EE 4715 Array Processing

2. Wave Propagation

April 2022

TU Delft

Wave propagation

Consider a position in 3D space: $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and a signal $s(\mathbf{x}, t)$.

From Maxwell follows the **wave equation** (RF, but a similar equation holds for acoustic waves):

$$\nabla^2 s(\mathbf{x}, t) = \frac{1}{c^2} \frac{\partial^2 s(\mathbf{x}, t)}{\partial t^2} \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- Coefficient c: later shown to be the propagation speed.
- This equation is for undamped (lossless) propagation.

Wave propagation

Wave equation: $\nabla^2 s(\boldsymbol{x}, t) = \frac{1}{c^2} \frac{\partial^2 s(\boldsymbol{x},t)}{\partial t^2}$.

- Typical solution (eigenfunction):

$$s(\boldsymbol{x}, t) = e^{j(\omega t - \boldsymbol{k} \cdot \boldsymbol{x})}$$

$\boldsymbol{k} \cdot \boldsymbol{x}$ denotes an inner product, same as $\boldsymbol{k}^{\mathsf{T}} \boldsymbol{x}$.

- To verify, insert into the wave equation:

$$k^2 s(\boldsymbol{x}, t) = \frac{\omega^2}{c^2} s(\boldsymbol{x}, t)\,, \qquad \text{where} \quad k = \|\boldsymbol{k}\|$$

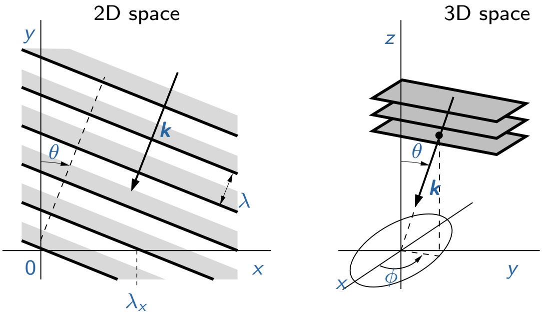
which is correct, under the constraint $k = \dfrac{\omega}{c}$.

k is called the **wavenumber** (or spatial frequency), in rad/m, and $\boldsymbol{k}$ is called the wavenumber vector.

Wave propagation

$s(\boldsymbol{x}, t) = e^{j(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}$ describes a **monochromatic plane wave**:

Pick a constant C, and consider $\omega t - \boldsymbol{k} \cdot \boldsymbol{x} = C$. These are (2D) lines or (3D) planes where $s(\boldsymbol{x}, t)$ is constant.



The wave propagates in the direction of $\boldsymbol{k}$. The wavefronts are planes orthogonal to $\boldsymbol{k}$.

Wave propagation

Consider a specific wavefront: $\omega t - \boldsymbol{k} \cdot \boldsymbol{x} = C$.

- The time period of a cycle is $T = \dfrac{2\pi}{\omega}$.
- $\omega T = 2\pi = \boldsymbol{k} \cdot \boldsymbol{x} \;\Rightarrow$ During time T, the wavefront moves in the direction $\boldsymbol{k}$ by a distance $\lambda = \dfrac{2\pi}{k}$.

Then, with $k = \dfrac{\omega}{c}$:

$$\text{velocity} = \frac{\lambda}{T} = \frac{\omega}{k} = c$$

showing c is the propagation speed, and λ is the wavelength.

$$\Rightarrow \quad k = \frac{\omega}{c} = \frac{2\pi}{\lambda} \qquad \text{“spatial frequency”}$$

Wave propagation

Let ζ be a unit-norm vector in the direction of $\boldsymbol{k}$, then

$$\omega t - \boldsymbol{k} \cdot \boldsymbol{x} = \omega \left(t - \frac{1}{c} \zeta \cdot \boldsymbol{x} \right)$$

This take care of the constraint.
If c is constant, then ω and ζ parametrize the wave.

Common parametrizations for the direction vector ζ:

- 2D: $\zeta = -\begin{bmatrix} \sin(\theta) \\ \cos(\theta) \end{bmatrix}$
- 3D: $\zeta = -\begin{bmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{bmatrix}, \qquad \boldsymbol{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \frac{\omega}{c}\zeta = \frac{2\pi}{\lambda}\zeta$

Wave propagation

Now generalize to a wave from direction $\boldsymbol{\zeta}$ with multiple frequencies (a wideband signal):

$$s\left(t - \frac{1}{c}\boldsymbol{\zeta} \cdot \mathbf{x}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{j\omega(t - \frac{1}{c}\boldsymbol{\zeta} \cdot \mathbf{x})} \mathrm{d}\omega$$

So we can turn any signal $s(t)$ into a propagating plane wave signal by taking $s(\mathbf{x}, t) = s(t - \frac{1}{c}\boldsymbol{\zeta} \cdot \mathbf{x})$.

Dispersion and diffraction

- If the medium is not lossless, waves **attenuate** as they propagate.
- Near field: point sources generate **spherical waves** (which attenuate as $1/R$ as they propagate)
- Non-homogeneous medium: different propagation speeds in different areas

  Reflection and refraction at interfaces (source of multipath)
- **Diffraction:** waves bend around objects (e.g. knife-edge diffraction)
- **Dispersion:** Frequency-dependent propagation velocities (e.g. prism; rainbow; ionosphere; salinity layers in ocean)
- **Doppler shifts** caused by a moving source, receiver, or objects in the medium

In most of the course, we consider only multipath effects.

Spatial Fourier transform

$$S(\boldsymbol{k},\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\boldsymbol{x},t)e^{-j(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})}\mathrm{d}\boldsymbol{x}\mathrm{d}t$$

$$s(\boldsymbol{x},t) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\boldsymbol{k},\omega)\, e^{j(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})}\mathrm{d}\boldsymbol{k}\mathrm{d}\omega$$

View $s(\boldsymbol{x},t)$ as a sum of monochromatic plane waves $e^{j(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})}$.

- Single monochromatic point source:

$$s(\boldsymbol{x},t) = e^{j(\omega_0 t-\boldsymbol{k}_0\cdot\boldsymbol{x})} \quad \Leftrightarrow \quad S(\boldsymbol{k},\omega) = (2\pi)^4\delta(\omega-\omega_0)\delta(\boldsymbol{k}-\boldsymbol{k}_0)$$

- Wideband point source:

$$S(\boldsymbol{k},\omega) = (2\pi)^3 S(\omega)\delta\left(\boldsymbol{k}-\boldsymbol{k}_0\right)$$

- Further extension: spatially extended sources (e.g., noise sources); not in this course

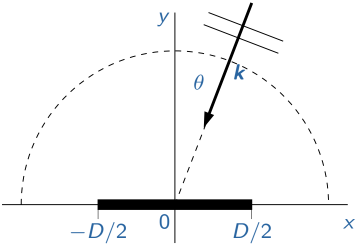
Apertures

Telescopes (dishes) and antennas have a finite **aperture** (collecting surface with dimension D). This is modeled by a window function $w(\boldsymbol{x})$:

$$y(\boldsymbol{x}, t) = w(\boldsymbol{x}) s(\boldsymbol{x}, t)$$

- **Linear aperture** (slit):

$$h(x) = \begin{cases} 1, & |x| < D/2 \\ 0, & \text{otherwise} \end{cases} \quad \Leftrightarrow \quad \boldsymbol{w}(\boldsymbol{x}) = h(x)\delta(y)\delta(z)\,,$$



Apertures

Apply the space-time Fourier transform to $y(\boldsymbol{x}, t) = w(\boldsymbol{x})s(\boldsymbol{x}, t)$:

$$Y(\boldsymbol{k}, \omega) = \frac{1}{(2\pi)^3} W(\boldsymbol{k}) * S(\boldsymbol{k}, \omega) = \frac{1}{(2\pi)^3} \int W(\boldsymbol{k} - \boldsymbol{p}) S(\boldsymbol{p}, \omega) \mathrm{d}\boldsymbol{p}$$

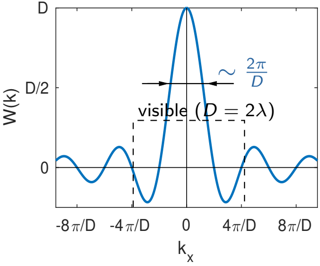
with

$$W(\boldsymbol{k}) = \int w(\boldsymbol{x})\, e^{j\, \boldsymbol{k} \cdot \boldsymbol{x}} \mathrm{d}\boldsymbol{x}$$

- Linear aperture:

$$W(\boldsymbol{k}) = \frac{\sin(k_x D/2)}{k_x/2}$$

- Zero crossings at multiples of $\frac{2\pi}{D}$



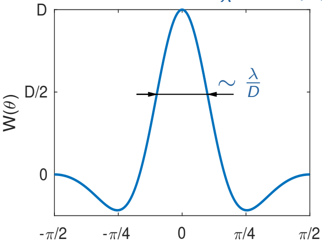
Apertures

- The observed signal for a point source from direction $\boldsymbol{k}_0$ has space-time spectrum:

$$Y(\boldsymbol{k},\omega) = W(\boldsymbol{k}) * S(\omega)\delta(\boldsymbol{k}-\boldsymbol{k}_0) = S(\omega)W(\boldsymbol{k}-\boldsymbol{k}_0)\,; \qquad \boldsymbol{k}_0 = \frac{\omega}{c}\zeta_0$$

  The source direction is smeared out over a range of directions: loss of resolution.

- Consider $W(\boldsymbol{k})$ with $k_x = -\frac{2\pi}{\lambda}\sin(\theta)$: this gives (with abuse of notation) $W(\theta) = D\,\dfrac{\sin(\frac{D}{\lambda}\pi\sin(\theta))}{\frac{D}{\lambda}\pi\sin(\theta)}$



(for $D = 2\lambda$)

Apertures

$$W(k_x) = \frac{\sin(k_x D/2)}{k_x/2} \text{ with } k_x = -\frac{2\pi}{\lambda}\sin(\theta)$$

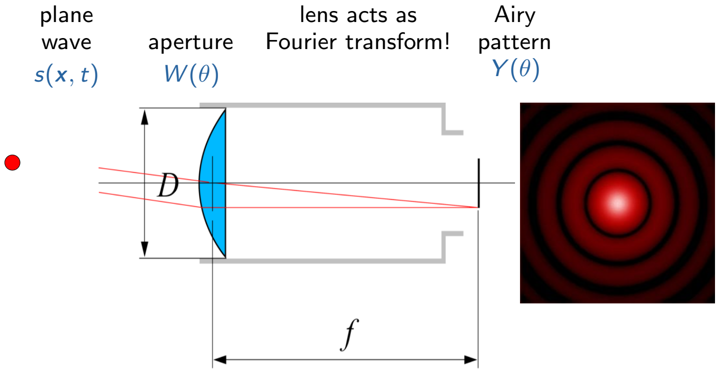
- $k_x \in [-\frac{2\pi}{\lambda}, \frac{2\pi}{\lambda}]$: not the entire $W(k_x)$ is "visible".
- The convolution $Y(\boldsymbol{k},\omega) = W(\boldsymbol{k}) * S(\omega)\delta(\boldsymbol{k}-\boldsymbol{k}_0)$ does not convert to a convolution in θ:

$$\begin{aligned} Y(\theta,\omega) &= S(\omega)W(\theta;\theta_0) \\ W(\theta;\theta_0) &= D\,\frac{\sin(\frac{D}{\lambda}\pi[\sin(\theta)-\sin(\theta_0)])}{\frac{D}{\lambda}\pi[\sin(\theta)-\sin(\theta_0)]}\,. \end{aligned}$$

Implication: if we plot $W(\theta)$ for $\theta_0 = 0$, then this is not the whole story!

Apertures

Analogy to optical systems – a simple telescope:

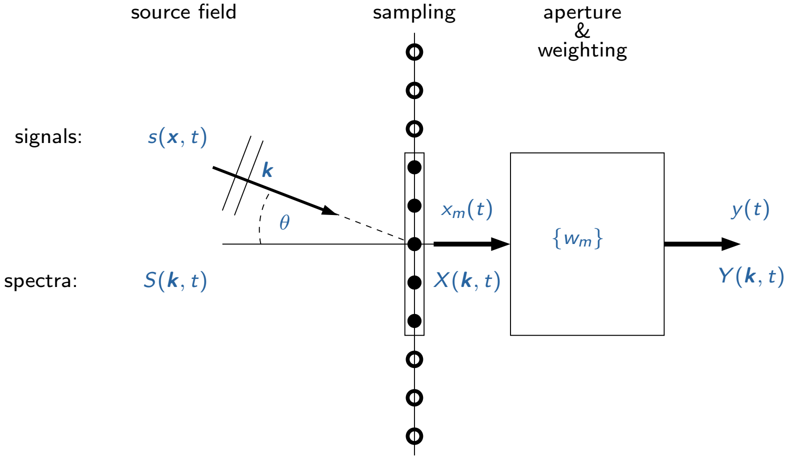


A circular aperture results in an "Airy" diffraction pattern. The main beam has size $\sim 1.22\frac{\lambda}{D}$. This determines the resolution in diffraction-limited systems.

Spatial sampling

We sample space with an antenna array (the elements are simple omnidirectional antennas, or dishes, or sometimes subarrays)

The array covers a certain aperture (so two effects play a role).



First: infinite number of sensors

Consider sampling in a 1D scenario, with antenna spacing d:

$$x_m(t) = s(md, t)\,, \qquad m = \cdots, -1, 0, 1, \cdots$$

where the original signal $s(x, t)$ has space-time spectrum

$$S(k, \omega) = \int\int s(x, t) e^{-j(\omega t - kx)} \mathrm{d}x\mathrm{d}t = \int \left[\int s(x, t) e^{jkx} \mathrm{d}x\right] e^{-j\omega t} \mathrm{d}t$$

- Note that the transformations over space and time are decoupled. For simplicity of notation, let us consider only the Fourier transformation over space:

$$S(k, t) = \int s(x, t) e^{jkx} \mathrm{d}x\,, \qquad s(x, t) = \frac{1}{2\pi} \int S(k, t) e^{-jkx} \mathrm{d}k$$

Infinite number of sensors

Define the spatial sampling frequency as $k_s = \dfrac{2\pi}{d}$.

$$
\begin{aligned}
s(x,t) &= \frac{1}{2\pi} \sum_n \int_{nk_s-k_s/2}^{nk_s+k_s/2} S(k,t) e^{-jkx} \mathrm{d}k \\
&= \frac{1}{2\pi} \sum_n \int_{-k_s/2}^{k_s/2} S(k-nk_s,t) e^{-jkx} e^{-jnk_s x} \mathrm{d}k
\end{aligned}
$$

Sampling x and using $nk_s md = 2\pi nm$,

$$
\begin{aligned}
x_m(t) = s(md,t) &= \frac{1}{2\pi} \sum_n \int_{-k_s/2}^{k_s/2} S(k-nk_s,t) e^{-jkdm} e^{-j2\pi nm} \mathrm{d}k \\
&= \frac{1}{2\pi} \int_{-k_s/2}^{k_s/2} \left[\sum_n S(k-nk_s,t) \right] e^{-jkdm} \mathrm{d}k
\end{aligned}
$$

Infinite number of sensors

The spatial spectrum is then defined from the samples $x_m(t)$ as (cf. DTFT)

$$X(k,t) = \sum_m x_m(t) e^{jkdm} \quad \Leftrightarrow \quad x_m(t) = \frac{d}{2\pi} \int_{-k_s/2}^{k_s/2} X(k,t) e^{-jkdm} \mathrm{d}k$$

Defined for $-k_s/2 \leq k \leq k_s/2$, and periodic elsewhere.

Then, comparing to the spectrum of the unsampled signal,

$$X(k,t) = \frac{1}{d} \sum_n S(k - nk_s, t), \qquad -\frac{1}{2}k_s \leq k \leq \frac{1}{2}k_s$$

and periodic elsewhere. The sum represents **spatial aliasing**.

Infinite number of sensors

No aliasing if $S(k,t)$ has limited support. The spatial Nyquist condition is

$$|k| < \frac{k_s}{2} \quad \Leftrightarrow \quad |\omega| < \frac{c}{d}\pi \quad \Leftrightarrow \quad |f| < \frac{c}{2d}$$

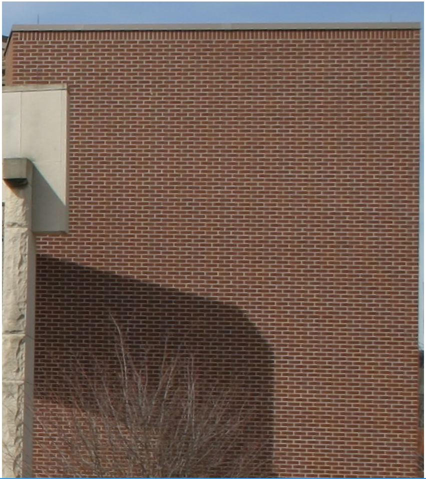
where $\omega = 2\pi f$, with f in Hz. Alternatively,

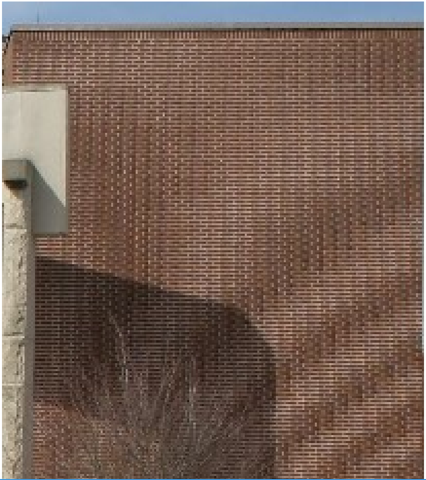
$$d < \frac{c}{2B} \quad \Leftrightarrow \quad d < \tfrac{1}{2}\lambda_{\min}$$

where $B = f_{\max}$ is the bandwidth of the signal in Hz, and $\lambda_{\min}$ the corresponding wavelength.

Moire effect

- In optics, spatial aliasing creates Moiré patterns, avoided using an optical low pass filter (blurring filter)





Finite number of sensors

Now, we combine sampling with an aperture function to obtain a finite number of antennas.

For M antennas on a line, the weights are

$$w_m = \begin{cases} 1, & m = 0, \cdots, M-1 \\ 0, & \text{elsewhere} \end{cases}$$

The spatial spectrum of the sampled signal using M sensors is

$$Y(k,t) = \sum_{m=0}^{M-1} x_m(t) e^{jk \cdot x_m} = \sum_{m=-\infty}^{\infty} w_m \, s(md, t) e^{jkmd}$$

Finite number of sensors

With a similar derivation as before, we find

$$Y(k,t) = \frac{d}{2\pi}\int_{-k_s/2}^{k_s/2} W(k-p)X(p,t)\mathrm{d}p$$

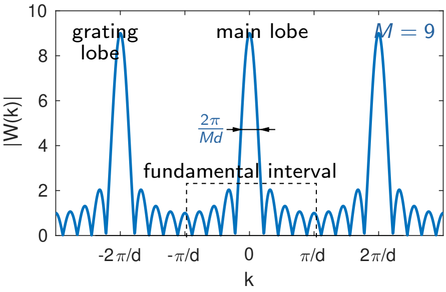
This is a circular convolution of $X(p,t)$ with the aperture function

$$W(k) = \sum_{m=-\infty}^{\infty} w_m e^{jkmd} = \sum_{m=0}^{M-1} e^{jkmd} = \frac{\sin(kMd/2)}{\sin(kd/2)} e^{jk(M-1)d/2}$$

The phase defines the array center (we drop it for now).

The amplitude $|W(k)|$ is periodic with period $k_s = \frac{2\pi}{d}$.

Finite number of sensors



$$W(k) = \frac{\sin(kMd/2)}{\sin(kd/2)}$$

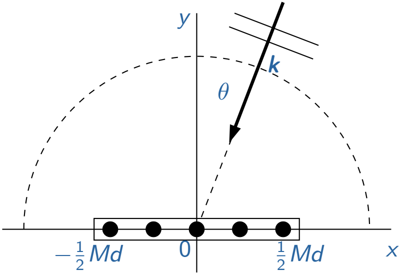
- $W(k)$ is a "periodic sinc function" (Dirichlet), with period $k_s = \frac{2\pi}{d}$
- Zero crossings at $k = \frac{2\pi}{Md} = k_s/M$: determines main lobe width
- Peak height M: the **array gain**

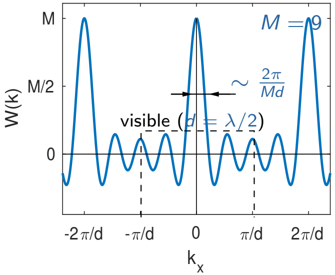
Finite number of sensors

- The convolution with $W(k)$ smears the observed spectrum; the side lobes create **confusion**: cannot know if a small peak is a source or a sidelobe.
- **Resolution** is the distance between two point sources that can still be discerned: about $\frac{2\pi}{Md}$.
  Note that $D = Md$ is the aperture: this determines resolution.
- Main lobe is repeated: **grating lobes** might appear in the spectrum.

1D array in a 2D propagation scenario

Now we extend to a uniform linear array in a 2D scenario. Only k_x is observed.



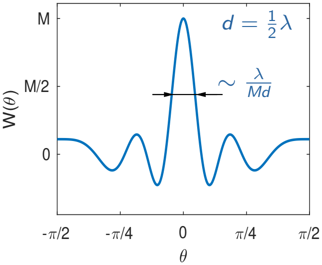


$$W(\boldsymbol{k}) = W(k_x) = \frac{\sin(k_x Md/2)}{\sin(k_x d/2)}$$

1D array in a 2D propagation scenario

With $k_x = -\frac{2\pi}{\lambda}\sin(\theta)$:

$$W(\theta) = \frac{\sin(\frac{Md}{\lambda}\pi\sin(\theta))}{\sin(\frac{d}{\lambda}\pi\sin(\theta))}$$

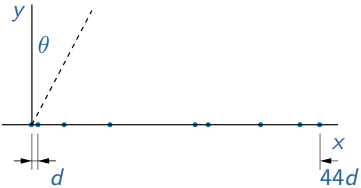


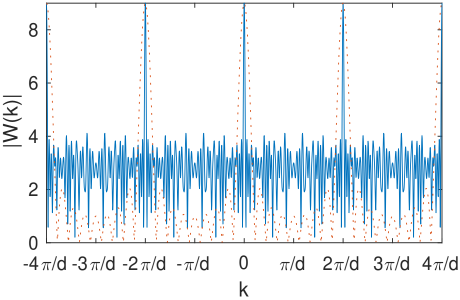
- The part of $W(k_x)$ visible in $W(\theta)$ is the interval $k_x \in [-\frac{2\pi}{\lambda}, \frac{2\pi}{\lambda}]$. If $d < \frac{1}{2}\lambda$, then the visible part is within one period of $W(k)$. For larger d, the grating lobes may become visible.
- Angular resolution: $\dfrac{\lambda}{Md}$, based on distance between zero crossings

Non-uniform sampling

We don't need to sample uniformly. We could sample randomly, or use specific antenna positions to guarantee both Nyquist (minimal spacing $d = \frac{1}{2}\lambda$) and maximal aperture: **sparse linear arrays**.

$$M = 9: \quad x_m = [1,\ 4,\ 7,\ 13,\ 2,\ 8,\ 6,\ 3] \cdot d\,.$$





Correlation processing

Sometimes we are not interested in $s(t)$:

- the source is random (not quite informative)
- interested in propagation parameters: direction, source powers

Then consider *statistics*, e.g.

mean:	$\mathrm{E}[s(t)]$
variance:	$\mathrm{E}\left[\lvert s(t) - \mathrm{E}[s(t)] \rvert^2 \right]$
autocorrelation:	$r_s(t, t') = \mathrm{E}[s(t) s^*(t')]$

* denotes a complex conjugate. We assume complex signals

Correlation processing

We often assume zero mean and wide-sense stationary processes:

$$\mathrm{E}[s(t)] = 0\,, \qquad r_s(\tau) = \mathrm{E}[s(t+\tau)s^*(t)]$$

Recall: the **power spectral density** is the Fourier transform of the autocorrelation function:

$$R_s(\omega) = \int r_s(\tau)e^{-j\omega\tau}\mathrm{d}\tau$$

White noise: $r_s(\tau) = \sigma_s^2\delta(\tau)$, and $R_s(\omega) = \sigma_s^2$ is constant.

We will now extend this to space-time stochastic processes, also called **random fields**.

Monochromatic plane wave

Consider a monochromatic plane wave with frequency ω_0, direction $\boldsymbol{k}_0$:

$$s(\boldsymbol{x}, t) = \alpha\, e^{j(\omega_0 t - \boldsymbol{k}_0 \cdot \boldsymbol{x})}$$

where α is a random (complex) amplitude with $\mathrm{E}[\,|\alpha|^2] = P$.

- At position $\boldsymbol{x}$, we measure the autocorrelation function

$$r_s(\tau) = \mathrm{E}[s(\boldsymbol{x}, t+\tau) s^*(\boldsymbol{x}, t)] = P\, e^{j\omega_0 \tau}$$

- Similarly, we can cross-correlate two locations $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$:

$$\begin{aligned} r_s(\boldsymbol{x}_0, \boldsymbol{x}_1, \tau) &= \mathrm{E}[s(\boldsymbol{x}_1, t+\tau) s^*(\boldsymbol{x}_0, t)] = \mathrm{E}\left[|\alpha|^2 e^{j(\omega_0 \tau - \boldsymbol{k}_0 \cdot (\boldsymbol{x}_1 - \boldsymbol{x}_0))}\right] \\ &= P\, e^{j\omega_0 \tau}\, e^{-j \boldsymbol{k}_0 \cdot \boldsymbol{b}} \end{aligned}$$

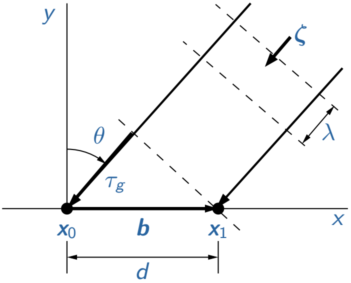
  This depends only on the **baseline:** $\boldsymbol{b} = \boldsymbol{x}_1 - \boldsymbol{x}_0$ (called a **homogeneous** random field)

Monochromatic plane wave

Recall: $\boldsymbol{k}_0 = \frac{\omega_0}{c}\zeta$ where ζ is a unit-norm direction vector. Also let $d = \|\boldsymbol{b}\|$, the baseline length

$$r_s(\boldsymbol{b}, \tau) = P\, e^{j\omega_0\tau} e^{-j\omega_0\tau_g}\,, \qquad \tau_g = \frac{\zeta \cdot \boldsymbol{b}}{c} = -\frac{d}{c}\sin(\theta)$$

- τ_g is the **geometric delay**; if we know τ_g we can find θ.
- We often try to estimate τ_g from the phase $e^{-j\omega_0\tau_g}$.



Wideband plane wave

For a wideband source with power spectral density $R_s(\omega)$, define

$$r_s(\boldsymbol{b}, \tau) = \frac{1}{2\pi} \int R_s(\omega) e^{j\omega(\tau - \tau_g)} \mathrm{d}\omega \,, \qquad \tau_g = \frac{\boldsymbol{\zeta} \cdot \boldsymbol{b}}{c}$$

Applying the temporal Fourier transform gives the cross power spectral density,

$$R_s(\boldsymbol{b}, \omega) = R_s(\omega) e^{-j\omega\tau_g} \,, \qquad \tau_g = \frac{\boldsymbol{\zeta} \cdot \boldsymbol{b}}{c}$$

Also,

$$r_s(\boldsymbol{b}, \tau) = r_s(\tau) * \delta(\tau - \tau_g)$$

i.e., the crosscorrelation between two sensors (spaced by $\boldsymbol{b}$) is the autocorrelation of the source, convolved with a delay.

Summary

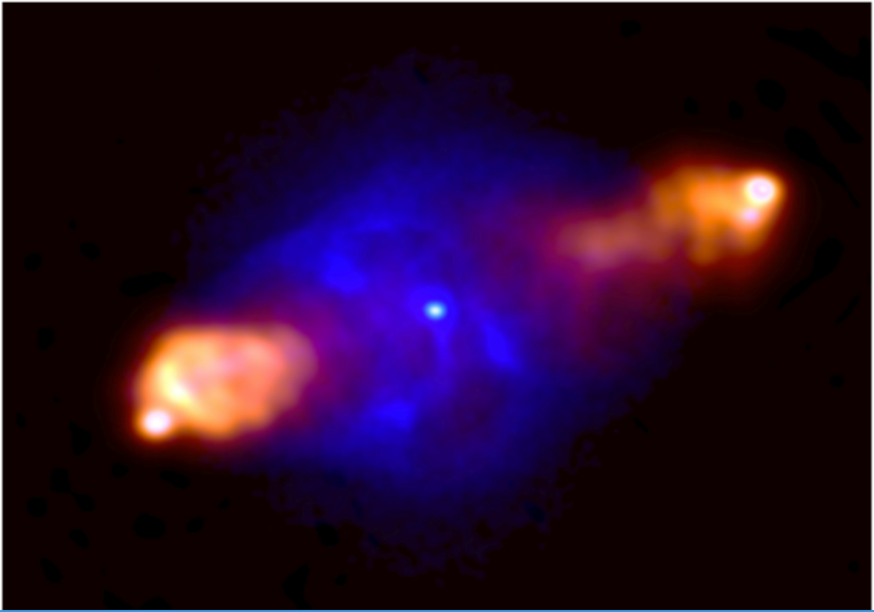
Today: mostly definitions

- An array samples propagating signals in space
- The antenna spacing and aperture are critical!
- A lot of concepts from DSP and SSP carry over and generalize. Thus, we can also think about spatial filters, spectrum analysis, $\cdots$

The theory is general and applies to wireless communication, radar, radio astronomy, underwater acoustics, geophysics, $\cdots$.

We will mostly look at a small subset of this, where the propagation is actually rather "simple".

Application: radio astronomy



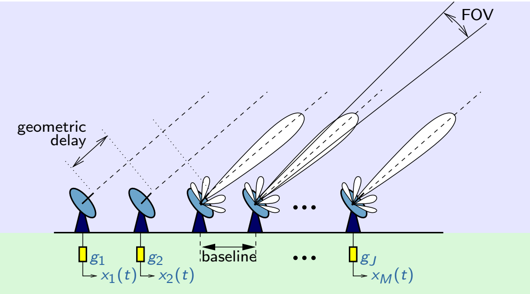
Application: radio astronomy

Westerbork Synthesis Radio Telescope (14 dishes)

Application: radio astronomy

LOFAR Low Frequency Radio Telescope (48 stations, each with 96 dipoles and subarrays))

Application: radio astronomy



An **interferometer** measures the spatial coherency of the incoming random field: the spatial correlations (called **visibilities**)

Application: radio astronomy

For a single source in direction ζ with **brightness** $R_s(\omega) = I(\omega, \zeta_0)$, we measure the visibility

$$V(\omega, \boldsymbol{b}) = I(\omega, \zeta_0) e^{-j\frac{\omega}{c}\zeta_0 \cdot \boldsymbol{b}}$$

For a superposition of sources:

$$V(\omega, \boldsymbol{b}) = \int I(\omega, \zeta) e^{-j\frac{\omega}{c}\zeta \cdot \boldsymbol{b}} \mathrm{d}\zeta$$

$I(\omega, \zeta)$ is the image (called the **map**).

If we could measure $V(\omega, \boldsymbol{b})$ for all possible baselines $\boldsymbol{b}$, then we can reconstruct the image via an inverse Fourier transform:

$$I(\omega, \zeta) = \frac{1}{(2\pi)^3} \int V(\omega, \boldsymbol{b})\, e^{j\frac{\omega}{c}\zeta \cdot \boldsymbol{b}} \mathrm{d}\boldsymbol{b}$$

Application: radio astronomy

But we only have a discrete set of baselines $\{\boldsymbol{b}_k\}$ (one for each telescope pair). In analogy, compute the **dirty image**

$$I_D(\omega, \zeta) = \frac{1}{(2\pi)^3} \sum_k V(\omega, \boldsymbol{b}_k) e^{j\frac{\omega}{c}\zeta\cdot\boldsymbol{b}_k}$$

Substitute the expression for $V(\omega, \boldsymbol{b})$, this gives

$$\begin{aligned} I_D(\omega, \zeta) &= \frac{1}{(2\pi)^3} \int I(\omega, \boldsymbol{n}) \left[\sum_k e^{j\frac{\omega}{c}(\boldsymbol{n}-\zeta)\cdot\boldsymbol{b}_k}\right] \mathrm{d}\boldsymbol{n} \\ &= \frac{1}{(2\pi)^3} W(\zeta) * I(\omega, \zeta) \end{aligned}$$

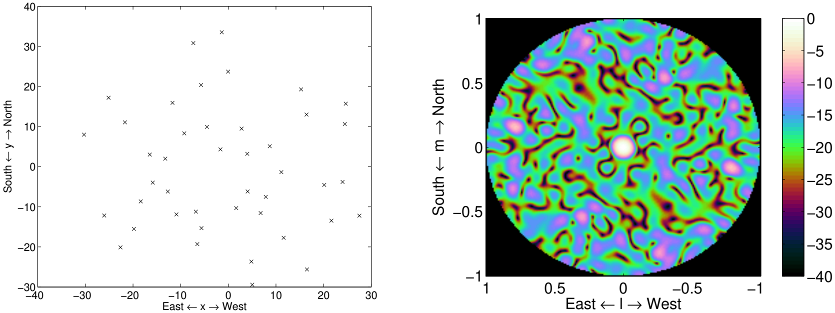
with

$$W(\zeta) = \sum_k e^{j\frac{\omega}{c}\zeta\cdot\boldsymbol{b}_k}$$

The dirty image is the true image convolved with the **dirty beam** $W(\zeta)$.

Application: radio astronomy

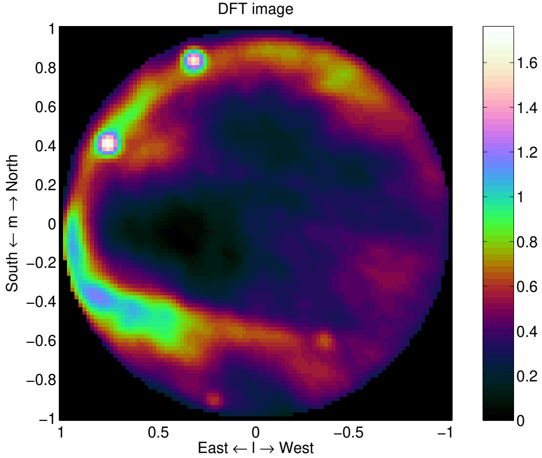
The dirty beams only depends on the telescope locations.



(a) Coordinates of the antennas in a LOFAR station, which defines the spatial sampling function; (b) the resulting *dirty beam*, plotted in dB.

Application: radio astronomy

LOFAR dirty image (single snapshot)



Application: radio astronomy

Image formation tries to recover the true image from the dirty image by deconvolution techniques (since we know $W(\zeta)$).

- We need calibration to know $W(\zeta)$: positions, gains, antenna responses, $\cdots$
- A single **snapshot image** is based on a few seconds of data. It does not show much: the noise is 20 dB stronger than the strongest source.
- To make a real image, astronomers observe for 8 hours or more (up to months). As the earth rotates, many more baselines are observed.