

# Signal Processing EE2S31

## Digital Signal Processing Lecture 7: Sigma-delta ADCs

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## Strategies to improve A/D conversion

- Oversampling
- Differential quantization
- Discrete-time model of sigma-delta ADCs
- Noise shaping

## Signal-to-quantization-noise ratio of the quantizer

$$SQNR = 10 \log_{10} \frac{\sigma_x^2}{\sigma_z^2}, \text{ with } \sigma_z^2 = \frac{\Delta^2}{12} \text{ and } \Delta = \frac{R}{2^{B+1}}$$

- the range of the input (which is proportional to  $\sigma_x$ ) should match the range of the quantizer  $\Delta$
- according to formula,  $\sigma_z$  is proportional to  $\Delta$   
     $\Downarrow$
- for a given  $B$ ,  $\sigma_z^2$  is proportional to  $\sigma_x^2$   
     $\Downarrow$
- reducing the variance of the input signal to the quantizer reduces the number of necessary bits

Strategies to improve SQNR:

- Differential quantization
- Oversampling

## Differential quantization

Idea: to reduce the range of the input to the quantizer, let us quantize the differential signal!

$$d[n] = x[n] - x[n-1]$$

What is the variance of the differential signal?

$$\begin{aligned}\sigma_d^2 &= E[d^2[n]] = E[x[n] - x[n-1]]^2 \\ &= E[x^2[n]] - 2E[x[n]x[n-1]] + E[x^2[n-1]] = \dots = \\ &= 2\sigma_x^2[1 - r_x[1]], \text{ with } r_x[1] = R_x[1]/R_x[0]\end{aligned}$$

- we need to obtain high autocorrelation at lag 1:  $r_x(1)$ , which can be achieved by oversampling (=well above Nyquist)!
- Alternative: Prediction!

## Differential predictive quantizer

Let us quantize:

$$d(n) = x(n) - ax(n-1)$$

What is the optimal choice of  $a$ ?

$$\begin{aligned}\sigma_d^2 &= E[d^2[n]] = E[x[n] - ax[n-1]]^2 \\ &= E[x^2[n]] - 2E[ax[n]x[n-1]] + E[a^2x^2[n-1]] = \\ &= (1 + a^2)\sigma_x^2 - 2aR_x[1]\end{aligned}$$

How do I find the optimal value of  $a$ ?

## Differential predictive quantizer

Let us quantize:

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What is the optimal choice of  $a$ ?

$$\begin{aligned}\sigma_d^2 &= E[d^2[n]] = E[x[n] - ax[n-1]]^2 \\ &= E[x^2[n]] - 2E[ax[n]x[n-1]] + E[a^2x^2[n-1]] = \\ &= (1 + a^2)\sigma_x^2 - 2aR_x[1]\end{aligned}$$

How do I find the optimal value of  $a$ ?

- setting the derivative of  $\sigma_d^2$  to 0 yields  $a = r_x(1)$ .
- then, by substituting back, we obtain  $\sigma_d^2 = \sigma_x^2(1 - r_x^2(1)) \leq \sigma_x^2$

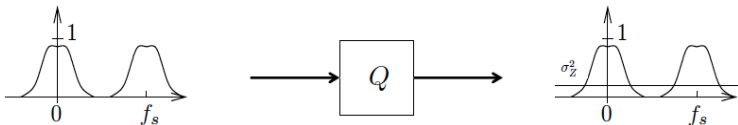
As such, differential predictive quantization is always better!

# Oversampling

Recall that the spectrum of a sampled signal equals

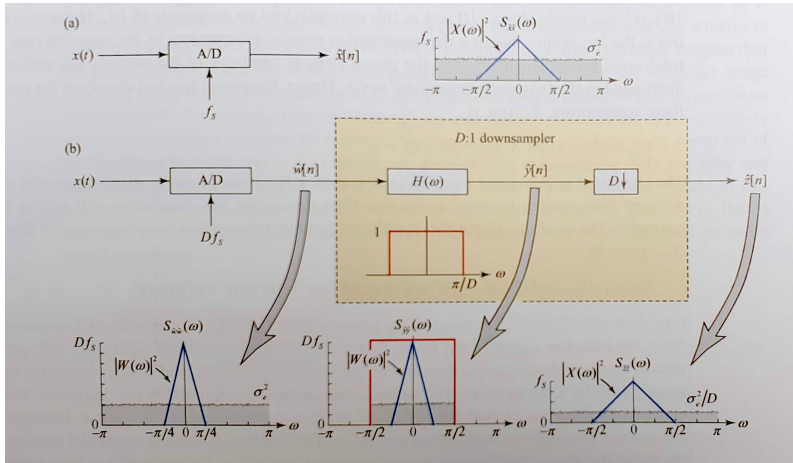
$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a(F + kF_s)$$

The effect of quantization is an addition of white quantization noise with power  $\sigma_z^2$ .



# Oversampling

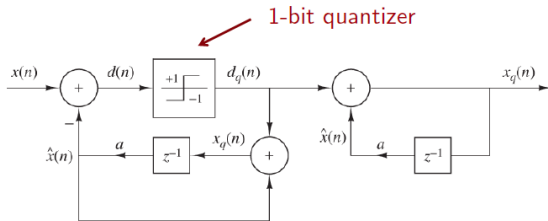
What happens if we oversample? ( $f_s \ll f_{Nyquist}$ )





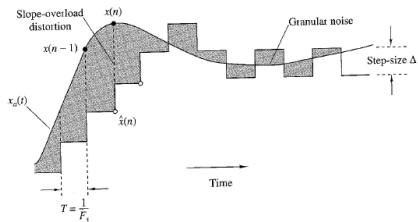
# Sigma-delta modulator

Let us combine the two principles, i.e. differential quantization and oversampling!

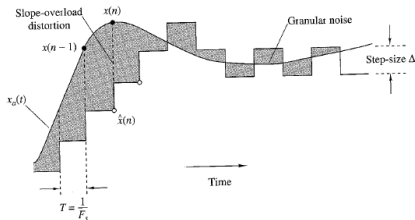


- coder consists of the predictor and the quantizer
- predictor provides an estimate  $\hat{x}[n]$  of  $x[n]$  using its past sample(s) in case of a first (or higher order) predictor
- with  $a = 1$  the predictor is a simple accumulator
- quantizer quantizes the difference  $d[n] = x[n] - \hat{x}[n]$
- oversampling allows the use of a 1-bit quantizer, providing a staircase approximation of the signal
- decoder reconstructs the signal from the differential quantized values

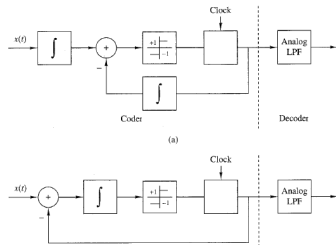
# Sigma-delta modulator



# Sigma-delta modulator



To reduce distortion, we can introduce an integrator before the differential quantizer. Uniting the two integrators yields a simpler circuit, where a single intergrator (*sigma*) is followed by a differentiator (*delta*).



## Sigma-delta modulator: discrete model

Let us model the SDM as a discrete system: the integrator as  $H(z)$  and the quantizer as an additive white noise!

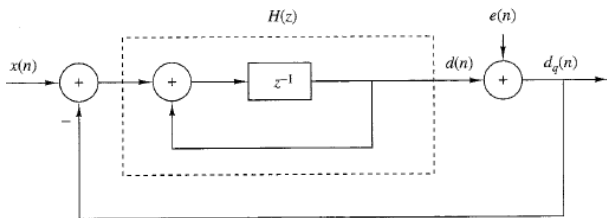


Figure 6.6.4 Discrete-time model of sigma-delta modulation.

The z-transform of the sequence  $d_q(n)$  can be separated into a response to the signal and a response to noise:

$$\begin{aligned} D_q(z) &= \frac{H(z)}{1 + H(z)} X(z) + \frac{1}{1 + H(z)} E(z) = \\ &= H_s(z) X(z) + H_n(z) E(z) \end{aligned}$$

## Noise shaping

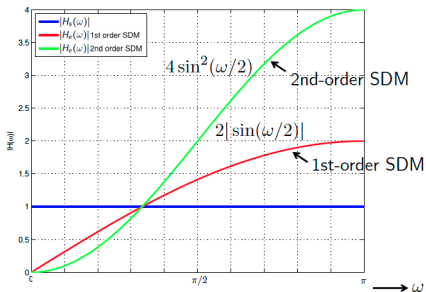
Let us work out the previous formula for  $D_q(z)$ , with  $H(z) = \frac{z^{-1}}{1-z^{-1}}$

$$\begin{aligned} D_q(z) &= \frac{H(z)}{1+H(z)}X(z) + \frac{1}{1+H(z)}E(z) = \dots \\ &= z^{-1}X(z) + (1-z^{-1})E(z) \end{aligned}$$

Therefore,

$$|H_s(\omega)| = 1 \text{ and } |H_e(\omega)| = 2|\sin(\omega/2)|$$

# Noise-shaping



- In case of oversampling, signal is restricted to the lowest (normalized) frequencies, and noise power is spread over a relatively larger frequency band (over the full range between 0 and  $2\pi$ )
- The signal is not distorted by the system  $H_S$ .
- The noise is attenuated at low frequencies.
- Therefore, higher SNR can be achieved by oversampling (see exercise 6.20)

# Sigma-delta modulation

## Summary

Oversampling A/D converters (=sigma-delta modulators) using 1-bit quantization can achieve very good accuracy at low cost, based on the principle of oversampling, differential quantization and noise shaping.