EE 4715 Array Processing

3. Narrowband Models

April 2022

Models

Signal processing follows these steps:

- Physics
- Model
- Method
- Algorithm
- Implementation

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- Physics
- Model
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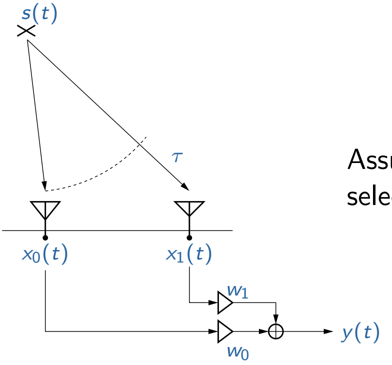
- Implementation

A **model** simplifies the physics to capture the essentials.

- Algorithms try to estimate model parameters: fewer is better!
- For the same physics, which model is suitable depends on the application.
- Whatever we don't model is put under the noise.

Spatial filtering

Using multiple antennas, we we can cancel an incoming interfering signal via **null steering**:



$$y(t) = w_0 s(t) + w_1 s(t - \tau)$$

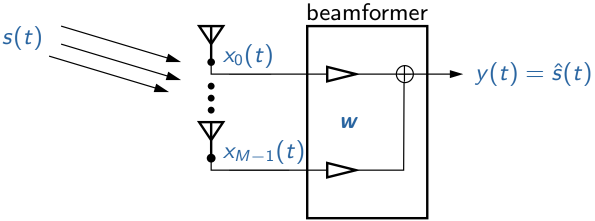
$$Y(\omega) = S(\omega)\left(w_0 + w_1 e^{-j\omega\tau}\right)$$

Assume we know τ, then $Y(\omega) = 0$ at a selected frequency ω_0 if

$$w_1 = -w_0 e^{j\omega_0\tau}$$

Beamforming

A **phased array** does linear beamforming by computing a weighted sum:



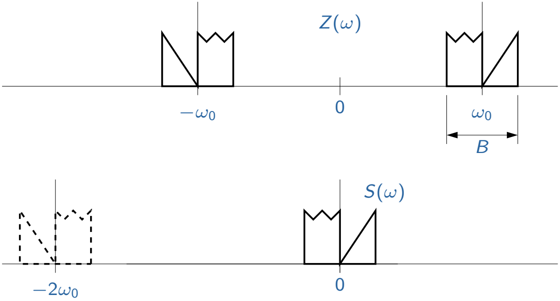
Using constant weights (not frequency-dependent), we can cancel a signal only at a specific frequency, but not at all frequencies.

Modulation and demodulation

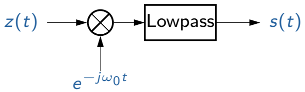
In digital communication systems, baseband signals $s(t)$ are modulated on a carrier before transmission:

$$z(t) = \text{real}\{s(t)e^{j\omega_0 t}\} = x(t)\cos(\omega_0 t) - y(t)\sin(\omega_0 t)$$

The baseband signal $s(t) = x(t) + jy(t)$ is the **complex envelope** of $z(t)$ and is recovered by demodulation



demodulation:



The narrowband condition

The bandpass signal $z(t)$ propagates through space. Suppose it is delayed by τ:

$$\begin{aligned} z_\tau(t) := z(t-\tau) &= \text{real}\{s(t-\tau)e^{j\omega_0(t-\tau)}\} \\ &= \text{real}\{\underbrace{s(t-\tau)e^{-j\omega_0\tau}}_{s_\tau(t)}\, e^{j\omega_0 t}\} \end{aligned}$$

- Let B be the bandwidth of $s(t)$. Then

$$s(t-\tau) = \frac{1}{2\pi}\int_{-B/2}^{B/2} S(\omega)e^{-j\omega\tau}e^{j\omega t}\mathrm{d}\omega \approx \frac{1}{2\pi}\int_{-B/2}^{B/2} S(\omega)e^{j\omega t}\mathrm{d}\omega = s(t)$$

  if $|\omega\tau| \ll \pi$ for all frequencies $|\omega| \le \frac{B}{2}$, so that $e^{-j\omega\tau} \approx 1$

- If the **narrowband condition** $B\tau \ll 2\pi$ holds, then

$$s_\tau(t) \approx s(t)e^{-j\omega_0\tau} \qquad \text{for } B\tau \ll 2\pi$$

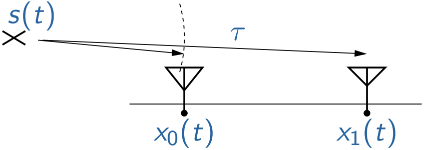
For narrowband signals, time delays smaller than the inverse bandwidth may be represented as phase shifts

The narrowband condition

The maximal delay depends on the situation!

Propagation across the antenna array: the maximum delay depends on the aperture D.

Let $W = \frac{B}{2\pi}$ be the bandwidth in Hz. Wavelength is $\lambda = c/f_0$.
Aperture is $\Delta = D/\lambda$ wavelengths.



- Maximal delay across the array is $\tau = D/c$
- The narrowband condition is satisfied if

$$B\tau \ll 2\pi \quad \Leftrightarrow \quad W\frac{D}{c} \ll 1 \quad \Leftrightarrow \quad W \ll \frac{f_0}{\Delta}$$

Examples

- Wireless communication: $f_0 \sim 1$ GHz, $\lambda \sim 30$ cm, $\Delta \sim 5$ wavelengths $\Rightarrow W \ll 30$ MHz.

  E.g., Bluetooth has $f_0 = 2400$ MHz, $W = 1$ MHz: narrowband across the array.

- Radio astronomy: e.g., $f_0 \sim 100$ MHz, $\lambda \sim 3$ m, $D \sim 100$ km, $\Delta \sim 33,000$ wavelengths $\Rightarrow W \ll 3$ kHz

If the narrowband condition is not satisfied, we can process signals in sufficiently narrow subbands.

Alternative interpretation

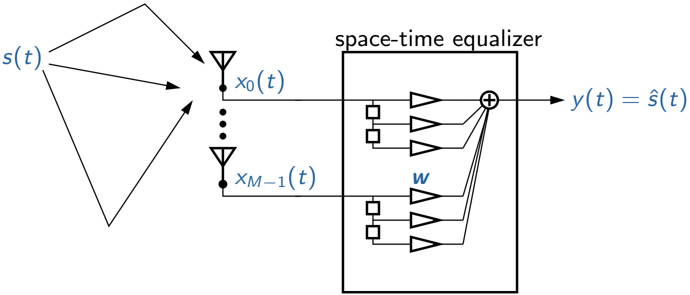
Sample at Nyquist: $f_s = W$, or $T_s = 1/W$. The narrowband condition is $\tau_{\max} \ll T_s$: **instantaneous** w.r.t. the sampling rate.

The narrowband condition

Due to multipath, e.g. reflections on distant buildings, delays can be much higher:

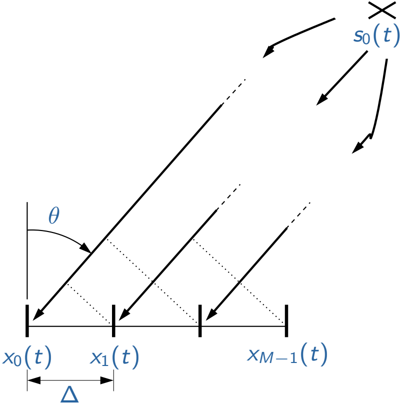
- Outdoor multipath creates pathlength differences of ~ 1 km, delays ~ 3 μs, $\Delta \sim 3000$ wavelengths $\Rightarrow W \ll 300$ kHz. This is barely satisfied by GSM (2G)

In general we require a **space-time equalizer** structure (Ch. 4).



Array response

Consider a uniform linear array with M elements.



$s_0(t)$: transmitted baseband signal, modulated at ω_0

$x_m(t)$: received baseband signal, after demodulation by ω_0

Array response

$$x_m(t) = a_m(\theta) \cdot s_0(t - T_m)e^{-j\omega_0 T_m}$$

- T_m: bulk delay, set $T_m = T_0 + \tau_m$

  Let $s(t) = s_0(t - T_0)e^{-j\omega_0 T_0}$ be the signal received at the first antenna. Then

  $$x_m(t) = a_m(\theta) \cdot s(t)e^{-\omega_0 \tau_m}$$

- $a_m(\theta)$: direction-dependent antenna response. We usually assume all antennas are identical ("uniform array").

Array response

Stack the received signals in a vector $\boldsymbol{x}(t)$:

$$\boldsymbol{x}(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \\ \vdots \\ x_{M-1}(t) \end{bmatrix} = \begin{bmatrix} 1 \\ e^{-j\omega_0\tau_1} \\ \vdots \\ e^{-j\omega_0\tau_{M-1}} \end{bmatrix} a_0(\theta)s(t)$$

- For a uniform linear array, all antennas have the same spacing d, so that $\tau_m = m\tau$

$$\omega_0\tau = -\omega_0\frac{d\sin(\theta)}{c} = -\frac{2\pi}{\lambda}d\sin(\theta) = -2\pi\Delta\sin(\theta)$$

where $\Delta = d/\lambda$ is the spacing in wavelengths.

$$\boldsymbol{x}(t) = \begin{bmatrix} 1 \\ e^{j2\pi\Delta\sin(\theta)} \\ \vdots \\ e^{j2\pi(M-1)\Delta\sin(\theta)} \end{bmatrix} a_0(\theta)s(t) =: \boldsymbol{a}(\theta)s(t)$$

Array response

$\boldsymbol{a}(\theta)$ is called the **array response vector**.

More in general,

$$\boldsymbol{a}(\theta) = \begin{bmatrix} 1 \\ e^{j\phi_1} \\ \vdots \\ e^{j\phi_{M-1}} \end{bmatrix} a_0(\theta)$$

With antenna positions $\boldsymbol{x}_m$,

$$\phi_m = -\omega_0 \tau_m = -\frac{\omega_0}{c} \boldsymbol{\zeta} \cdot \boldsymbol{x}_m = -\frac{2\pi}{\lambda} \boldsymbol{\zeta} \cdot \boldsymbol{x}_m$$

- With a direction vector in 2D $\boldsymbol{\zeta} = -\begin{bmatrix} \sin(\theta) \\ \cos(\theta) \end{bmatrix}$, the phase factors are

$$\phi_m = 2\pi \frac{x_m}{\lambda} \sin(\theta) + 2\pi \frac{y_m}{\lambda} \cos(\theta)$$

Array manifold

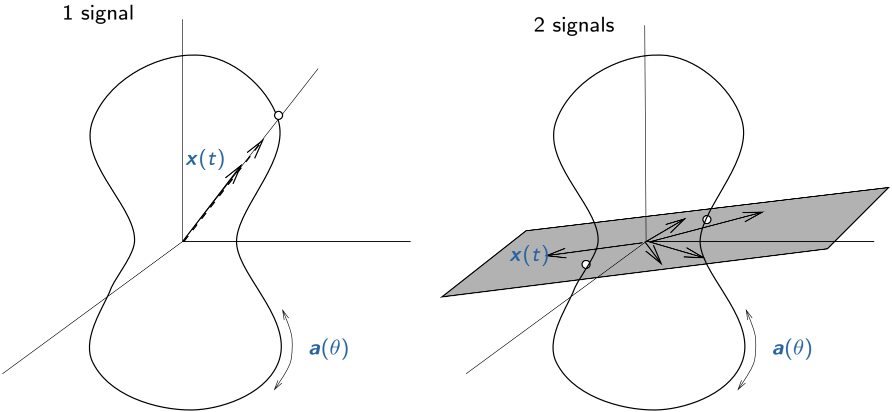
$$\boldsymbol{x}(t) = \boldsymbol{a}(\theta)s(t)$$

- Vary $s(t)$, then $\boldsymbol{x}(t)$ traces a line in $\mathbb{C}^M$
- Two independent sources: $\boldsymbol{x}(t) = \boldsymbol{a}(\theta_1)s_1(t) + \boldsymbol{a}(\theta_2)s_2(t)$ spans a plane

The **array manifold** $\mathcal{A}$ is the curve that $\boldsymbol{a}(\theta)$ describes in the M-dimensional complex vector space $\mathbb{C}^M$ when θ is varied over the domain of interest:

$$\mathcal{A} = \{\boldsymbol{a}(\theta) : 0 \leq \theta < 2\pi\}.$$

Array manifold



⇒ we can find θ from the intersections!

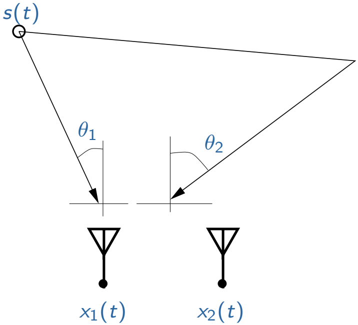
Array manifold

However, this has limitations:

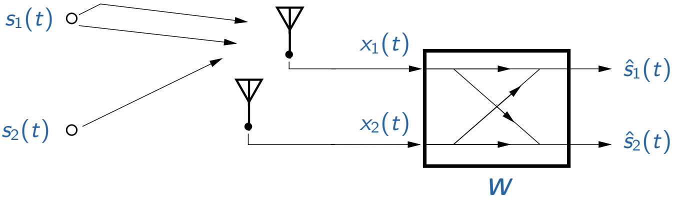
- One source with multipath:

$$\mathbf{x}(t) = \mathbf{a}(\theta_1)s(t) + \mathbf{a}(\theta_2)\beta s(t) = \{\mathbf{a}(\theta_1) + \beta\mathbf{a}(\theta_2)\}\, s(t) \quad = \mathbf{a}\, s(t)$$

  In general $\mathbf{a}$ is **not** on the manifold!



Beamforming



Consider two narrowband sources:

$$\mathbf{x}(t) = \mathbf{a}_1 s_1(t) + \mathbf{a}_2 s_2(t) = \mathbf{A}\mathbf{s}(t)\,, \qquad \mathbf{A} = [\mathbf{a}_1 \ \ \mathbf{a}_2],\ \mathbf{s}(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix}$$

- Source separation: estimate beamformers $\mathbf{w}_1, \mathbf{w}_2$ to separate and recover the sources:

$$y_1(t) = \mathbf{w}_1^{\mathrm{H}}\mathbf{x}(t) = \hat{s}_1(t)\,, \qquad y_2(t) = \mathbf{w}_2^{\mathrm{H}}\mathbf{x}(t) = \hat{s}_2(t)$$

This is **beamforming:** (linearly) combine the antenna outputs

Beamforming

- With $\boldsymbol{W} = [\boldsymbol{w}_1 \quad \boldsymbol{w}_2]$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \boldsymbol{W}^{\mathrm{H}} \boldsymbol{x}(t) = \boldsymbol{s}(t) \quad \Leftrightarrow \quad \boldsymbol{W}^{\mathrm{H}} \boldsymbol{A} = \boldsymbol{I} \quad \Leftrightarrow \quad \boldsymbol{W} = \boldsymbol{A}(\boldsymbol{A}^{\mathrm{H}} \boldsymbol{A})^{-1}$$

  The separating beamformers are found from a pseudo-inverse of $\boldsymbol{A}$, as the rows of $\boldsymbol{W}^{\mathrm{H}}$.

If we know $\boldsymbol{A}$, we can find the beamformers by inversion. This is a "zero-forcing beamformer".

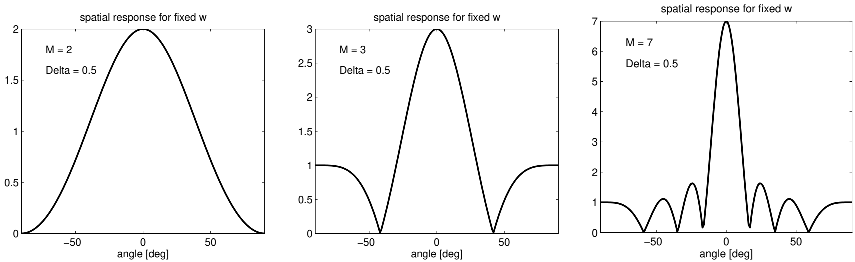
With noise added, we might have noise enhancement. MMSE (Wiener) beamformers take this into account (Ch.5).

Beamforming

Take a single unit-power source from direction θ, and look at the beamformer output $y(t) = \boldsymbol{w}^{\mathsf{H}}\boldsymbol{x}(t) = \boldsymbol{w}^{\mathsf{H}}\boldsymbol{a}(\theta)s(t)$

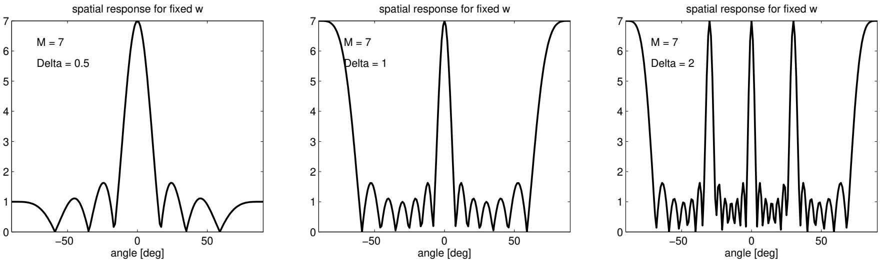
Compute the output power $P_y(\theta) = \mathrm{E}[\,|y(t)|^2] = |\boldsymbol{w}^{\mathsf{H}}\boldsymbol{a}(\theta)|^2$

- First consider a **fixed** beamformer $\boldsymbol{w} = [1 \cdots 1]^{\mathsf{T}}$.



This beamformer is tuned to receive from $\theta = 0^\circ$ (broadside).

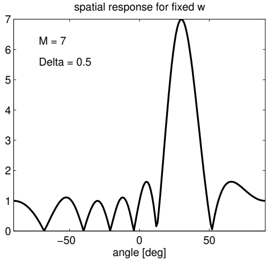
Beamforming



Grating lobes appear if the antenna spacing is larger than $\lambda/2$. Then there can be a response also for sources from directions $\neq 0^\circ$.

Beamforming

Let's take $\boldsymbol{w} = \boldsymbol{a}(30^\circ)$:

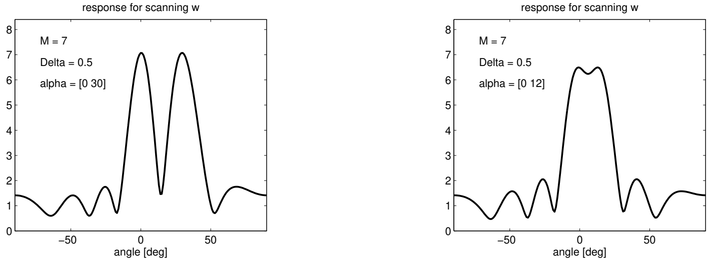


The array now has a maximal response to signals from 30°.

Beamforming

To do direction finding, we take a *varying* beamformer of the form $\mathbf{w}(\theta) = \mathbf{a}(\theta)$ and we vary θ to scan space.

- With one source in direction θ_1, this peaks at the source direction.
- With two sources, $\mathbf{x}(t) = \mathbf{a}(\theta_1)s_1(t) + \mathbf{a}(\theta_2)s_2(t)$, the output power has peaks at θ_1 and θ_2



The resolution is determined by the beam width, $\frac{2\pi}{M}$.

Narrowband correlation models

Consider again the model $\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{s}(t)$, and sample $\boldsymbol{x}(t)$:

$$\boldsymbol{x}_n = \boldsymbol{A}\boldsymbol{s}_n\,, \quad n = 0, \cdots, N-1$$

We consider a stochastic model:

$$\mathrm{E}[\boldsymbol{s}_n] = 0\,, \qquad \boldsymbol{R}_{\boldsymbol{s}} = \mathrm{E}[\boldsymbol{s}_n\boldsymbol{s}_n^{\mathrm{H}}]$$

Then the data satisfies

$$\mathrm{E}[\boldsymbol{x}_n] = 0\,, \qquad \boldsymbol{R}_{\boldsymbol{x}} = \mathrm{E}[\boldsymbol{x}_n\boldsymbol{x}_n^{\mathrm{H}}] = \boldsymbol{A}\boldsymbol{R}_{\boldsymbol{s}}\boldsymbol{A}^{\mathrm{H}}$$

- With d sources, generally, $\boldsymbol{R}_{\boldsymbol{s}}$ is a full $d \times d$ matrix.
- if $M > d$ then $\boldsymbol{R}_{\boldsymbol{x}}$ is rank deficient (rank d)

Narrowband correlation models

Two important special cases are:

- **Independent sources:** the source covariance is diagonal,

$$\boldsymbol{R_s} = \boldsymbol{\Sigma_s} = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_d^2 \end{bmatrix}$$

- **Independent sources with equal variances:**

$$\boldsymbol{R_s} = \sigma_{\boldsymbol{s}}^2 \boldsymbol{I} \qquad \Rightarrow \qquad \boldsymbol{R_x} = \sigma_{\boldsymbol{s}}^2 \boldsymbol{A}\boldsymbol{A}^{\mathsf{H}} \,,$$

Narrowband correlation models

With additive noise (zero mean, WSS, independent from the signals):

$$\boldsymbol{x}_n = \boldsymbol{A}_n \boldsymbol{s}_n + \boldsymbol{n}_n \qquad \Rightarrow \qquad \boldsymbol{R}_{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{R}_{\boldsymbol{s}}\boldsymbol{A}^{\mathrm{H}} + \boldsymbol{R}_{\boldsymbol{n}}$$

- **Independent noise:** the noise covariance $\boldsymbol{R}_n$ is diagonal
- **Spatially white noise:** $\boldsymbol{R}_n = \sigma_{\boldsymbol{n}}^2 \boldsymbol{I}$

Sample correlation matrices

The sample covariance matrix is

$$\hat{\boldsymbol{R}}_{\boldsymbol{x}} = \frac{1}{N} \sum_n \boldsymbol{x}_n \boldsymbol{x}_n^{\mathrm{H}}$$

It estimates the data covariance matrix $\boldsymbol{R}_{\boldsymbol{x}}$. Then

$$\hat{\boldsymbol{R}}_{\boldsymbol{x}} = \boldsymbol{A}\hat{\boldsymbol{R}}_{\boldsymbol{s}}\boldsymbol{A}^{\mathrm{H}} + \hat{\boldsymbol{R}}_{\boldsymbol{n}} + \text{(cross terms)}$$

This is an unbiased estimate:

$$\mathrm{E}[\hat{\boldsymbol{R}}_{\boldsymbol{x}}] = \boldsymbol{R}_{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{R}_{\boldsymbol{s}}\boldsymbol{A}^{\mathrm{H}} + \boldsymbol{R}_{\boldsymbol{n}} \,.$$

- Often we stack the samples $\boldsymbol{x}_n$ in a matrix $\boldsymbol{X} = [\boldsymbol{x}_0, \cdots, \boldsymbol{x}_{N-1}]$:

  $$\boldsymbol{X} = \boldsymbol{A}\boldsymbol{S}$$

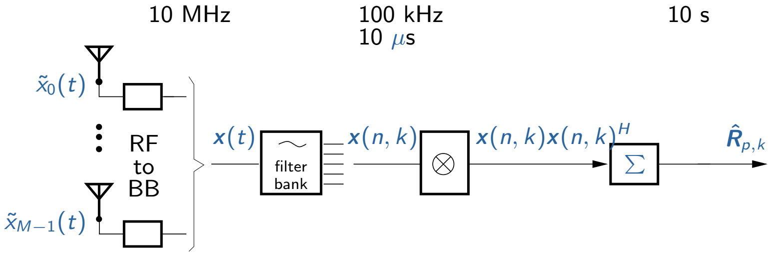
  Then $\hat{\boldsymbol{R}}_{\boldsymbol{x}} = \frac{1}{N}\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}$

Application: radio astronomy

Recall the measurement equation in radio astronomy:

$$V(\omega, \boldsymbol{b}) = \int I(\omega, \zeta) e^{-j\frac{\omega}{c}\zeta \cdot \boldsymbol{b}} \mathrm{d}\zeta$$

- $V(\omega, \boldsymbol{b})$: received cross power spectral density over baseline $\boldsymbol{b}$
- $I(\omega, \zeta)$: sky brightness in look direction ζ



Application: radio astronomy
Data acquisition

- The received signals $\tilde{x}_m(t)$ are moved to baseband ($x_m(t)$), sampled and split into narrow subbands ($x_m[n,k]$)

  Stack these into a vector $\boldsymbol{x}[n,k]$ with n: time index, k: frequency bin

- Correlate over short time intervals with index p (order seconds):

$$\hat{\boldsymbol{R}}_{p,k} = \frac{1}{N} \sum_{n=(p-1)N}^{pN-1} \boldsymbol{x}[n,k]\boldsymbol{x}^{\mathrm{H}}[n,k]$$

  N is about 1,000 to 1,000,000. The maximum correlation time depends on stationarity: rotation of the earth vs baseline length.

Application: radio astronomy

Basic covariance data model

- One source without noise:

$$\mathbf{x}[n,k] = \mathbf{a}_q[n,k]s_q[n,k]$$

where the array response vector $\mathbf{a}_q[n,k]$ has entries

$$a_m = e^{j\phi_m}\,, \qquad \phi_m = -\frac{\omega}{c}\zeta_q \cdot \mathbf{x}_m \qquad \begin{cases} \mathbf{x}_m\text{: antenna position} \\ \zeta_q\text{: source direction} \end{cases}$$

- Full point source model: $\mathbf{x}[n,k] = \sum_{q=1}^{Q} \mathbf{a}_q[n,k]s_q[n,k] + \mathbf{n}[n,k]$

Application: radio astronomy

The sky sources and the noise are modeled as independent zero mean complex Gaussians (WSS). We drop the frequency index k.

Let $\boldsymbol{R}_p = \mathrm{E}\left[\boldsymbol{x}[n]\,\boldsymbol{x}^{\mathrm{H}}[n]\right]$

$$\boldsymbol{R}_p = \boldsymbol{A}_p \boldsymbol{\Sigma}_{\boldsymbol{s}} \boldsymbol{A}_p^{\mathrm{H}} + \boldsymbol{\Sigma}_{\boldsymbol{n}}\,, \qquad p = 0, 1, 2, \cdots$$

where $\boldsymbol{\Sigma}_{\boldsymbol{s}}$ and $\boldsymbol{\Sigma}_{\boldsymbol{n}}$ are diagonal matrices with the sky source and noise powers, resp.

$\boldsymbol{\Sigma}_{\boldsymbol{s}}$ is a diagonal containing the source brightness $I(\zeta_q)$

$\boldsymbol{A}_p$ has columns $\boldsymbol{a}_q(p)$ with entries $e^{-j\boldsymbol{z}_m(p)\cdot\zeta_q}$

$\boldsymbol{z}_m(p)$: the normalized location of antenna m at time interval p

ζ_q: the direction of the qth source

Application: radio astronomy

Image formation

Direct beamforming (matched filter): to measure power in direction ζ, take $\boldsymbol{w}(\zeta, p)$ with entries $e^{-j\boldsymbol{z}_m(p)\cdot\zeta}$

- Output power of the beamformer:

$$\hat{I}_D(\zeta) = \sum_p \boldsymbol{w}(\zeta, p)^{\mathrm{H}} \hat{\boldsymbol{R}}_p \boldsymbol{w}(\zeta, p)$$

- Insert the model:

$$\begin{aligned} I_D(\zeta) &= \sum_p \boldsymbol{w}(\zeta, p)^{\mathrm{H}} \boldsymbol{A}_p \Sigma_{\boldsymbol{s}} \boldsymbol{A}_p^{\mathrm{H}} \boldsymbol{w}(\zeta, p) \\ &= \sum_p \sum_q \boldsymbol{w}(\zeta, p)^{\mathrm{H}} \boldsymbol{a}_q(p)\, I(\zeta_q)\, \boldsymbol{a}_q(p)^{\mathrm{H}} \boldsymbol{w}(\zeta, p) \\ &= \sum_q I(\zeta_q) B(\zeta - \zeta_q) \end{aligned}$$

where $B(\zeta) := \sum_{i,j,p} e^{-j(\boldsymbol{z}_i(p) - \boldsymbol{z}_j(p))\cdot\zeta}$ is the dirty beam.

Summary

- The **narrowband condition** allows to translate delays into phases
  If it does not hold, we could split the received data into narrow subbands, or consider space-time processing
- The **narrowband data model** $\boldsymbol{X} = \boldsymbol{AS} + \boldsymbol{N}$ represents an instantaneous mixture of sources
- The corresponding **covariance model** is $\boldsymbol{R_x} = \boldsymbol{AR_sA}^{\mathrm{H}} + \boldsymbol{R_n}$
- A beamformer computes $y(t) = \boldsymbol{w}^{\mathrm{H}}\boldsymbol{x}(t)$.
  The output power is $P_y(\boldsymbol{w}) = \mathrm{E}[|y(t)|^2] = \boldsymbol{w}^{\mathrm{H}}\boldsymbol{R_x}\boldsymbol{w}$
  If we scan $\boldsymbol{w}$ we obtain a spatial spectrum.

A variety of beamforming methods exist to compute various spectrum estimates. They are used for **image formation** and **direction finding**.