x$$

$$= x^T P_H x + x(I - P_H)x$$

$$= x^T x$$

Hence proved!

⑦ Let $\underline{x} = [x[0], x[1] \ldots \; x[N]]^T$, then.

$$p(\underline{x}|\theta) = \exp\left[-N(\bar{x}-\theta)\right] \quad \forall \, x[n] > \theta$$

$$p(\theta) = \exp(-\theta)$$

Using Bayes,

$$p(\theta|\underline{x}) = \frac{p(\underline{x}|\theta)\,p(\theta)}{p(\underline{x})} = \frac{p(\underline{x}|\theta)\,p(\theta)}{\int_0^K p(\underline{x}|\theta)\,p(\theta)\,d\theta}$$

where $K = \max[x[n]]$

$$= \frac{\exp\left[-N(\bar{x}-\theta)\right]\exp\left[-\theta\right]}{\int_0^K \exp\left[-N(\bar{x}-\theta)\right]\exp\left[-\theta\right]d\theta}$$

$$= \frac{\exp\left[-N\bar{x} + N\theta - \theta\right]}{\int_0^K \exp\left[-N\bar{x} + N\theta - \theta\right]d\theta}$$

$$= \frac{\cancel{\exp\left[-N\bar{x}\right]}\exp\left[N\theta - \theta\right]}{\cancel{\exp\left[-N\bar{x}\right]}\int_0^K \exp\left[-N\theta - \theta\right]d\theta}$$

$$= \frac{\exp\left[\theta(N-1)\right]}{\left.\frac{\exp \theta(N-1)}{(N-1)}\right|_0^K}$$

$$= \frac{(N-1)\exp\left[\theta(N-1)\right]}{\exp\left[(N-1)K\right]-1} \quad \forall\, 0<\theta<K$$

The MMSE estimator is then.

$$\hat{\theta}_{MMSE} = \mathbb{E}\{\theta | x\}$$

$$= \frac{\int_0^K \theta(N-1)\exp\left[\theta(N-1)\right] d\theta}{\exp\left[(N-1)K\right]-1}$$

$$= \frac{N-1}{\exp[N-1]K-1}\left[\frac{\theta(N-1)-1}{(N-1)^2}\exp\theta\{N-1\}\right]_0^K$$

$$= \frac{(K(N-1)-1)\exp\left[(N-1)K\right]+1}{(N-1)\exp\left[(N-1)K-1\right]}$$

8. Let $z = [x, y]$

$$p(z) = p(x,y) = \frac{1}{(2\pi)^{1/2} |C|^{1/2}} \exp\left[-\frac{1}{2} z^T C^{-1} z\right]$$

$$g(y) = p(x_0, y)$$

$$= \frac{1}{(2\pi)^{1/2} |C|^{1/2}} \exp\left[-\frac{1}{2} Q(y)\right]$$

where

$$Q(y) = [x_0 \quad y] \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ y \end{bmatrix}$$

$$= \frac{[x_0 \quad y] \begin{bmatrix} 1 & -\rho \\ - & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y \end{bmatrix}}{(1-\rho^2)}$$

$$= \frac{[x_0^2 + y^2 - 2\rho x_0 y]}{(1-\rho^2)}$$

To maximize $g(y)$, we minimize $Q(y)$ i.e. taking partial derivatives of $Q(y)$ w.r.t y & setting to 0, we have.

$$\frac{\partial Q(y)}{\partial y} = \frac{\partial}{\partial y} \frac{(y^2 - 2\rho x_0 y)}{(1-\rho^2)} = 0$$

$$\Rightarrow \quad y = \rho x_0$$

(9)

Given.

$$x[n] = A\cos(2\pi f_0 n + \phi) + w[n]$$

where

$$A = \sqrt{a^2+b^2}$$

$$\phi = \arctan\left(\frac{-b}{a}\right)$$

and

$$\theta = [a, b]^T \sim \mathcal{N}(0, \sigma^2 I)$$

Solution:

$$p(A) = \int_0^{2\pi} p(A, \phi)\, d\phi$$

$$p(\phi) = \int_0^{\infty} p(A, \phi)\, dA$$

To find $p(A, \phi)$ in terms of θ, we use the transformation property of Random variables i.e.,

$$p(A, \phi) = \frac{p(a, b)}{|J|}$$

where J is the Jacobian matrix.

$$J = \frac{\partial [A, \phi]^T}{\partial \theta^T} = \begin{bmatrix} \frac{\partial A}{\partial a} & \frac{\partial A}{\partial b} \\ \frac{\partial \phi}{\partial a} & \frac{\partial \phi}{\partial b} \end{bmatrix}$$

Substituting the expressions for A and Φ,

$$J = \begin{bmatrix} \frac{a}{A} & \frac{b}{A} \\ \frac{b}{A^2} & -\frac{a}{A^2} \end{bmatrix}, \quad |J| = \frac{1}{A}$$

Note,

$$p(a,b) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{1}{2\sigma^2}[a \;\; b]^T[a \;\; b]\right]$$

$$= \frac{1}{2\pi\sigma^2} \exp\left[-\frac{1}{2\sigma^2}(a^2+b^2)\right]$$

Thus.

$$p(A,\phi) = \frac{p(a,b)}{|J|} = \frac{A}{2\pi\sigma^2} \exp\left[\frac{-1}{2\sigma^2}(a^2+b^2)\right]$$

Solving for $p(A)$ + $p(\phi)$, we have.

$$p(A) = \int_0^{2\pi} p(A,\phi)\, d\phi = \frac{A}{\sigma^2} \exp\left[-\frac{1}{2\sigma^2} A^2\right]$$

$$p(\phi) = \int_0^{2\pi} p(A,\phi)\, dA = \frac{1}{2\pi}$$

Rewriting $p(A,\phi)$, we have.

$$p(A,\phi) = \underbrace{\frac{1}{2\pi}}_{p(\phi)} \cdot \underbrace{\frac{A}{\sigma^2} \exp\left[-\frac{1}{2\sigma^2} A^2\right]}_{p(A)}$$

Hence proven!

(10) DATA MODEL:

$$\underline{x} = A\underline{h} + \underline{w}$$

where

$$\underline{x} = [x[0], x[1], \ldots x[N-1]]^T$$

$$\underline{h} = [1, r, \ldots r^{N-1}]^T$$

$$\underline{w} \sim \mathcal{N}(0, C) \quad \text{where } C = \sigma^2 I$$

$$A \sim \mathcal{N}(\mu_A, C_A) \quad \text{where } C_A = \sigma_A^2$$

(a) The LMMSE estimator of the above Bayesian linear model is.

$$\hat{A}_{LMMSE} = \mathbb{E}(A) + C_A h^T [h C_A h^T + C_w]^{-1} (\underline{x} - h\mathbb{E}(A)]$$

$$= \mu_A + \sigma_A^2 \underline{h}^T [\sigma_A^2 \underline{h}\underline{h}^T + \sigma^2 I]^{-1} (\underline{x} - h\mu_A)$$

$$= \mu_A + \underline{h}^T \left[\alpha\, \underline{h}\underline{h}^T + I\right]^{-1} (\underline{x} - h\mu_A)$$

where $\alpha = \sigma_A^2 / \sigma^2$

→ Using Sherman-Morrison property,

$$\hat{A}_{LMMSE} = \mu_A + h^T \left[I + \frac{\alpha h h^T}{1 + \alpha h^T h} \right] (x - h\mu_A)$$

(b) The Bayesian MSE is.

$$\text{Bmse}(\hat{A}_{LMMSE}) = (C_A^{-1} + h^T C_w^{-1} h)^{-1}$$

$$= (\sigma_A^{-2} + \sigma^{-2} h^T h)^{-1}$$

$$= \frac{\sigma^2}{(\alpha^{-1} + h^T h)}$$

(11) Given $\underline{x} = [x_1 \ x_2]^T \sim \mathcal{N}(0, C_{xx})$, we need to estimate x_2 linearly from x_1, i.e.,

$$x_2 = \alpha x_1 + \beta$$

and.

$$C_{xx} = \begin{bmatrix} \mathbb{E}(x_1^2) & \mathbb{E}_1(x_1, x_2) \\ \mathbb{E}(x_2, x_1) & \mathbb{E}_1(x_2^2) \end{bmatrix}$$

(a) The LMMSE estimator for x_2 is.

$$\hat{x}_2 = \mathbb{E}(x_2) + C_{x_2x_1} C_{xx}^{-1} (x_1 - \mathbb{E}(x_1))$$

$$= \frac{\mathbb{E}(x_1, x_2)}{\mathbb{E}(x_1^2)} (x_1)$$

(b) $$\text{BMSE}(\hat{x}_2) = C_{x_2x_2} - C_{x_2x_1} C_{xx}^{-1} C_{x_1x_2}$$

$$= \mathbb{E}(x_2^2) - \frac{\mathbb{E}^2(x_1 x_2)}{\mathbb{E}(x_1^2)}$$