Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Total points (both blocks): 55 pt. The grade has been computed relative to 38 pt = grade 10.

Partial exam EE2S31 SIGNAL PROCESSING
Part 1: May 27, 2020
Block 1: Stochastic Processes (13:30-14:30)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.
Upload answers during 14:25–14:35

This block consists of three questions (30 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (12 points)

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c(x+y) & \text{for} \quad x \geq 0,\ y \geq 0,\ x+y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant c.

(b) Find $f_X(x)$, $f_Y(y)$, and $f_{X|Y}(x|y)$.

(c) Find $\mathrm{E}[X]$ and $\mathrm{E}[Y]$.

(d) Find $\hat{X}_M(Y)$, the minimum mean square error (MMSE) estimator for X given a single sample of Y.

(e) Find the PDF of $W = X + Y$.

Solution

(a) 2 pnt

$$\begin{aligned}
\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx &= \int_{x=0}^{1}\int_{y=0}^{1-x} c(x+y)\,dy\,dx \\
&= c\int_0^1 \left[xy + \frac{1}{2}y^2\right]_{y=0}^{1-x} dx \\
&= c\int_0^1 \left[x(1-x) + \frac{1}{2}(1-x)^2\right] dx \\
&= c\int_0^1 \left(\frac{1}{2} - \frac{1}{2}x^2\right) dx \\
&= c\left[\frac{1}{2}x - \frac{1}{6}x^3\right]_0^1 \\
&= \frac{1}{3}c \quad = \quad 1
\end{aligned}$$

Hence $c = 3$.

(b) 3 pnt For $0 \leq x \leq 1$,

$$f_X(x) = \int_{y=0}^{1-x} c(x+y)\,dy = c\left(\frac{1}{2} - \frac{1}{2}x^2\right)$$

The complete PDF is

$$f_X(x) = \begin{cases} \frac{3}{2}(1-x^2) & \text{for} \quad 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Based on symmetry, $f_Y(y) = f_X(y)$, hence

$$f_Y(y) = \begin{cases} \frac{3}{2}(1-y^2) & \text{for} \quad 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 2\dfrac{x+y}{1-y^2} & 0 \leq x \leq 1-y \\ 0 & \text{otherwise} \end{cases}$$

(c) 2 pnt

$$\begin{aligned} \mathrm{E}[X] &= \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \frac{3}{2}\int_0^1 x(1-x^2)\,dx = \frac{3}{2}\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = \frac{3}{8} \\ \mathrm{E}[Y] &= \mathrm{E}[X] = \frac{3}{8} \qquad \text{(based on symmetry)} \end{aligned}$$

(d) 3 pnt The MMSE is always $\hat{X}_M(Y) = \mathrm{E}[X|Y]$. Using $f_{X|Y}(x|y)$,

$$\begin{aligned} \mathrm{E}[X|y] &= \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx \\ &= 2\int_0^{1-y} x\frac{x+y}{1-y^2}\,dx \\ &= \frac{2}{1-y^2}\left[\frac{1}{3}x^3 + \frac{1}{2}x^2 y\right]_{x=0}^{1-y} \\ &= \frac{2}{1-y^2}\left(\frac{1}{3}(1-y)^3 + \frac{1}{2}(1-y)^2 y\right) \\ &= \frac{1-y}{1+y}\left(\frac{2}{3}(1-y) + y\right) \\ &= \frac{1}{3}\,\frac{(1-y)(2+y)}{1+y}. \end{aligned}$$

Then $\hat{X}_M(Y) = \mathrm{E}[X|Y] = \dfrac{1}{3}\,\dfrac{(1-Y)(2+Y)}{1+Y}$.

(e) 2 pnt For $0 \le w \le 1$,

$$\begin{aligned} f_W(w) &= \int_{x=-\infty}^{\infty} f_{X,Y}(x, w-x)\, dx \\ &= \int_{x=0}^{w} 3(x + w - x)\, dx \\ &= 3\left[wx\right]_{x=0}^{w} \quad = 3w^2 \end{aligned}$$

and $f_W(w) = 0$, otherwise.

Question 2 (8 points)

In a test of a circuit, the probability that it functions correctly is $p = 0.8$. We test n circuits; the outcomes of the tests are independent. Let X be a RV that describes the outcome of a single test (0 or 1), and let K be the number of circuits that successfully pass the test.

(a) Derive that the moment generating function (MGF) of X is $\phi_X(s) = 1 - p + p\, e^s$.

(b) What is the MGF of K?

(c) Use the MGF to derive $\mathrm{E}[K]$, $\mathrm{E}[K^2]$, and compute from this $\mathrm{Var}[K]$.

(d) Use the central limit theorem to estimate the probability of finding at least 500 acceptable circuits in a batch of 600 circuits.

(e) Using the Chebyshev inequality, what is an upper bound on the probability of finding at least 500 acceptable circuits in a batch of 600 circuits?

Note: You may express your answer for (d) in terms of $\Phi(z)$ or $Q(z)$.

Solution

(a)

$$\begin{aligned} P_X(x) &= \begin{cases} 1-p & x = 0, \\ p & x = 1, \\ 0 & \text{otherwise.} \end{cases} \\ \phi_X(s) &= \sum_x e^{sx} P_X(x) = e^{s\,0}(1-p) + e^{s\,1}p = 1 - p + p\, e^s \end{aligned}$$

(b) Since the tests are independent,

$$\phi_K(s) = (\phi_X(s))^n = (1 - p + p\, e^s)^n$$

(c) Using the MGF,

$$\begin{aligned} \mathrm{E}[K] &= \left.\frac{d\Phi_K(s)}{ds}\right|_{s=0} = n(1-p+p\,e^s)^{n-1} p\, e^s\big|_{s=0} = np \\ \mathrm{E}[K^2] &= \left.\frac{d}{ds}\frac{d\Phi_K(s)}{ds}\right|_{s=0} = n(n-1)(1-p+p\,e^s)^{n-2}p^2\, e^{2s} + n(1-p+p\,e^s)^{n-1}p\, e^s\big|_{s=0} \\ &= n(n-1)p^2 + np \\ \mathrm{Var}[K] &= \mathrm{E}[K^2] - (\mathrm{E}[K])^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p) \end{aligned}$$

This result is consistent with the mean and variance of a binomial distribution.

(d) For $n = 600$, we find $\mathrm{E}[K] = 480$, $\mathrm{Var}[K] = 96$. Compute a normalized variable

$$z = \frac{K - 480}{\sqrt{96}}$$

which will be approximated by a Gaussian distribution.

$$\begin{aligned} \mathrm{P}[K \geq 500] &= \mathrm{P}\left[\frac{K-480}{\sqrt{96}} \geq \frac{500-480}{\sqrt{96}}\right] \\ &= \mathrm{P}[z \geq 2.04] \approx Q(2.04) \approx 0.0206 \end{aligned}$$

(e) The Chebyshev inequality gives

$$\begin{aligned} \mathrm{P}[K \geq 500] &= \mathrm{P}[K - \mathrm{E}[K] \geq 20] \\ &\leq \mathrm{P}[\,|K - \mathrm{E}[K]| \geq 20] \leq \frac{\mathrm{Var}[K]}{(20)^2} = \frac{96}{400} = 0.24 \end{aligned}$$

(This appears not very accurate. The first inequality can be improved if we assume the PDF is nearly symmetric.)

Question 3 (10 points)

A continuous random variable X has PDF

$$f_X(x) = \begin{cases} \frac{1}{12} & \text{for} \quad -6 \leq x \leq 6, \\ 0 & \text{otherwise.} \end{cases}$$

We observe a noisy version of X, namely $Y = X + N$, where N is an independent zero mean Gaussian random variable with variance 3.

Let $\hat{X}_{\mathrm{ML}}(Y)$ be the maximum likelihood (ML) estimate of X based on a single observation of Y, and let $\hat{X}_L(Y) = aY + b$ be the linear minimum mean square error (LMMSE) estimate of X.

(a) What is $\mathrm{E}[Y]$, $\mathrm{Var}[Y]$, and $\mathrm{Cov}[X, Y]$?

(b) What is $\hat{X}_{\mathrm{ML}}(Y)$?

(c) Is $\hat{X}_{\mathrm{ML}}(Y)$ an unbiased estimator?

(d) Compute e_{ML}, the mean square error of the ML estimator.

(e) Find a and b, the optimum coefficients for the LMMSE estimate.

(f) Is $\hat{X}_L(Y)$ an unbiased estimator?

(g) What is e_L, the mean square error of the optimum linear estimate.

Solution

(a)

$$\begin{aligned}
\mathrm{E}[X] &= 0\,, & \mathrm{Var}[X] &= \frac{(12)^2}{12} = 12 \\
\mathrm{E}[Y] &= 0 + 0 = 0\,, & \mathrm{Var}[Y] &= \mathrm{Var}[X] + \mathrm{Var}[N] = 12 + 3 = 15\,,
\end{aligned}$$

since X and N are independent.

Since $\mathrm{E}[Y] = \mathrm{E}[X] = 0$ (and using independence),

$$\mathrm{Cov}[X, Y] \;=\; \mathrm{E}[XY] = \mathrm{E}[X(X + N)] = \mathrm{E}[X^2] = \mathrm{Var}[X] = 12\,.$$

(b) The likelihood function (conditional PDF seen as function of X) is

$$f_{Y|X}(y|x) = f_N(y - x)$$

Since it is zero mean Gaussian, the maximum of $f_N(n)$ occurs for $n = 0$. Therefore, maximizing $f_{Y|X}(y|x)$ over x gives $x = y$, and

$$\hat{X}_{\mathrm{ML}}(y) = \arg\max_x f_{Y|X}(y|x) = y$$

Thus, $\hat{X}_{\mathrm{ML}}(Y) = Y$.

(c) Unbiased, because $\mathrm{E}[\hat{X}_{\mathrm{ML}}(Y)] = \mathrm{E}[Y] = 0$.

(d) The MSE of the ML estimator is

$$\mathrm{E}[(X - \hat{X}(Y))^2] = \mathrm{E}[(X - Y)^2] = \mathrm{E}[N^2] = 3\,.$$

(e) The LMMSE estimator given Y is

$$\begin{aligned}
\hat{X}_L(Y) &= \frac{\mathrm{Cov}[X, Y]}{\mathrm{Var}[Y]}(Y - \mathrm{E}[Y]) + \mathrm{E}[X] \\
&= \frac{12}{15}Y \;=\; \frac{4}{5}Y\,.
\end{aligned}$$

Hence $a = 4/5$ and $b = 0$.

(f) Unbiased, because $\mathrm{E}[\hat{X}_L(Y)] = \frac{4}{5}\mathrm{E}[Y] = 0$.

(g) The correlation coefficient is

$$\rho_{X,Y} = \frac{\mathrm{Cov}[X, Y]}{\sqrt{\mathrm{Var}[Y]\mathrm{Var}[X]}} = \sqrt{\frac{12}{15}}$$

Then $e_L = \mathrm{Var}[X](1 - \rho_{X,Y}^2) = 12(1 - \frac{12}{15}) = 12/5 = 2.4$.

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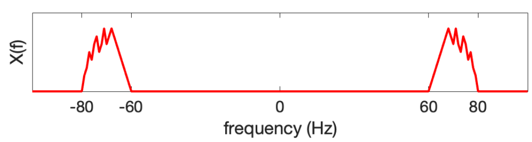
Partial exam EE2S31 SIGNAL PROCESSING
Part 1: May 27th 2020
Block 2: Digital Signal Processing (14:35-15:35)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted. Upload answers during 15:30-15:40.

This block consists of three questions (25 points), more than usual. This will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. No points will be awarded for results without a derivation. Write your name and student number on each sheet.

Question 4 (9 points)

Given an analog signal $x_a(t)$ with a spectrum shown below. I want to digitize the signal.



(1 p) (a) What is the bandwidth of the signal?

(1 p) (b) What is the minimum sampling rate F_s to avoid aliasing?

(2 p) (c) I have 2 A/D converters available, one operating at 50Hz, the other operating at 60Hz. Can I use any of the two to sample the signal without aliasing? Sketch and explain!

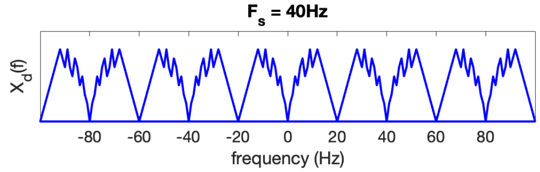
(3 p) (d) I would like to down-convert the signal $x_a(t)$, i.e. reconstruct the equivalent baseband signal. Is that possible given any of the sampling rates in point (b) and (c), assuming ideal interpolation? Give the equation to reconstruct the baseband signal using ideal interpolation!

(2 p) (e) What changes if I use sample-and-hold interpolation for the same purpose as in (d)? Sketch and explain in case 60Hz sampling rate is chosen!

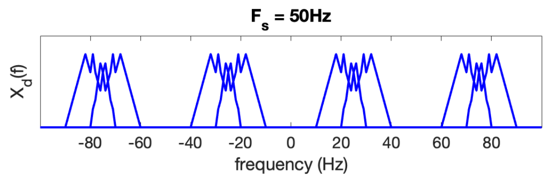
Solution

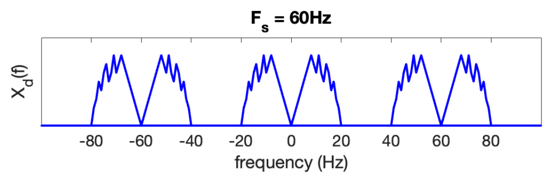
(a) $B = 80 - 60 = 20$ Hz.

(b) It is an integer band positioned signal, therefore $F_s^{(min)} = 2B = 40Hz$. To verify this, the sampled spectrum is shown below.



(c) As shown in the figures below, sampling at 50Hz causes aliasing, while sampling at 60Hz is possible.



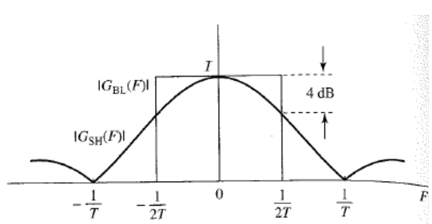


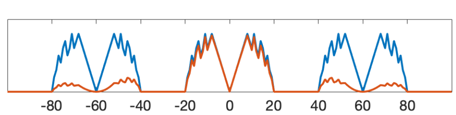
(d) After inspection of the sampled spectra, we can say that (1) it is possible for 60Hz: the baseband spectrum (0-20Hz) is the perfect replica of the original spectrum. (2) it is not possible for 50Hz (3) it is not directly possible for 40Hz: the baseband spectrum is the inverted image of the original spectrum (there is an easy fix, but we did not cover this during the course).
The reconstruction formula is (an ideal low-pass filter can be used):

$$x_a(t) = \sum_{n=-\infty}^{\infty} x[n] g_{id}(t - nT_s), \text{ with } T_s = \frac{1}{F_s}, \text{ where } F_s = 60\text{Hz, and}$$

$$g_{id}(t) = \frac{sin(\pi t/T_s)}{\pi t/T_s}$$

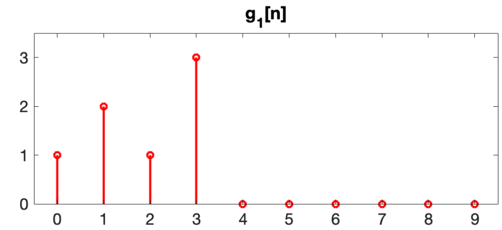
(e) As evident from the frequency response of the sample-and-hold interpolator shown below (in thick black line), it will let through frequencies above $F_s/2 = 30Hz$, i.e. there will be some aliasing from the shifted copies at higher frequencies. Moreover, it also attenuates frequencies below but close to 20Hz which is also undesirable. The spectrum of the sampled signal and its multiplication with the S/H frequency response are also shown below (in blue and red, respectively).





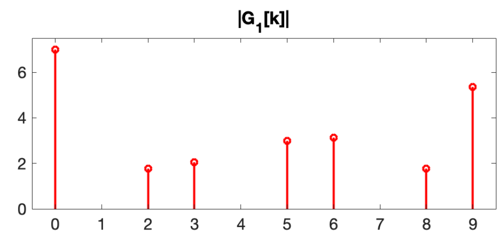
Question 5 (7 points)

Given a discrete sequence $g_1[n]$ as shown below.



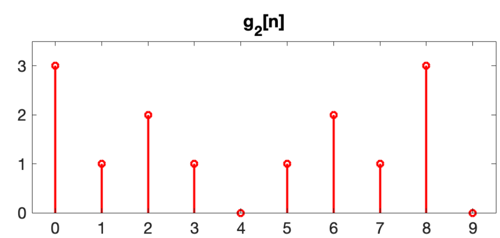
(1 p) (a) How is $G_1[k]$, the 10-point DFT of $g_1[n]$ related to the $G_1(\omega)$, the DTFT of $g_1[n]$? Explain!

(2 p) (b) The disk where we stored $G_1[k]$ was corrupted. We did not store $g_1[n]$ We could retrieve the magnitude of some of the DFT coefficients, these are shown in the figure below. Is it possible to reconstruct the missing values $|G_1[1]|$, $|G_1[4]|$ and $|G_1[7]|$? Explain and sketch!



(1 p) (c) What is the 5-point DFT of $g_1[n]$? Explain and sketch/express in terms of $G_1[k]$!

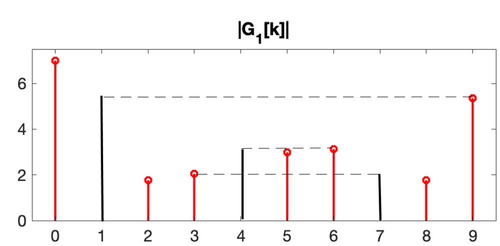
(3 p) (d) Given $g_2[n]$ as shown below. What is its 10-point DFT, denoted by $G_2[k]$, expressed in terms of $G_1[k]$?



Solution

(a) The 10-point DFT consists of 10 equidistant samples of the DTFT, at frequencies $2\pi k/10$ with $k = 0, .., 9$.

(b) For real sequences $|X(k)| = |X(N-k)|$, therefore, $|G_1[1]| = |G_1[9]|, |G_1[4]| = |G_1[6]| and |G_1[7]| = |G_1[3]|$.



(c) The 5-point DFT (let's denote with $G_1^{(5)}$) consists of 5 equidistant samples of the DTFT, at frequencies $2\pi k/5$, with $k = 0, ..., 4$ as such, the 5 coefficients are equal to every second coefficient of the 10-point DFT: $G_2^{(5)}[0] = G_1[0]$, $G_2^{(5)}[1] = G_1[2]$, $G_2^{(5)}[2] = G_1[4]$, $G_2^{(5)}[3] = G_1[6]$, $G_2^{(5)}[4] = G_1[8]$.

(d) $g_2[n]$ is equal to a circularly shifted version of $g_1[n]$ plus a time-reversed and circularly shifted version it, as follows:
$g_2[n] = (g_1[n-5])_{10} + (g_1[3-n])_{10}$
Using the DFT properties for linearity, time reversal and circular time shift, we get: $G_2[k] = G_1[k]e^{-5j2\pi\frac{k}{N}} + G_1[10-k]e^{-3j2\pi\frac{k}{N}}$

Question 6 (9 points)

Given a short discrete sequence $x[n]$ and a filter with impulse response $h[n]$, as shown below:
$x[n] = [1, 2, 1, 0, -1, -2]$ $h = [1, 2, 3, 3, 2, 1]$

(2 p) (a) Describe the 4 consecutive steps of performing the filtering using DFT. Be as specific as possible, but no need to make any computations at this point!

(1 p) (b) What is the value of the 6th sample of the filtered signal?

(2 p) (c) Now, consider the circular convolution of $x[n]$ and $h[n]$! What is the value of the 6th sample? Is it different or identical to the solution in (b)? Explain!

(4 p) (d) Given the 3-point DFTs of the sequences $x_1[n] = [1, 1, -1]$ and $x_2[n] = [2, 0, -2]$ as $X_1[n] = [1, 1 - 1.73j, 1 + 1.73j]$ and $X_2[n] = [0, 3 - 1.73j, 3 + 1.73j]$. Explain how to compute $X[k]$, the 6-point DFT of $x[n]$ using these 3-point DFTs! Be as specific as possible. Compute the coefficients $X[0]$ and $X[3]$!

Solution

(a) **1.** zero-pad each sequence such that they both have 6+6-1=11 samples **2.** take the 11-point DFT of each sequence **3.** multiply their DFTs **4.** take the inverse DFT of the product.

(b) It is easier to compute this using linear convolution, rather than using the DFT. $y[5] = \sum_{k=0}^{5} x[k]h[5-k] = 1 \cdot 1 + 2 \cdot 2 + 1 \cdot 3 + 0 + (-1) \cdot 2 + (-2) \cdot 1 = 4$

(c) The value of the 6th sample of the circular convolution is 4 as well. Explanation: Linear convolution of the two sequences gives a sequence of 11 samples. We also know that circular convolution of two sequencs is equivalent to multiplying their 6-point DFTs directly, and taking the IDFT of that. We know that linear and circular convolution are not equivalent: using a 6-point DFT to reconstruct an 11-sample signal will result in temporal aliasing. However, temporal aliasing does not affect all samples. Only samples beyond the 6th sample will be aliased onto the first samples of the sequence due to the circular shifting. Therefore, only the first 11-6=5 samples are affected, the 6th sample is unaffected.

(d) Using the divide and conquer approach, a 6-point DFT can be computed using a combination of 2-point and 3-point DFTs by properly arranging the samples of the sequence in a matrix. Steps of the computation:
1. We obtain the sequences $x_1[n]$ and $x_2[n]$ if we store $x[n]$ in a 2-by-3 matrix column-wise:
$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -2 \end{bmatrix}$$
2. Compute the 3-point DFT of each row.
3. Multiply each element with a phase factor $W_6^{lq} = e^{-j2\pi lq/6}$, with l being the row-index and q the column-index of each element.
3. Compute the 2-point DFT of each row. The resulting matrix gives the 6-point DFT in the following arrangement: $\begin{bmatrix} X[0] & X[1] & X[2] \\ X[3] & X[4] & X[5] \end{bmatrix}$
Accordingly, to obtain X[0] and X[3] (unnecessary computations are omitted and indicated

with a '?'):

$$
\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -2 \end{bmatrix}
\begin{matrix} \xrightarrow{\text{3-point DFT}} \\ \xrightarrow{\text{3-point DFT}} \end{matrix}
\begin{bmatrix} 1 & ? & ? \\ 0 & ? & ? \end{bmatrix}
\begin{matrix} \xrightarrow{\cdot W_6^{0\cdot 0}} \\ \xrightarrow{\cdot W_6^{1\cdot 0}} \end{matrix}
\begin{bmatrix} 1 & ? & ? \\ 0 & ? & ? \end{bmatrix}
\;\downarrow\; \text{2-point DFT}\;
\begin{bmatrix} 1 & ? & ? \\ 1 & ? & ? \end{bmatrix}
$$

This is a very general algorithm that works for any sequence of length M=LN . For this specific case (M=2N), it is also possible to use the the decimation-in-time algorithm described in chapter 8.1.3 or 8.2.2.