EE 4715 Array Processing

4. Wideband Models

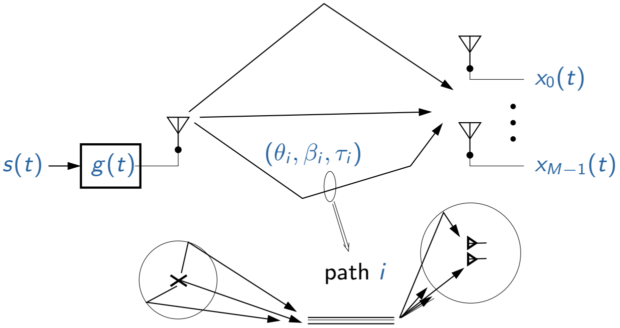
April 2022

Models

Wideband models are used in wireless communication, acoustics (audio, ultrasound, underwater, geophysics), and elsewhere. In this lecture we take a communications perspective.

- Physical channel
- Source model
- Receiver model

Multipath channel model



Jakes' multipath model

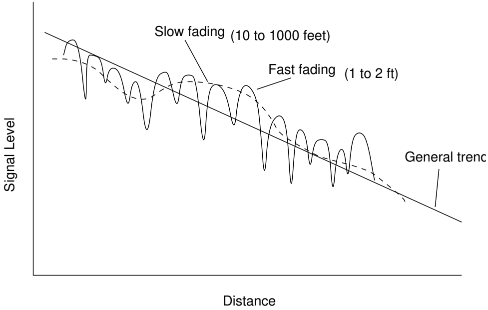
- scatterers local to mobile (nearby buildings, sensitive to motion)
- remote scatterers (remote buildings, hills: dominant scatterers)
- scatterers local to base (static effect)

Multipath channel model

$$\boldsymbol{x}(t) = \left[\sum_{i=1}^{r} \boldsymbol{a}(\theta_i)\beta_i g(t-\tau_i)\right] * s(t)$$

- $g(t)$ is a pulse shape function (where $s(t)$ contains dirac pulses)
- $\boldsymbol{a}(\theta)$ is the array response vector
- Scattering local to mobile causes fluctuations in path gains β_i:
  - Each ray has a (small) angular and delay spread: phase changes
  - Adding the rays in/out of phase creates major effects on the gains: **fading**

Fading

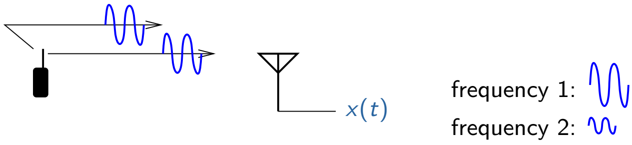


- **General trend:** $\approx$ 35 - 50 dB / decade (path loss)
- **Slow fading:** Caused by shadowing.
- **Fast Fading:** Caused by local scatterers near mobile. Typically Rayleigh distributed

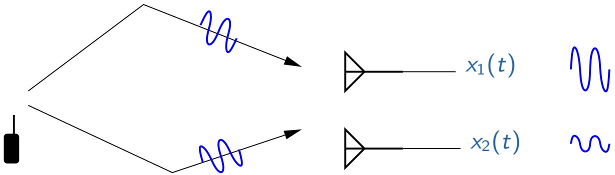
Fading

$\phi = \omega_0 \tau$: a phase can change due to changes in frequency or delay

- Time-selective fading (due to motion/Doppler)
- Frequency-selective fading (due to long delays)



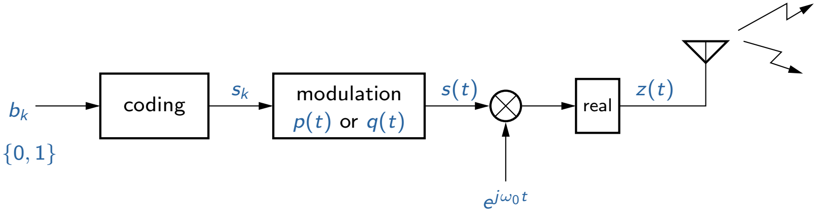
- Space-selective fading (large angles)



Typical channel parameter values

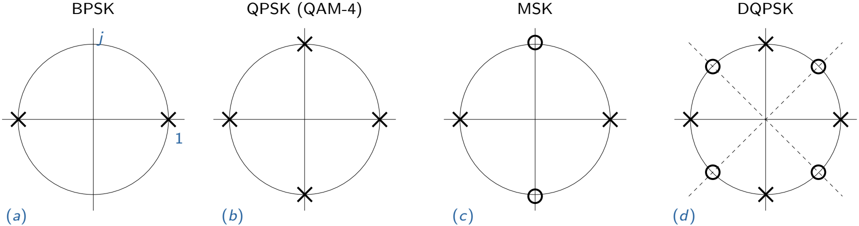
Environment	delay spread	angle spread	Doppler spread
Flat rural (macro)	0.5 μs	1°	190 Hz
Urban (macro)	5 μs	20°	120 Hz
Hilly (macro)	20 μs	30°	190 Hz
Mall (micro)	0.3 μs	120°	10 Hz
Indoors (pico)	0.1 μs	360°	5 Hz

Signal modulation



- Digital alphabets
- Modulation
  - Linear modulation (e.g. with a raised-cosine pulse)
  - Phase modulation (not discussed here)

Digital constellations



$b_k \in \{0, -1\}$ $\rightarrow$	s_k chosen from (up to scaling):
BPSK	$\{1, -1\}$
QPSK (QAM-4)	$\{1, j, -1, -j\}$
m-PSK	$\{1, e^{j2\pi/m}, e^{j2\pi 2/m}, \cdots, e^{j2\pi(m-1)/m}\}$
m-PAM	$\{\pm 1, \pm 3, \cdots, \pm(2m-1)\}$
m-PAM	$\{\pm 1, \pm 3, \cdots, \pm(2m-1)\}$
MSK	$\{\{1, -1\}$, k even; $\{j, -j\}$, k odd

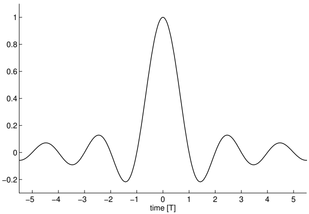
Linear modulation

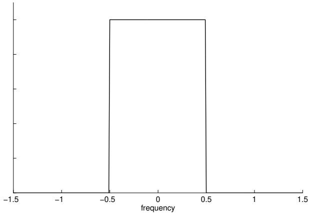
- Amplitude is modulated (possibly complex):

$$s_\delta(t) = \sum_k s_k \delta(t-k) \qquad s(t) = p(t) * s_\delta(t) = \sum_k s_k p(t-k)$$

- Optimum waveform is localized in time and frequency (contradiction)
- One possible choice: sinc pulse shape

$$p(t) = \frac{\sin \pi t}{\pi t}, \qquad P(f) = \begin{cases} 1, & |f| < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

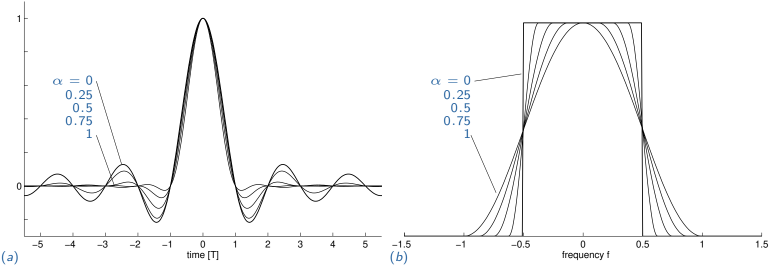




Linear modulation

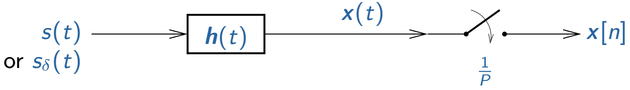
- Modification: *raised-cosine* pulseshapes

$$p(t) = \frac{\sin \pi t}{\pi t} \cdot \frac{\cos \alpha \pi t}{1 - 4\alpha^2 t^2}, \; P(f) = \begin{cases} 1\,, & |f| < \frac{1}{2}(1-\alpha) \\ \frac{1}{2} - \frac{1}{2}\sin(\frac{\pi}{\alpha}(|f| - \frac{1}{2})), & \frac{1}{2}(1-\alpha) < |f| < \frac{1}{2}(1+\alpha) \\ 0\,, & \text{otherwise} \end{cases}$$



α: excess bandwidth (or rolloff) parameter; common choice is $\alpha = 0.35$

FIR channel model



- We collect all temporal effects in $\boldsymbol{h}(t)$
  - Filtering effects at the transmitter and the receiver
  - Propagation channel (array response, multipath delays)
  - Optional: pulse shape $g(t)$ for linear modulation
- Resulting data model (M antennas):

$$\boldsymbol{x}(t) = \boldsymbol{h}(t) * s(t), \qquad \boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_M(t) \end{bmatrix}, \quad \boldsymbol{h}(t) = \begin{bmatrix} h_1(t) \\ \vdots \\ h_M(t) \end{bmatrix}$$

Oversampling

If the symbol rate is below the Nyquist rate, we can **oversample** beyond the symbol rate, by a factor P. Often, $P = 2$.

Define $P \times 1$ vectors spanning 1 symbol period:

$$\boldsymbol{x}[n] = \begin{bmatrix} x(n) \\ x(n + \frac{1}{P}) \\ \vdots \\ x(n + \frac{P-1}{P}) \end{bmatrix}, \qquad \boldsymbol{h}[k] = \begin{bmatrix} h(k) \\ h(k + \frac{1}{P}) \\ \vdots \\ h(k + \frac{P-1}{P}) \end{bmatrix}$$

In the received signal, over 1 symbol period, the same L transmitted symbols play a role:

$$\boldsymbol{x}[n] = \boldsymbol{h}[n] * s[n] = \sum_{k=0}^{L-1} \boldsymbol{h}[k] s[n-k]$$

- This is the same as we had before, but now using sample vectors consisting of the P samples that fall within one sample period.

Matrix representation of convolution

Collect samples in a matrix $\boldsymbol{X} = [\boldsymbol{x}[0] \quad \boldsymbol{x}[1] \quad \cdots \quad \boldsymbol{x}[L-1]]$.

The convolution model $\boldsymbol{x}[n] = \boldsymbol{h}[n] * s[n]$ gives

$$\boldsymbol{X} = \boldsymbol{H}\boldsymbol{S}$$

with

$$\boldsymbol{H} = [\boldsymbol{h}[0] \quad \boldsymbol{h}[1] \quad \cdots \quad \boldsymbol{h}[L-1]]$$

$$\boldsymbol{S} = \left[\begin{array}{ccc|cccc|ccc} s_0 & s_1 & \cdots & s_{L-1} & \cdots & s_{N_s-2} & s_{N_s-1} & & & \mathbf{0} \\ & s_0 & \cdots & \cdots & \cdots & \cdots & s_{N_s-2} & s_{N_s-1} & & \\ & & \ddots & s_1 & \cdots & \cdots & \cdots & \ddots & \ddots & \\ \mathbf{0} & & & s_0 & \cdots & \cdots & s_{N_s-L} & \cdots & s_{N_s-2} & s_{N_s-1} \end{array}\right]$$

Stacking

Create an augmented data matrix by stacking m samples of $\boldsymbol{x}[n]$:

$$\mathcal{X} = \begin{bmatrix} \boldsymbol{x}[0] & \boldsymbol{x}[1] & \iddots & \boldsymbol{x}[N-m] \\ \boldsymbol{x}[1] & \boldsymbol{x}[2] & \iddots & \iddots \\ \iddots & \iddots & \iddots & \boldsymbol{x}[N-2] \\ \boldsymbol{x}[m-1] & \iddots & \boldsymbol{x}[N-2] & \boldsymbol{x}[N-1] \end{bmatrix}.$$

Due to the convolution model we can factor $\mathcal{X}$ as

$$\mathcal{X} = \mathcal{H}\mathcal{S} = \begin{bmatrix} \mathbf{0} & & \boxed{\boldsymbol{H}} \\ & \iddots \;\; \iddots & \\ & \boxed{\boldsymbol{H}} & \\ \boxed{\boldsymbol{H}} & & \mathbf{0} \end{bmatrix} \begin{bmatrix} s_{m-1} & \ddots & s_{N-2} & s_{N-1} \\ \ddots & \ddots & \ddots & s_{N-2} \\ s_{-L+2} & s_{-L+3} & \ddots & \ddots \\ s_{-L+1} & s_{-L+2} & \ddots & s_{N-L-m+1} \end{bmatrix}$$

Space-time equalizer

A space-time equalizer is a vector $\boldsymbol{w}$ which combines the rows of $\mathcal{X}$:

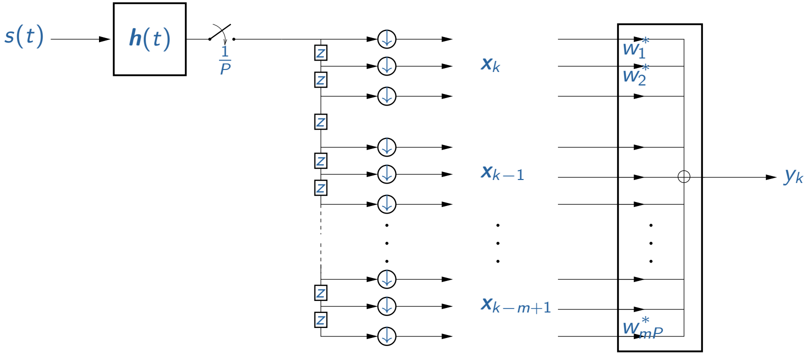
$$\boldsymbol{w}^{\mathsf{H}}\mathcal{X} = [\hat{s}_{k_0}\ \hat{s}_{k_0+1}\ \cdots]$$

From the factorization $\mathcal{X} = \mathcal{H}\mathcal{S}$ we see that

$$\boldsymbol{w}^{\mathsf{H}}\mathcal{H} = [0, \cdots, 0, 1, 0, \cdots, 0]$$

- $\boldsymbol{w}^{\mathsf{H}}$ is one row of a left inverse $\mathcal{H}^{\dagger}$ of $\mathcal{H}$.
- We can choose which row of $\mathcal{S}$ we reconstruct.

Space-time equalizer

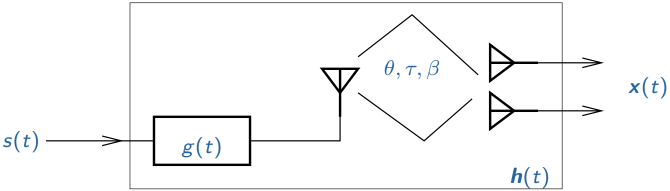


Connection to the multiray model

The multiray propagation model is (for specular multipath)

$$\boldsymbol{h}(t) = \begin{bmatrix} h_1(t) \\ \vdots \\ h_M(t) \end{bmatrix} = \sum_{i=1}^{r} \boldsymbol{a}(\theta_i)\, \beta_i\, g(t - \tau_i)$$

where $g(t)$: pulse shape function (e.g., raised-cosine),
θ_i: direction-of-arrival (DOA),
τ_i: path delay,
$\beta_i \in \mathbb{C}$: complex path attenuation (fading).



Connection to the multiray model

Collect the samples of $g(t)$ into a row vector

$$\boldsymbol{g} = [g(0) \quad g(\tfrac{1}{P}) \quad \cdots \quad g(L - \tfrac{1}{P})]$$

Similar for $g(t - \tau_i)$: gives $\boldsymbol{g}_i = [g(k - \tau_i)]_{k=0,1/P,\cdots,L-1/P}$

The channel model can be written as

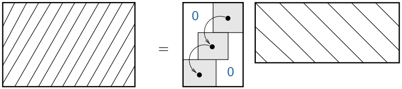
$$\boldsymbol{H}' = \begin{bmatrix} -\boldsymbol{h}_1- \\ \vdots \\ -\boldsymbol{h}_M- \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_r \\ | & & | \end{bmatrix} \begin{bmatrix} \beta_1 & & 0 \\ & \ddots & \\ 0 & & \beta_r \end{bmatrix} \begin{bmatrix} -\boldsymbol{g}_1- \\ \vdots \\ -\boldsymbol{g}_r- \end{bmatrix} = \boldsymbol{ABG}$$

where $\boldsymbol{a}_i = \boldsymbol{a}(\theta_i)$, and $\boldsymbol{B}$ is diagonal (fading matrix)

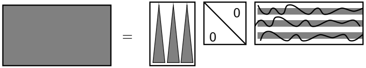
- $\boldsymbol{H}$ and $\boldsymbol{H}'$ are "the same", but reorganized: $\boldsymbol{H}$ is $MP \times L$ and $\boldsymbol{H}'$ is $M \times LP$.

Summary of properties

$\mathcal{X} = \mathcal{H}\mathcal{S}$



$\boldsymbol{H}' = \boldsymbol{ABG}$



Properties		$\mathcal{H}$	$\mathcal{S}$
macro	matrix	block Hankel $\text{col}(\mathcal{H}) = \text{col}(\mathcal{X})$	block Toeplitz $\text{row}(\mathcal{S}) = \text{row}(\mathcal{X})$
	modulation temporal	 cyclostationarity	constellation, non-Gaussian independence
parametric	temporal spatial	known $\boldsymbol{g}(\tau)$ known $\boldsymbol{a}(\theta)$	training: known $\{s_k\}$