# Microphone Array Processing

- **GEVD for estimation of the ATF**
- **Beamformer representations using the GEVD**

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# Summary of Previous Lecture

# Problem formulation

$$\mathbf{x}(k,l) = \sum_{i=1}^{d} \mathbf{a}_i(k,l) s_i(k,l) + \mathbf{n}(k,l)$$

- Assuming a single target and considering remaining point sources as interferers, abusing notation we can write

$$\mathbf{x}(k,l) = \underbrace{\mathbf{a}_1(k,l) s_1(k,l)}_{target} + \underbrace{\sum_{i=2}^{d} \mathbf{a}_i(k,l) s_i(k,l) + \mathbf{n}'(k,l)}_{interferers+noise}$$

$$= \mathbf{a}(k,l) s(k,l) + \mathbf{n}(k,l)$$

- Goal: Estimate $s(k,l)$ given $\mathbf{x}(k,l)$: e.g. $\hat{s}(k,l) = E[s(k,l)|\mathbf{x}(k,l)]$

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# Summary of Previous Lecture

- Delay and sum beamformer

$$\mathbf{w}(k,l) = \frac{\mathbf{a}(k,l)}{\mathbf{a}^H(k,l)\mathbf{a}(k,l)}$$

- MVDR beamformer

$$\mathbf{w}(k,l) = \frac{\mathbf{R}_{\mathbf{x}}^{-1}(k,l)\mathbf{a}(k,l)}{\mathbf{a}^H(k,l)\mathbf{R}_{\mathbf{x}}^{-1}(k,l)\mathbf{a}(k,l)} = \frac{\mathbf{R}_{\mathbf{n}}^{-1}(k,l)\mathbf{a}(k,l)}{\mathbf{a}^H(k,l)\mathbf{R}_{\mathbf{n}}^{-1}(k,l)\mathbf{a}(k,l)}$$

- Multi-Channel Wiener

$$\mathbf{w}(k,l) = \underbrace{\frac{\sigma_s^2(k,l)}{\sigma_s^2(k,l) + (\mathbf{a}^H(k,l)\mathbf{R}_{\mathbf{n}}^{-1}(k,l)\mathbf{a}(k,l))^{-1}}}_{\text{Single-channel Wiener}} \underbrace{\frac{\mathbf{R}_{\mathbf{n}}^{-1}(k,l)\mathbf{a}(k,l)}{\mathbf{a}^H(k,l)\mathbf{R}_{\mathbf{n}}^{-1}(k,l)\mathbf{a}(k,l)}}_{MVDR}$$



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# Cross Power Spectral Density Matrices

Assuming that all sources ($s_i[n]$ and $n[n]$) are realizations of random processes, we can define the cross power spectral density matrix per frequency band $k$ and time frame $l$:

$$E[\mathbf{x}(k,l)\mathbf{x}^H(k,l)] = \underbrace{E[\mathbf{s}(k,l)\mathbf{s}^H(k,l)]}_{\text{target source}} + \underbrace{E[\mathbf{n}(k,l)\mathbf{n}^H(k,l)]}_{\text{interferers/noise}}$$

often written as

$$\mathbf{R}_{\mathbf{x}}(k,l) = \mathbf{R}_{\mathbf{s}}(k,l) + \mathbf{R}_{\mathbf{n}}(k,l)$$

# Estimating $R_s$ – Pre-Whitening

**Estimation of $\mathbf{R_s}$:**

1. Compute $\mathbf{R}_\mathbf{n}^{\frac{1}{2}}$ and pre-whiten the data: $\tilde{\mathbf{x}} = \mathbf{R}_\mathbf{n}^{-\frac{1}{2}}\mathbf{x}$
2. Compute the EVD $\mathbf{R}_{\tilde{\mathbf{x}}} = \tilde{\mathbf{U}}\left(\tilde{\mathbf{\Lambda}} + \mathbf{I}_M\right)\tilde{\mathbf{U}}^H$, truncate the $M - r$ smallest eigenvalues and reduce the remaining ones by one.
3. Estimate $\hat{\mathbf{R}}_{\tilde{\mathbf{s}}} = \tilde{\mathbf{U}}_1\tilde{\mathbf{\Lambda}}_1\tilde{\mathbf{U}}_1^H$
4. De-whiten the result thus obtained so that

$$\hat{\mathbf{R}}_\mathbf{s} = \mathbf{R}_\mathbf{n}^{\frac{1}{2}}\tilde{\mathbf{U}}_1\tilde{\mathbf{\Lambda}}_1\tilde{\mathbf{U}}_1^H\mathbf{R}_\mathbf{n}^{\frac{1}{2}}$$

If $rank\left(\mathbf{R_s}(k,l)\right) = 1$, the ATF for spatially non-white noise can thus be obtained by selecting the principle eigenvector from $\hat{\mathbf{R}}_\mathbf{s}$ or from $\mathbf{R}_\mathbf{n}^{1/2}\tilde{\mathbf{U}}_1$



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# Generalised eigenvalue decomposition

Remarks:

- The explicit use of $\mathbf{R}_{\mathbf{n}}^{\frac{1}{2}}$ may result in a loss of accuracy in the data
- Can be avoided by working directly with $\mathbf{R}_{\mathbf{x}}$ and $\mathbf{R}_{\mathbf{n}}$
- In addition, when $\mathbf{R}_{\mathbf{n}}$ and/or $\mathbf{R}_{\mathbf{x}}$ are updated in a recursive way, it is generally very complicated to update $\mathbf{R}_{\tilde{\mathbf{x}}}$, while it is much simpler to calculate updates of $\mathbf{R}_{\mathbf{n}}^{-1}$ (using the matrix inversion lemma)

Another (in theory equivalent) method is given by the *generalised eigenvalue decomposition*



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# Generalised eigenvalue decomposition

Given the Hermitian matrices $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ with $\mathbf{B} \succ 0$, there exists a non-singular $\mathbf{U} =(\mathbf{u}_1, \ldots, \mathbf{u}_n), \mathbf{u}_i \in \mathbb{C}^n$, such that

$$\mathbf{U}^H\mathbf{A}\mathbf{U} = \mathrm{diag}(a_1, \ldots, a_n) \quad \text{and} \quad \mathbf{U}^H\mathbf{B}\mathbf{U} = \mathrm{diag}(b_1, \ldots, b_n).$$

Hence, we have $\mathbf{B}\mathbf{U} = \mathbf{U}^{-H}\mathbf{\Lambda}_B$ so that

$$\mathbf{A}\mathbf{U} = \mathbf{U}^{-H}\mathbf{\Lambda}_A = \mathbf{U}^{-H}\mathbf{\Lambda}_B\mathbf{\Lambda}_B^{-1}\mathbf{\Lambda}_A = \mathbf{B}\mathbf{U}\mathbf{\Lambda}$$

That is, $\mathbf{A}\mathbf{u}_i = \lambda_i\mathbf{B}\mathbf{u}_i$ for $i = 1, \ldots, n$ where $\lambda_i = a_i/b_i$.

This decomposition is known as the *generalised eigenvalue decomposition (GEVD)*.

# Generalised eigenvalue decomposition

Note that since $\mathbf{B} \succ 0$ ($\mathbf{B}$ is invertible), we have

$$\mathbf{B}^{-1}\mathbf{A}\mathbf{u}_i = \lambda_i \mathbf{u}_i$$

Hence, the generalised eigenvalues and eigenvectors of $(\mathbf{A}, \mathbf{B})$ are the (ordinary) eigenvalues and eigenvectors of the matrix $\mathbf{B}^{-1}\mathbf{A}$.

Note that $\mathbf{B}^{-1}\mathbf{A}$ is not Hermitian and thus $\mathbf{U}^{-1} \neq \mathbf{U}^H$.

# Generalised eigenvalue decomposition

Further, we can write $\mathbf{A} = \mathbf{U}^{-H}\mathbf{\Lambda}_{\mathbf{A}}\mathbf{U}^{-1}$ and $\mathbf{B} = \mathbf{U}^{-H}\mathbf{\Lambda}_{\mathbf{B}}\mathbf{U}^{-1}$ If we then let $\mathbf{Q} = \mathbf{U}^{-H} = (\mathbf{q}_1, \ldots, \mathbf{q}_M)$, $\mathbf{q}_i \in \mathbb{C}^M$, then we can write

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}_{\mathbf{A}}\mathbf{Q}^H \text{ and } \mathbf{B} = \mathbf{Q}\mathbf{\Lambda}_{\mathbf{B}}\mathbf{Q}^H.$$

From this it follows that

$$\mathbf{Q}^H\mathbf{B}^{-1}\mathbf{A} = \mathbf{\Lambda}_{\mathbf{B}}^{-1}\mathbf{\Lambda}_{\mathbf{A}}\mathbf{Q}^H$$

and thus

$$\mathbf{q}_i^H\mathbf{B}^{-1}\mathbf{A} = \mathbf{q}_i^H\lambda_i$$

Hence, $\mathbf{q}_1, \ldots, \mathbf{q}_M$ are the left eigenvectors of $\mathbf{B}^{-1}\mathbf{A}$.

# Generalised eigenvalue decomposition

Application to $\mathbf{R_s}$ and $\mathbf{R_n}$, and setting $b_i = 1$ for all $i$, we have

$$\mathbf{U}^H \mathbf{R_s} \mathbf{U} = \mathbf{\Lambda} \quad \text{and} \quad \mathbf{U}^H \mathbf{R_n} \mathbf{U} = \mathbf{I}_M,$$

where $\mathbf{\Lambda} \succeq 0$. Hence, the pair $(\mathbf{\Lambda}, \mathbf{U})$ are the eigenvalues/vectors of the matrix $\mathbf{R_n}^{-1}\mathbf{R_s}$ and $\mathbf{Q}$ the left eigenvectors of $\mathbf{R_n}^{-1}\mathbf{R_s}$.

Again, since $\mathbf{R_x} = \mathbf{R_s} + \mathbf{R_n}$, we have

$$\mathbf{U}^H \mathbf{R_x} \mathbf{U} = \mathbf{\Lambda} + \mathbf{I}_M \quad \Leftrightarrow \quad \mathbf{R_x} = \mathbf{U}^{-H} (\mathbf{\Lambda} + \mathbf{I}_M) \mathbf{U}^{-1}.$$



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# Generalised eigenvalue decomposition

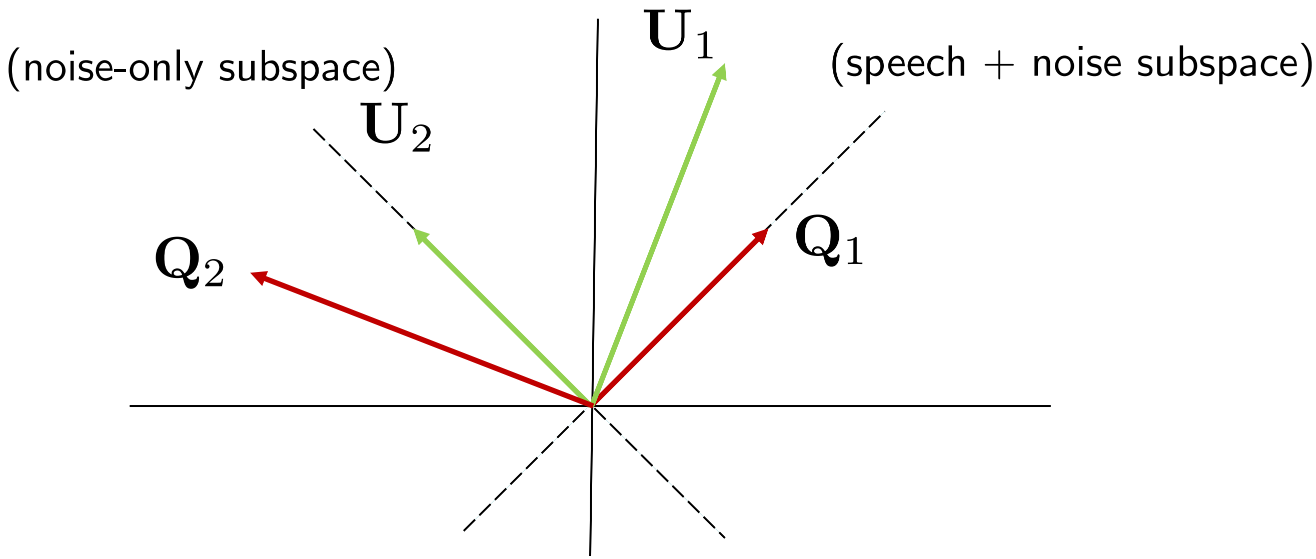
Again, if we assume that $\mathrm{rank}(\mathbf{R_s}) = r < M$, we can partition $\mathbf{R_x}$ as

$$\mathbf{R_x} = (\mathbf{Q}_1 \; \mathbf{Q}_2) \begin{pmatrix} \mathbf{\Lambda}_1 + \mathbf{I}_r & O \\ O & \mathbf{I}_{M-r} \end{pmatrix} \begin{pmatrix} \mathbf{Q}_1^H \\ \mathbf{Q}_2^H \end{pmatrix},$$

where $\mathbf{Q}_1 \in \mathbb{C}^{M \times r}$ and $\mathbf{Q}_2 \in \mathbb{C}^{M \times (M-r)}$.

# Geometric interpretation

Since $\mathbf{R_x} = \mathbf{Q}_1(\boldsymbol{\Lambda}_1 + \mathbf{I}_r)\mathbf{Q}_1^H + \mathbf{Q}_2\mathbf{Q}_2^H$, the vectors $\mathbf{q}_1, \ldots, \mathbf{q}_r$ span the *speech (+ noise) subspace*. Since $\mathbf{Q}^H\mathbf{U} = \mathbf{I}_M$ we conclude that $\mathbf{Q}_1^H\mathbf{U}_2 = \mathbf{O}$ so that the vectors $\mathbf{u}_{r+1}, \ldots, \mathbf{u}_M$ span the orthogonal subspace containing *noise* only.



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# Estimation of $R_S$

Similar to what we did before, we can compute (estimate) $\mathbf{R_s}$ from the GEVD of $\mathbf{R_x}$ as

$$\hat{\mathbf{R}}_{\mathbf{s}} = \mathbf{Q}_1(\mathbf{\Lambda}_1 + \mathbf{I}_r)\mathbf{Q}_1^H$$

or, by reducing the remaining eigenvalues by one,

$$\hat{\mathbf{R}}_{\mathbf{s}} = \mathbf{Q}_1\mathbf{\Lambda}_1\mathbf{Q}_1^H$$

If $rank\,(\mathbf{R_s}(k,l)) = 1$, the ATF for spatially non-white noise can thus also be obtained by selecting the $\mathbf{q}_1$, the principle generalized eigenvector between $\mathbf{R_s}(k,l)$ and $\mathbf{R_n}(k,l)$



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# GEVD versus pre-whitening

We have $\mathbf{R_x} = \mathbf{Q}(\mathbf{\Lambda} + \mathbf{I}_M)\mathbf{Q}^H$ so that

EVD of $\mathbf{R}_{\tilde{\mathbf{x}}}$

$$\mathbf{R}_{\tilde{\mathbf{x}}} = \mathbf{R}_{\mathbf{n}}^{-\frac{1}{2}}\mathbf{R_x}\mathbf{R}_{\mathbf{n}}^{-\frac{1}{2}} = \mathbf{R}_{\mathbf{n}}^{-\frac{1}{2}}\mathbf{Q}(\mathbf{\Lambda} + \mathbf{I}_M)\mathbf{Q}^H\mathbf{R}_{\mathbf{n}}^{-\frac{1}{2}} = \tilde{\mathbf{U}}(\tilde{\mathbf{\Lambda}} + \mathbf{I}_M)\tilde{\mathbf{U}}^H$$

from which we conclude that $\tilde{\mathbf{\Lambda}} = \mathbf{\Lambda}$ and $\tilde{\mathbf{U}} = \mathbf{R}_{\mathbf{n}}^{-\frac{1}{2}}\mathbf{Q}$, and thus $\mathbf{Q} = \mathbf{R}_{\mathbf{n}}^{\frac{1}{2}}\tilde{\mathbf{U}}$.

The approximation of $\mathbf{R_X}$ obtained by the GEVD is thus given by

$$\hat{\mathbf{R}}_{\mathbf{x}} = \mathbf{Q}_1\mathbf{\Lambda}_1\mathbf{Q}_1^H = \mathbf{R}_{\mathbf{n}}^{\frac{1}{2}}\tilde{\mathbf{U}}_1\tilde{\mathbf{\Lambda}}_1\tilde{\mathbf{U}}_1^H\mathbf{R}_{\mathbf{n}}^{\frac{1}{2}},$$

which is identical to the result obtained by pre-whitening.

# Beamforming

Recall that if $\mathrm{rank}(\mathbf{R}_\mathbf{x}) = r < M$, we can express $\mathbf{R}_\mathbf{x}$ as

$$\mathbf{R}_\mathbf{x} = \mathbf{Q}_1(\mathbf{\Lambda}_1 + \mathbf{I}_r)\mathbf{Q}_1^H + \mathbf{Q}_2\mathbf{Q}_2^H$$

Since the beamformer takes linear combinations of the microphone signals ($\hat{\mathbf{s}} = \mathbf{w}^H\mathbf{x}$), we have that

$$\mathbf{R}_{\hat{\mathbf{s}}} = \mathbf{w}^H\mathbf{R}_\mathbf{x}\mathbf{w} = \mathbf{w}^H\mathbf{Q}_1(\mathbf{\Lambda}_1 + \mathbf{I}_r)\mathbf{Q}_1^H\mathbf{w} + \mathbf{w}^H\mathbf{Q}_2\mathbf{Q}_2^H\mathbf{w}$$

Since we know that $\mathbf{U}_1^H\mathbf{Q}_1 = \mathbf{I}_r$ and $\mathbf{U}_1^H\mathbf{Q}_2 = \mathbf{0}$, we expect that a "good" beamformer can be expressed as a linear combination of the columns of $\mathbf{U}_1$. That is, $\mathbf{w} = \mathbf{U}_1\mathbf{b}$, where $\mathbf{b} \in \mathbb{C}^r$.



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# Beamformer performance measures

Beamformer performance measures:

- Output signal-to-noise ratio (SNR)
- Means square error (MSE)
- Noise reduction
- Speech distortion
- $\cdots$

# Output SNR

We can consider the output SNR, given by

$$\mathrm{SNR}_{\mathrm{out}}(\mathbf{w}) = \frac{E|\mathbf{w}^H\mathbf{s}|^2}{E|\mathbf{w}^H\mathbf{n}|^2} = \frac{\mathbf{w}^H\mathbf{R_s}\mathbf{w}}{\mathbf{w}^H\mathbf{R_n}\mathbf{w}}.$$

Note that the SNR is a real-valued function of the complex vector variable $\mathbf{w}$.

# Output SNR

**Theorem:** Let $f : \mathbb{C}^n \mapsto \mathbb{R}$ be a real valued function of a complex variable $z$. Let $f(z) = g(z, \bar{z})$, where $g : \mathbb{C}^n \times \mathbb{C}^n \mapsto \mathbb{R}$ is a function of two complex variables such that $g(z, a)$ and $g(b, z)$, $a, b \in \mathbb{C}$, are analytic functions of $z$. Then a necessary and sufficient condition for $f$ to have a stationary point is that $\nabla_z g = 0$, where the partial derivative with respect to $z$ treats $\bar{z}$ as a constant, or $\nabla_{\bar{z}} g = 0$.

**Theorem:** Let $f$ and $g$ be defined as above. Then the gradient $\nabla_{\bar{z}} g(z)$ defines the direction of steepest descent of $f$ at $z$.

[1] D.H. Brandwood, "A complex gradient operator and its application in adaptive array theory ", *IEE Proceedings*, vol. 130, no. 1, pp. 11-16, February 1983.

# Output SNR

Taking the derivative of $\mathrm{SNR}_{\mathrm{out}}$ with respect to $\mathbf{w}^H$, we find that

$$\nabla_{\mathbf{w}^H} \mathrm{SNR}_{\mathrm{out}}(\mathbf{w}) = \mathbf{R_s w} - \frac{\mathbf{w}^H \mathbf{R_s w}}{\mathbf{w}^H \mathbf{R_n w}} \mathbf{R_n w} = 0,$$

where $\mathbf{w}$ is a stationary point of $\mathrm{SNR}_{\mathrm{out}}$. Hence, we have $\mathbf{R_s w} = \lambda \mathbf{R_n w}$ where $\mathbf{w}$ is a generalised eigenvector with corresponding eigenvalue

$$\lambda = \frac{\mathbf{w}^H \mathbf{R_s w}}{\mathbf{w}^H \mathbf{R_n w}},$$

and we conclude that

$$\mathrm{SNR}_{\mathrm{out}}(\mathbf{w}) \leq \max_{\mathbf{w}} \frac{\mathbf{w}^H \mathbf{R_s w}}{\mathbf{w}^H \mathbf{R_n w}} = \lambda_1.$$

# Output SNR

We conclude that the choice $\mathbf{w} = \mathbf{w}_1$ results in maximising the output SNR.

Note that this result is unique up to a scaling. Indeed, if $\mathbf{z} = \alpha \mathbf{u}_1$ for any $\alpha \neq 0$, we have

$$\frac{\mathbf{z}^H \mathbf{R_s} \mathbf{z}}{\mathbf{z}^H \mathbf{R_n} \mathbf{z}} = \frac{\mathbf{w}^H \mathbf{R_s} \mathbf{w}}{\mathbf{w}^H \mathbf{R_n} \mathbf{w}}$$

which is obvious since the eigenvectors are unique up to an arbitrary scaling $\alpha \neq 0$.

In addition, this result is independent of $r = \mathrm{rank}(\mathbf{R_s})$.



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# Mean squared-error

Consider the mean squared-error (MSE) between the beamformer output and the desired target signal at the reference microphone, which we will assume, without loss of generality, to be microphone $1$.

We have

$$\begin{aligned} \mathrm{E}|\mathbf{w}^H\mathbf{x} - s_1|^2 &= \mathrm{E}|\mathbf{w}^H\mathbf{s} + \mathbf{w}^H\mathbf{n} - s_1|^2 \\ &= \mathrm{E}|\mathbf{w}^H\mathbf{s} - s_1|^2 + \mathrm{E}|\mathbf{w}^H\mathbf{n}|^2, \end{aligned}$$

where we used the property $\mathrm{E}(\mathbf{s}\mathbf{n}^H) = 0$. The term $\mathrm{E}|\mathbf{w}^H\mathbf{x} - s_1|^2$ represents the *signal distortion*, whereas the term $\mathrm{E}|\mathbf{w}^H\mathbf{n}|^2$ represents the *residual noise variance*

# Mean squared-error

We can compromise between signal distortion and noise reduction by defining the constraint optimisation problem

$$\begin{aligned} \text{minimise} \quad & \mathrm{E}|\mathbf{w}^H\mathbf{s} - s_1|^2 \\ \text{subject to} \quad & \mathrm{E}|\mathbf{w}^H\mathbf{n}|^2 \leq c, \end{aligned}$$

where $0 \leq c \leq \sigma_{n_1}^2$ and $\sigma_{n_1}^2$ the noise variance at the reference microphone before beamforming.



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# MMSE solution

In order to find the expressions for the different beamformers, we express the beamformers weights in terms of the generalised eigenvectors. That is, we have $\mathbf{w} = \mathbf{U}\mathbf{b}$ with $\mathbf{b} \in \mathbb{C}^M$.

Let $\mathbf{e}_1 = (1, 0, \ldots, 0)^T \in \mathbb{C}^M$. With this we have $s_1 = \mathbf{e}_1^H \mathbf{s}$ so that we can express the objective function as

$$\begin{aligned} \mathrm{E}|\mathbf{w}^H \mathbf{s} - s_1|^2 &= \mathrm{E}|\mathbf{w}^H \mathbf{s} - \mathbf{e}_1^H \mathbf{s}|^2 \\ &= \mathbf{b}^H \mathbf{U}^H \mathbf{R}_\mathbf{s} \mathbf{U} \mathbf{b} + \sigma_{s_1}^2 - 2\mathrm{Re}\{\mathbf{b}^H \mathbf{U}^H \mathbf{R}_\mathbf{s} \mathbf{e}_1\} \\ &= \mathbf{b}^H \boldsymbol{\Lambda} \mathbf{b} + \sigma_{s_1}^2 - 2\mathrm{Re}\{\mathbf{b}^H \mathbf{U}^H \mathbf{R}_\mathbf{s} \mathbf{e}_1\}, \end{aligned}$$

and the feasible set becomes $\{\mathbf{b} \in \mathbb{C}^M : \mathbf{b}^H \mathbf{b} \le c\}$.

# MMSE solution

The corresponding Lagrangian is given by

$$L(\mathbf{b}, \mu) = \mathbf{b}^H \mathbf{\Lambda} \mathbf{b} + \sigma_{s_1}^2 - 2\mathrm{Re}\{\mathbf{b}^H \mathbf{U}^H \mathbf{R_s} \mathbf{e}_1\} + \mu(\mathbf{b}^H \mathbf{b} - c),$$

with $\mu \geq 0$ a Lagrange multiplier.

Let $\mathbf{b}^*$ denote the (unique) minimiser. The optimality conditions (KKT conditions) for $\mathbf{b}^*$ to be optimal are then given by[1]

$$\nabla_{\bar{\mathbf{b}}} L(\mathbf{b}^*, \mu) = \mathbf{\Lambda} \mathbf{b}^* - \mathbf{U}^H \mathbf{R_s} \mathbf{e}_1 + \mu \mathbf{b}^* = 0.$$

[1]Since the minimum of our minimisation problem is attained on the boundary of the feasible set $\{\mathbf{b} \in \mathbb{C}^m : \mathbf{b}^H \mathbf{b} \leq c\}$, we can replace the inequality constraint by an equality one.



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# MMSE solution

Hence,

$$\mathbf{b}^* = (\mathbf{\Lambda} + \mu\mathbf{I}_M)^{-1}\mathbf{U}^H\mathbf{R}_{\mathbf{s}}\mathbf{e}_1,$$

and thus

$$\begin{aligned}\mathbf{w}^* &= \mathbf{U}\mathbf{b}^* \\ &= \mathbf{U}(\mathbf{\Lambda} + \mu\mathbf{I}_M)^{-1}\mathbf{U}^H\mathbf{R}_{\mathbf{s}}\mathbf{e}_1\end{aligned}$$

where the Lagrange multiplier $\mu \geq 0$ is chosen such that $\mathbf{b}^H\mathbf{b} = c$.



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# MMSE solution

As mentioned before, in many applications we have $\mathrm{rank}(\mathbf{R_s}) = r < M$ and we have $\mathbf{R_s} = \mathbf{Q}_1\mathbf{\Lambda}_1\mathbf{Q}_1^H$.

In those cases the optimal filter weights $\mathbf{w}^*$ become

$$\begin{aligned}\mathbf{w}^* &= \mathbf{U}(\mathbf{\Lambda} + \mu\mathbf{I}_M)^{-1}\mathbf{U}^H\mathbf{Q}_1\mathbf{\Lambda}_1\mathbf{Q}_1^H\mathbf{e}_1 \\ &= \mathbf{U}_1(\mathbf{\Lambda}_1 + \mu\mathbf{I}_r)^{-1}\mathbf{\Lambda}_1\mathbf{Q}_1^H\mathbf{e}_1\end{aligned}$$

since $\mathbf{U}_1^H\mathbf{Q}_1 = \mathbf{I}_r$ and $\mathbf{U}_2^H\mathbf{Q}_1 = \mathbf{O}$.

We indeed conclude that MMSE optimal beamformers can be expressed as a linear combination of the columns of $\mathbf{U}_1$



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# MMSE solution

Note that since $\mathbf{R_s} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^H$ and $\mathbf{R_n} = \mathbf{Q}\mathbf{Q}^H$ we have

$$\begin{aligned}\mathbf{U}(\boldsymbol{\Lambda} + \mu \mathbf{I}_M)^{-1}\mathbf{U}^H &= (\mathbf{U}^{-H}(\boldsymbol{\Lambda} + \mu \mathbf{I}_M)\mathbf{U}^{-1})^{-1} \\ &= (\mathbf{Q}(\boldsymbol{\Lambda} + \mu \mathbf{I}_M)\mathbf{Q}^H)^{-1} \\ &= (\mathbf{R_s} + \mu \mathbf{R_n})^{-1}\end{aligned}$$

and we conclude that

$$\mathbf{w}^* = (\mathbf{R_s} + \mu \mathbf{R_n})^{-1}\mathbf{R_s}\mathbf{e}_1.$$

This solution is referred to as the *signal-distortion weighted* (SDW) Wiener filter



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# Multi-channel Wiener filter

The case $\mu = 1$ gives the classical multi-channel Wiener filter:

$$\mathbf{w}_{MWF} = \mathbf{R}_{\mathbf{x}}^{-1}\mathbf{R}_{\mathbf{s}}\mathbf{e}_1.$$

In the case we have $\mathbf{R}_{\mathbf{s}} = \sigma_{s_1}^2 \mathbf{a}\mathbf{a}^H$ this reduces to

$$\mathbf{w}_{MWF} = \sigma_{s_1}^2 \mathbf{R}_{\mathbf{x}}^{-1}\mathbf{a}.$$

In fact, the parameter $\mu$ can be seen as a trade-off parameter that controls the signal distortion and noise reduction.



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# MVDR beamformer

The choice $\mu = 0$ and rank $r$ will lead to the MVDR beamformer.

Recall that

$$\mathrm{E}|\mathbf{w}^H\mathbf{s} - s_1|^2 = \mathbf{b}^{*H}\mathbf{\Lambda}_1\mathbf{b}^* + \sigma_{s_1}^2 - 2\mathrm{Re}\{\mathbf{b}^{*H}\mathbf{U}_1^H\mathbf{R}_\mathbf{s}\mathbf{e}_1\},$$

where

$$\begin{aligned}\mathbf{b}^* &= \mathbf{\Lambda}_1^{-1}\mathbf{U}_1^H\mathbf{R}_\mathbf{s}\mathbf{e}_1 \\ &= \mathbf{\Lambda}_1^{-1}\mathbf{U}_1^H\mathbf{Q}_1\mathbf{\Lambda}_1\mathbf{Q}_1^H\mathbf{e}_1 \\ &= \mathbf{Q}_1^H\mathbf{e}_1\end{aligned}$$

and thus, we get the rank $r$ mvdr as $\mathbf{w}^* = \mathbf{U}\mathbf{b}^* = \mathbf{U}\mathbf{Q}_1^H\mathbf{e}_1 = \mathbf{U}_1\mathbf{Q}_1^H\mathbf{e}_1$



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# MVDR beamformer

With this we have

$$\mathbf{b}^{*H}\mathbf{\Lambda}_1\mathbf{b}^* = \mathbf{e}_1^H \underbrace{\mathbf{Q}_1\mathbf{\Lambda}_1\mathbf{Q}_1^H}_{\mathbf{R_s}} \mathbf{e}_1 = \sigma_{s_1}^2,$$

and

$$\mathbf{b}^{*H}\mathbf{U}_1^H\mathbf{R_s}\mathbf{e}_1 = \mathbf{e}_1^H \underbrace{\mathbf{Q}_1\mathbf{U}_1^H}_{\mathbf{I}_r} \mathbf{Q}_1\mathbf{\Lambda}_1\mathbf{Q}_1^H\mathbf{e}_1 = \sigma_{s_1}^2,$$

so that

$$\mathrm{E}|\mathbf{w}^H\mathbf{s} - s_1|^2 = \mathbf{b}^{*H}\mathbf{\Lambda}_1\mathbf{b}^* + \sigma_{s_1}^2 - 2\mathrm{Re}\{\mathbf{b}^{*H}\mathbf{U}_1^H\mathbf{R_s}\mathbf{e}_1\} = 0,$$

and we conclude that the response is distortionless.



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# MVDR beamformer

As a special case, consider $r = 1$ so that $\mathbf{R_s}$ can be expressed as $\sigma_{s_1}^2 \mathbf{a}\mathbf{a}^H$.

We have $\mathbf{w}^* = \mathbf{u}_1 \mathbf{b}^* = \mathbf{u}_1 \mathbf{q}_1^H \mathbf{e}_1$ so that

$$
\begin{aligned}
\mathbf{w}^* &= \mathbf{u}_1 \mathbf{u}_1^H \mathbf{q}_1 \mathbf{q}_1^H \mathbf{e}_1 \\
&= \mathbf{U}\mathbf{U}^H \mathbf{q}_1 \mathbf{q}_1^H \mathbf{e}_1 \\
&= \mathbf{R_n}^{-1} \mathbf{q}_1 \mathbf{q}_1^H \mathbf{e}_1 \\
&= \lambda_1^{-1} \mathbf{R_n}^{-1} \mathbf{R_s} \mathbf{e}_1 \\
&= \lambda_1^{-1} \sigma_{s_1}^2 \mathbf{R_n}^{-1} \mathbf{a}\mathbf{a}^H \mathbf{e}_1 \\
&= \lambda_1^{-1} \sigma_{s_1}^2 \mathbf{R_n}^{-1} \mathbf{a}
\end{aligned}
$$

# MVDR beamformer

To find an expression for $\lambda_1$, we note that $\mathbf{w}^*$ is a scaled version of $\mathbf{u}_1$ and, therefore, maximises the output SNR:

$$\lambda_1 = \frac{\mathbf{w}^{*H}\mathbf{R_s}\mathbf{w}^*}{\mathbf{w}^{*H}\mathbf{R_n}\mathbf{w}^*}$$

$$= \frac{\mathbf{a}^H\mathbf{R_n}^{-1}\left(\sigma_{s_1}^2\mathbf{a}\mathbf{a}^H\right)\mathbf{R_n}^{-1}\mathbf{a}}{\mathbf{a}^H\mathbf{R_n}^{-1}\mathbf{a}}$$

$$= \sigma_{s_1}^2\mathbf{a}^H\mathbf{R_n}^{-1}\mathbf{a}$$

and we conclude that

$$\mathbf{w}^* = \frac{\mathbf{R_n}^{-1}\mathbf{a}}{\mathbf{a}^H\mathbf{R_n}^{-1}\mathbf{a}}.$$



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# Multi-channel Wiener filter

Recall that in general we have $\mathbf{w}^* = \mathbf{U}(\boldsymbol{\Lambda} + \mu \mathbf{I}_M)^{-1}\mathbf{U}^H \mathbf{R}_\mathbf{s}\mathbf{e}_1$. Using the same arguments as before, we have for $r = 1$ that

$$
\begin{aligned}
\mathbf{w}^* &= \frac{\sigma_{s_1}^2}{\lambda_1 + \mu}\mathbf{R}_\mathbf{n}^{-1}\mathbf{a} \\
&= \frac{\sigma_{s_1}^2}{\sigma_{s_1}^2 \mathbf{a}^H \mathbf{R}_\mathbf{n}^{-1}\mathbf{a} + \mu}\mathbf{R}_\mathbf{n}^{-1}\mathbf{a} \\
&= \frac{\sigma_{s_1}^2}{\sigma_{s_1}^2 + \mu(\mathbf{a}^H \mathbf{R}_\mathbf{n}^{-1}\mathbf{a})^{-1}} \frac{\mathbf{R}_\mathbf{n}^{-1}\mathbf{a}}{\mathbf{a}^H \mathbf{R}_\mathbf{n}^{-1}\mathbf{a}}
\end{aligned}
$$

which shows that the (SDW) MWF can be implemented as an MVDR beamformer, followed by a single-channel Wiener filter.