

# EE2S31 Signal Processing – Stochastic Processes

## Lecture 8: Frequency Domain Relationships – Suppl. 7, 8

Alle-Jan van der Veen

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## Summarizing: Power spectral density

For WSS random processes we use the power spectral density to provide a frequency domain description.

**Time-continuous:**

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} df$$

**Time-discrete:**

$$S_X(\phi) = \sum_{k=-\infty}^{\infty} R_X[k] e^{-j2\pi\phi k}$$

$$R_X[k] = \int_{-1/2}^{1/2} S_X(\phi) e^{j2\pi\phi k} d\phi$$

- $S_X(\phi) \geq 0$  for all  $f$
- $\int_{-1/2}^{1/2} S_X(\phi) d\phi = E[X_n^2] = R_X[0]$
- $S_X(-\phi) = S_X(\phi)$
- for any integer  $n$ ,  $S_X(\phi + n) = S_X(\phi)$  (periodic)

## Cross power spectral density

- The **cross-correlation** between two stochastic processes is defined as

$$R_{XY}(t, \tau) = E[X(t)Y(t + \tau)]$$

Two random processes  $X(t)$  and  $Y(t)$  are jointly wide sense stationary, if  $X(t)$  and  $Y(t)$  are wide sense stationary, and

$$R_{XY}(t, \tau) = R_{XY}(\tau).$$

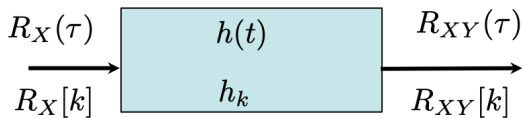
If  $X(t)$  and  $Y(t)$  are jointly WSS, then  $R_{XY}(\tau) = R_{YX}(-\tau)$ .

- Define the **cross power spectral density** for jointly WSS processes  $X(t)$  and  $Y(t)$ :

$$S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j2\pi f\tau} d\tau$$

(Similar for time-discrete processes)

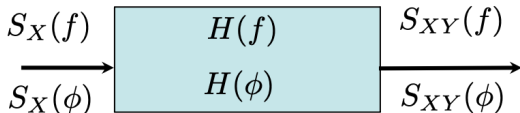
# Frequency Domain Relationships I



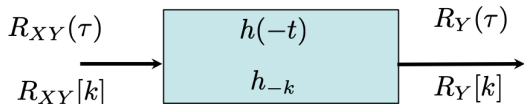
We know:  $R_{XY}(\tau) = h(\tau) * R_X(\tau)$  and  $R_{XY}[k] = h[k] * R_X[k]$

$\Leftrightarrow$ Fourier transform $\Rightarrow$

$$S_{XY}(f) = H(f)S_X(f) \text{ and } S_{XY}(\phi) = H(\phi)S_X(\phi)$$



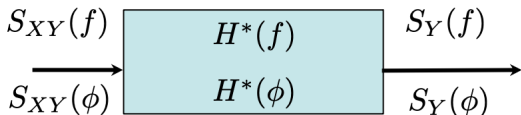
## Frequency Domain Relationships II



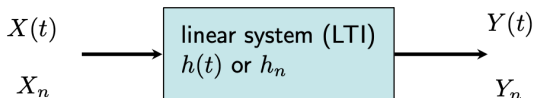
We know:  $R_Y(\tau) = h(-\tau) * R_{XY}(\tau)$  and  $R_Y[k] = h[-k] * R_{XY}[k]$

$\Leftarrow$ Fourier transform $\Rightarrow$

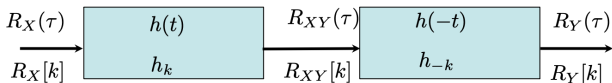
$$S_Y(f) = H^*(f)S_{XY}(f) \text{ and } S_Y(\phi) = H^*(\phi)S_{XY}(\phi)$$



# Summary



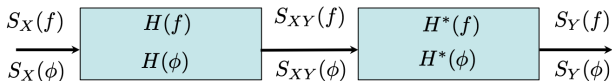
## Time domain:



$$S_Y(f) = H^*(f)S_{XY}(f) = |H(f)|^2 S_X(f)$$

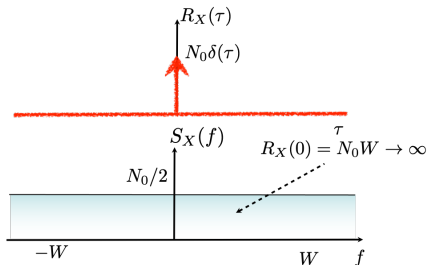
$$S_Y(\phi) = H^*(\phi)S_{XY}(\phi) = |H(\phi)|^2 S_X(\phi)$$

## Frequency domain:



# Continuous Time White Noise Process

- Let  $X(t)$  be a white noise process (i.e., zero mean and uncorrelated), with  $R_X(\tau) = N_0\delta(\tau)$
- Then  $S_X(f) = N_0$ : constant for all  $f$
- What is the average power of  $X(t)$ ?



The average power is  
 $R_X(0) = N_0W \rightarrow \infty$  as the  
bandwidth  $W \rightarrow \infty$

This process cannot be physically  
realized (infinite average power)

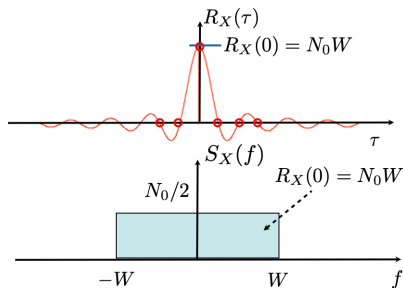
# Continuous Time Bandlimited White Noise Process

- What happens if we bandlimit the process?

From Table 1:

$$N_0 W \text{sinc}(2W\tau) \Leftrightarrow \frac{N_0}{2} \text{rect}\left(\frac{f}{2W}\right)$$

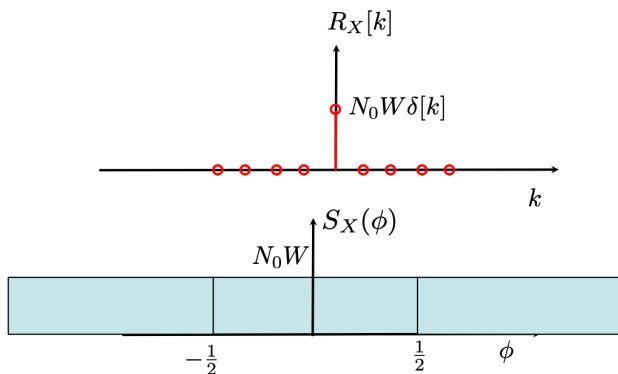
zero for  $\tau = \frac{k}{2W}$  ( $k \in \mathbb{Z} \setminus \{0\}$ )





## White Noise Process for Discrete-time signals

Sampling at  $f_s = 2W$ , the resulting discrete-time random process is truly white, with  $R_X[k] = N_0 W \delta[k]$



(Recall sampling:  $X_s(\Omega) = \frac{1}{T_s} \sum X(\Omega - k\Omega_s)$ )

## Problem 8.2 (really the same... with $B = 2W$ )

Let  $W(t)$  denote a WSS Gaussian noise process with  $\mu_W = 0$  and power spectral density  $S_W(f) = 1$ .

- (a) What is  $R_W(\tau)$ , the autocorrelation of  $W(t)$ ?
- (b)  $W(t)$  is the input to a linear time-invariant filter with impulse response

$$H(f) = \begin{cases} 1 & |f| \leq B/2 \\ 0 & \text{otherwise} \end{cases}$$

The filter output is  $Y(t)$ . What is the power spectral density function of  $Y(t)$ ?

- (c) What is the average power of  $Y(t)$ ?
  - (d) What is the expected value of the filter output?
-

## Problem 8.2 (really the same... with $B = 2W$ )

- (a)  $R_W(\tau) = \delta(\tau)$  is the autocorrelation function whose Fourier transform is  $S_W(f) = 1$ .
- (b) The output  $Y(t)$  has power spectral density

$$S_Y(f) = |H(f)|^2 S_W(f) = |H(f)|^2.$$

- (c) Since  $H(f) = 1$  for  $f \in [-\frac{1}{2}B, \frac{1}{2}B]$ , the average power of  $Y(t)$  is

$$E[Y^2(t)] = \int_{-\infty}^{\infty} S_Y(f) df = \int_{-B/2}^{B/2} df = B$$

- (d) Since the white noise  $W(f)$  has zero mean, the expected value of the filter output is

$$E[Y(t)] = E[W(t)]H(0) = 0$$

## Problem 8.7

A white Gaussian noise process  $N(t)$  with power spectral density of  $10^{-15}$  W/Hz is the input to a lowpass filter  $H(f) = 10^3 e^{-10^{-6}|f|}$ . Find the following properties of the output  $Y(t)$ :

- (a) The expected value  $\mu_Y$
  - (b) The output power spectral density  $S_Y(f)$
  - (c) The average power  $E[Y^2(t)]$
  - (d)  $P[Y(t) > 0.01]$
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  - (c) The average power  $E[Y^2(t)]$
  - (d)  $P[Y(t) > 0.01]$
- 

(a)  $E[N(t)] = \mu_N = 0 \Rightarrow \mu_Y = \mu_N H(0) = 0$

(b)  $S_Y(f) = |H(f)|^2 S_N(f) = 10^{-9} e^{-2 \cdot 10^{-6}|f|}$

(c)  $E[Y^2(t)] = \int_{-\infty}^{\infty} S_Y(f) df = \int_{-\infty}^{\infty} 10^{-9} e^{-2 \cdot 10^{-6}|f|} df =$   
 $2 \cdot 10^{-9} \int_0^{\infty} e^{-2 \cdot 10^{-6} f} df = 10^{-3} \text{ W}$

## Problem 8.7

- (d) Since  $N(t)$  is a Gaussian process, Theorem 3 says  $Y(t)$  is a Gaussian process. Thus the random variable  $Y(t)$  is Gaussian with

$$E[Y(t)] = 0, \quad \text{var}[Y(t)] = E[Y^2(t)] = 10^{-3}$$

Thus we can use Table 4.2 to calculate

$$\begin{aligned} P[Y(t) > 0.01] &= P \left[ \frac{Y(t)}{\sqrt{\text{var}[Y(t)]}} > \frac{0.01}{\sqrt{\text{var}[Y(t)]}} \right] \\ &= 1 - \Phi \left( \frac{0.01}{\sqrt{0.001}} \right) \\ &= 1 - \Phi(0.32) = 0.3745 \end{aligned}$$

## Problem 8.9

Let  $M(t)$  be a WSS random process with average power  $E[M^2(t)] = q$  and power spectral density  $S_M(f)$ . The Hilbert transform of  $M(t)$  is  $\hat{M}(t)$ , a signal obtained by passing  $M(t)$  through a linear time-invariant filter with frequency response

$$H(f) = -j \operatorname{sgn}(f) = \begin{cases} -j & f \geq 0, \\ j & f < 0. \end{cases}$$

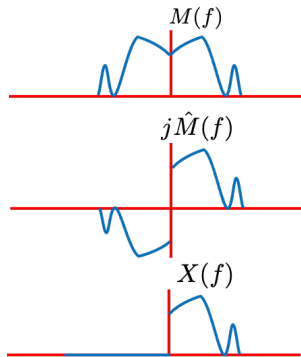
- (a) Find the power spectral density  $S_{\hat{M}}(f)$  and the average power  $\hat{q} = E[\hat{M}^2(t)]$ .
-

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- (a) Find the power spectral density  $S_{\hat{M}}(f)$  and the average power  $\hat{q} = E[\hat{M}^2(t)]$ .
- 



$x(t) = M(t) + j\hat{M}(t)$  is the “analytic signal”:  $X(f) = 0$  for  $f < 0$



## Problem 8.9 (cont'd)

- (a) Note that  $|H(f)| = 1$ . This implies  $S_{\hat{M}}(f) = S_M(f)$ . Thus the average power of  $\hat{M}(t)$  is

$$\hat{q} = \int_{-\infty}^{\infty} S_{\hat{M}}(f)df = \int_{-\infty}^{\infty} S_M(f)df = q$$

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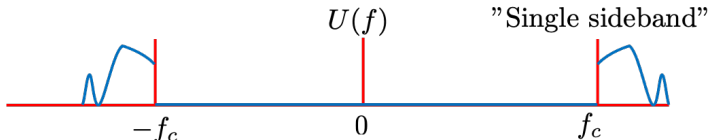
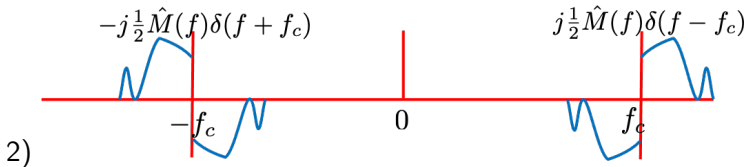
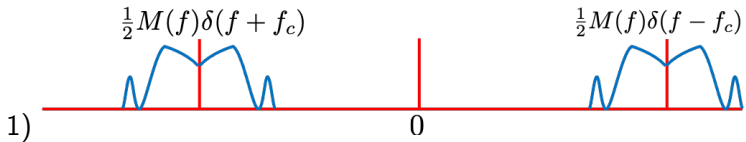
$$\hat{q} = \int_{-\infty}^{\infty} S_{\hat{M}}(f) df = \int_{-\infty}^{\infty} S_M(f) df = q$$

- 
- (b) In a single sideband communication system, the upper sideband signal is

$$U(t) = M(t) \cos(2\pi f_c t + \Theta) - \hat{M}(t) \sin(2\pi f_c t + \Theta)$$

where  $\Theta$  has a uniform PDF over  $[0, 2\pi)$ , independent of  $M(t)$  and  $\hat{M}(t)$ . What is the average power  $E[U^2(t)]$ ?

## Problem 8.9 (cont'd) – Idea



$$\Rightarrow E[U^2(t)] = q$$

## Problem 8.9 (cont'd) – Idea

(b) The average power of the upper sideband signal is

$$\begin{aligned} E[U^2(t)] &= E[M^2(t) \cos^2(2\pi f_c t + \Theta)] \\ &\quad - E\left[2M(t)\hat{M}(t) \cos(2\pi f_c t + \Theta) \sin(2\pi f_c t + \Theta)\right] \\ &\quad + E\left[\hat{M}^2(t) \sin^2(2\pi f_c t + \Theta)\right] \end{aligned}$$

Use:

$$\begin{aligned} E[\cos^2(2\pi f_c t + \Theta)] &= \frac{1}{2} \\ E[\sin^2(2\pi f_c t + \Theta)] &= \frac{1}{2} \\ E[2 \sin(2\pi f_c t + \Theta) \cos(2\pi f_c t + \Theta)] &= E[\sin(4\pi f_c t + 2\Theta)] = 0 \end{aligned}$$

## Problem 8.9 (cont'd)

Since  $M(t)$  and  $\hat{M}(t)$  are independent of  $\Theta$ , the average power of the upper sideband signal is

$$\begin{aligned} E[U^2(t)] &= E[M^2(t)] E[\cos^2(2\pi f_c t + \Theta)] \\ &\quad - E[M(t)\hat{M}(t)] E[2 \cos(2\pi f_c t + \Theta) \sin(2\pi f_c t + \Theta)] \\ &\quad + E[\hat{M}^2(t)] E[\sin^2(2\pi f_c t + \Theta)] \\ &= q/2 + 0 + q/2 = q \end{aligned}$$

## To do:

- Study Sections 7 and 8
- Check old exams for related exercises

Next lecture, we'll go over some old exams.