Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

partial exam EE2S31 SIGNAL PROCESSING
Part 2: July 2, 2019 (13:30–15:30)

Closed book; two sides A4 of handwritten notes permitted

This exam consists of four questions (38 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 1 (9 points)

Given is the stochastic process $X(t) = A|\cos(2\pi ft)|$ with random variable A uniformly distributed in the (continuous) interval $[-0.5, 0.5]$.

(1 p) (a) Take the frequency $f = 1$ Hz and sketch three different realizations of process $X(t)$.

(3 p) (b) Consider only the case $|\cos(2\pi ft)| \neq 0$, and calculate the cumulative distribution function (CDF) $F_X(t)(x)$, as well as the probability density function (PDF) $f_X(t)(x)$. Argue whether $X(t)$ is stationary.

(3 p) (c) Calculate the expected value $E[X(t)]$ and the autocorrelation function $R(t, \tau)$, and argue whether or not process $X(t)$ is wide sense stationary.

Now consider the stochastic process $Y(t) = A\cos(2\pi ft + \Phi)$ with random variable A again uniformly distributed in the (continuous) interval $[-0.5, 0.5]$, and random variable Φ uniformly distributed in the (continuous) interval $[0, 2\pi]$.

(2 p) (d) Determine whether or not process $Y(t)$ is wide sense stationary (WSS).

Hint: make use of the relationship $\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$.

Solution

(1 p) (a)

3 p) (b) For $|\cos(2\pi ft)| \neq 0$:

$$\begin{aligned}
F_{X(t)}(x) &= P(X(t) \leq x) \\
&= P(A|\cos(2\pi ft)| \leq x) \\
&= P(A \leq \frac{x}{|\cos(2\pi ft)|}) \\
&= \int_{-1/2}^{\frac{x}{|\cos(2\pi ft)|}} 1 da \\
&= [a]_{-1/2}^{\frac{x}{|\cos(2\pi ft)|}} \\
&= \frac{x}{|\cos(2\pi ft)|} + 1/2\,,
\end{aligned}$$

for $-1/2|\cos(2\pi ft)| \leq x \leq 1/2|\cos(2\pi ft)|$.

Altogether,

$$F_{X(t)}(x) = \begin{cases} 1\,, & \text{for } x \geq 1/2|\cos(2\pi f t)| \\ \frac{x}{|\cos(2\pi f t)|} + 1/2\,, & \text{for } \quad -1/2|\cos(2\pi f t)| \leq x \leq 1/2|\cos(2\pi f t)| \\ 0\,, & \text{otherwise.} \end{cases}$$

$$f_{X(t)}(x) = \frac{dF_{X(t)}(x)}{dx} = \begin{cases} \frac{1}{|\cos(2\pi f t)|}\,, & \text{for } \quad -1/2|\cos(2\pi f t)| \leq x \leq 1/2|\cos(2\pi f t)| \\ 0\,, & \text{otherwise.} \end{cases}$$

The pdf is not shift invariant and thus the process $X(t)$ is not stationary.

(3 p) (c)

$$E[X(t)] = E[A|\cos(2\pi f t)|] = |\cos(2\pi f t)|E[A] = 0$$

as $E[A] = 0$.

$$R(t,\tau) = E[X(t)X(t+\tau)] = E[A^2]|\cos(2\pi f t)||\cos(2\pi f(t+\tau))|\,.$$

Although $E[X(t)]$ is time independent, $R(t,\tau)$ does depend on the actual time and thus is not time shift invariant. Therefore, the process is also not wide sense stationary.

(2 p) (d)

$$E[X(t)] = E[A\cos(2\pi f t + \Phi)] = E[A]E[\cos(2\pi f t + \Phi)] = 0$$

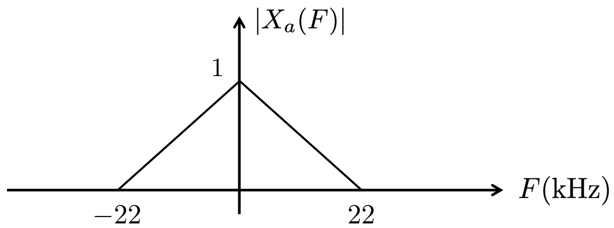
as $E[A] = 0$.

$$\begin{aligned} R(t,\tau) &= E[X(t)X(t+\tau)] \\ &= E[A^2]E[\cos(2\pi f t + \Phi)\cos(2\pi f(t+\tau) + \Phi)] \\ &= \frac{1}{2}E[A^2]\cos(2\pi f\tau) + \frac{1}{2}E[A^2]E[\cos(2\pi f(2t+\tau) + 2\Phi) \\ &= \frac{1}{2}E[A^2]\cos(2\pi f\tau) + \frac{1}{2}E[A^2]\int_0^{2\pi}\cos(2\pi f(2t+\tau) + 2\Phi)d\Phi \\ &= \frac{1}{2}E[A^2]\cos(2\pi f\tau) + \frac{1}{4}E[A^2]\left[\sin(2\pi f(2t+\tau) + 2\Phi)\right]_0^{2\pi} \\ &= \frac{1}{2}E[A^2]\cos(2\pi f\tau)\,. \end{aligned}$$

Thus, the process is wide sense stationary.

Question 2 (9 points)

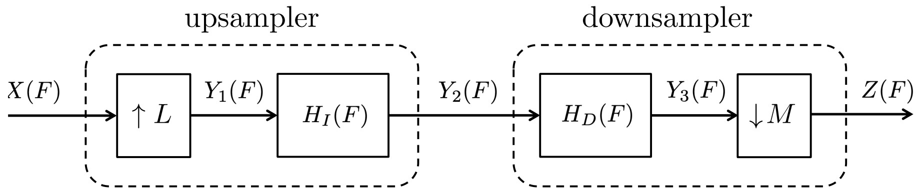
Consider an analogue audio signal, $x_a(t)$, with the following magnitude spectrum $|X_a(F)|$:



Assume that $x(n)$ is a discrete-time version of $x_a(t)$. That is, $x(n) = x_a(nT_s)$ where T_s is the sampling period. Assume that $x_a(t)$ is sampled at its Nyquist rate so that the sampling frequency is given by $F_s = 44$ kHz.

(1 p) (a) Give a sketch of the magnitude spectrum $|X(F)|$ of $x(n)$.

Suppose we have to resample the discrete-time signal $x(n)$ to a rate of 16 kHz. Since the sampling rate needs to be converted by a rational factor, we will do this in two stages. First we will upsample the signal $x(n)$ by a factor of L, after which we downsample the result thus obtained by a factor M. The block diagram of this sample rate convertor is as follows:



Note that the signal $y_2(n)$ is a discrete-time version of $x_a(t)$ sampled at a rate LF_s.

(2 p) (b) Explain in words the purpose of the individual blocks in the figure.

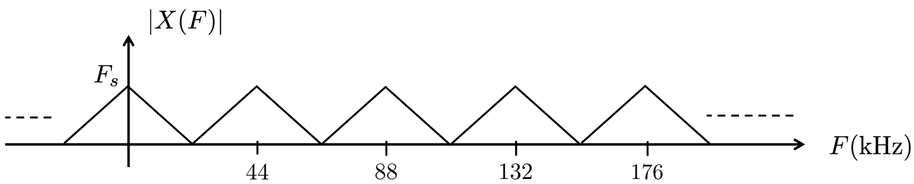
(1 p) (c) Give values for L and M such that the output signal $z(n)$ is sampled at a rate of 16 kHz.

(2 p) (d) What are the specifications of the filters $H_I(F)$ and $H_D(F)$ in terms of pass, stop, and transition band? Motivate your answer.

(3 p) (e) Sketch the spectra of the signals $y_1(n), y_2(n), y_3(n)$ and $z(n)$, both as a function of the normalised angular frequency ω (dimensionless) and the frequency F expressed in Hz.

Solution

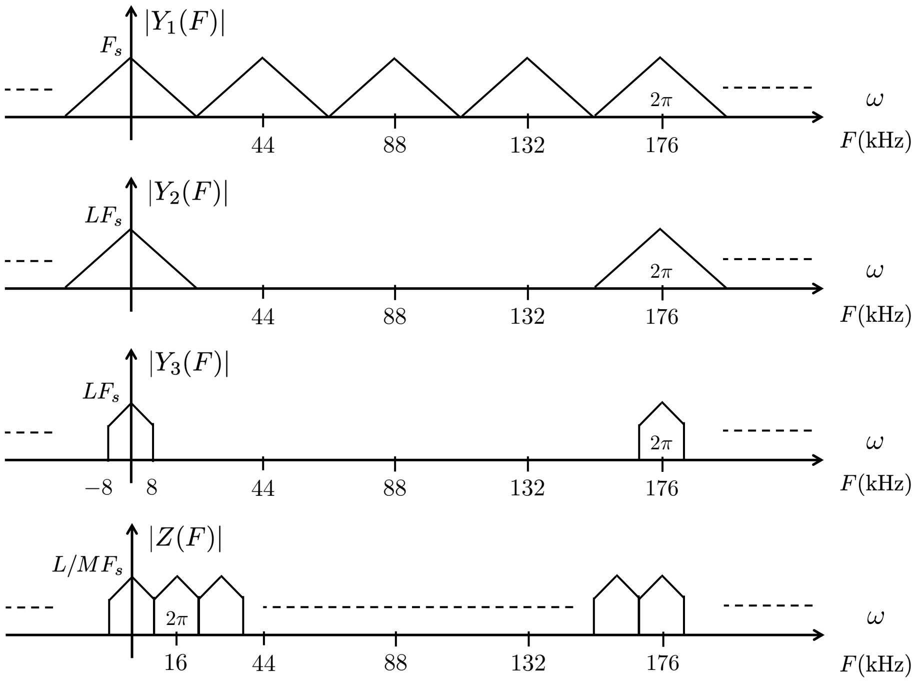
a)



b) The first block, called the expander, increases the sample rate by a factor L by inserting $L-1$ zeros in between every two samples of the input signal $x(n)$. The interpolation filter $H_I(F)$ interpolates the signal thus obtained between the nonzero samples such that the result is an LF_s sampled version of $x_a(t)$. The filter $H_D(F)$ is a decimation filter which reduces the bandwidth of the signal to π/M in order to avoid aliasing introduced in the last block, the downsampler, which leaves out every $M-1$ samples, thereby converting the sampling frequency to a rate of L/MF_s.

c) $L = 4, M = 11$.

d) Both filters are ideal brickwall low-pass filters. The interpolation filter has a bandwidth of 22 kHz in order to reject high-frequency images. The decimation filter should have a bandwidth of $\pi/M = 88/11 = 8$ kHz in order to avoid aliasing.

e) 

Question 3 (10 points)

In this question you might want to use Table 1, shown at the end of the exam.

We are interested in calculating the autocorrelation function at the output of a filter. Let $X[n]$ be a zero-mean uncorrelated Gaussian process with variance σ_X^2. Let the input-output relation of the filter be given by

$$Y[n] = \frac{1}{3}Y[n-1] + X[n].$$

(1 p) (a) Explain how the stochastic process $Y[n]$ is called.

(1 p) (b) Give the autocorrelation function $R_X[k]$ of the input and make a plot of $R_X[k]$.

(1 p) (c) Will the output $Y[n]$ be wide sense stationary? Motivate your answer.

(2 p) (d) Determine the system function $H(z)$ and show that the impulse response is given by $h[n] = \left(\frac{1}{3}\right)^n u[n]$, with the function $u[n]$ denoting the unit-step function.

(2 p) (e) Calculate the cross-correlation $R_{XY}[k]$.

(3 p) (f) Calculate the autocorrelation function $R_Y[k]$.

Hint: Depending on the approach that you take, you might want to use the expression for the geometric series for $|r| < 1$, given by

$$\sum_{k=0}^{b} r^k = \frac{1 - r^{b+1}}{1 - r},$$

and/or Table 1

Solution

(1 p) (a) This is an IIR filter as the output depends on previous outputs. If the input signal for this filter is white Gaussian noise, process $Y[n]$ is called an AR process.

(1 p) (b) $R_X[k]$ is a deltapulse at zero with height σ_X^2, i.e., $R_X[k] = \sigma_X^2 \delta[k]$.

(1 p) (c) The input is WSS and the filter is linear and time-invariant. As a consequence, the output is also WSS.

(2 p) (d) The system function is given by $H(z) = \frac{1}{1-1/3z^{-1}}$. The inverse Z-transform of this is given by $h[n] = \left(\frac{1}{3}\right)^n u[n]$.

(2 p) (e) $P_{XY}(\phi) = H(\phi)P_X(\phi) = \dfrac{\sigma_X^2}{1-1/3e^{-j2\pi\phi}}$, leading to $R_{XY}[k] = \sigma_X^2(\dfrac{1}{3})^n u[n]$.

(3 p) (f)

$$R_Y[k] = \sum_n (\frac{1}{3})^{-n} u[-n] R_{XY}[k-n] = \sum_n (\frac{1}{3})^{-n} u[-n] R_{XY}[-(n-k)]\,.$$

We can split this into a part for which $k \geq 0$ and a part for which $k < 0$.

For $k < 0$:

$$\begin{aligned}
R_Y[k] &= \sum_{n=-\infty}^{k} (\frac{1}{3})^{-n} \sigma_X^2 (\frac{1}{3})^{k-n} \\
&= \frac{\sigma_X^2}{3^k} \sum_{n=-\infty}^{k} (\frac{1}{9})^{-n} \\
&= \frac{\sigma_X^2}{3^k} \sum_{-k}^{\infty} (\frac{1}{9})^{n} \\
&= \frac{\sigma_X^2}{3^k} \left(\frac{1}{1-1/9} - \frac{1-(1/9)^{-k}}{1-1/9} \right) \\
&= \frac{\sigma_X^2}{3^k} \frac{(1/3)^{-2k}}{1-(1/3)^2} \\
&= \sigma_X^2 \frac{(1/3)^{-k}}{1-(1/3)^2}\,.
\end{aligned}$$

For $k \geq 0$:

$$\begin{aligned}
R_Y[k] &= \sum_{n=-\infty}^{0} (\frac{1}{3})^{-n} \sigma_X^2 (\frac{1}{3})^{k-n} \\
&= \frac{\sigma_X^2}{3^k} \sum_{n=-\infty}^{0} (\frac{1}{9})^{-n} \\
&= \frac{\sigma_X^2}{3^k} \sum_{0}^{\infty} (\frac{1}{9})^{n} \\
&= \frac{\sigma_X^2}{3^k} \frac{1}{1-1/9} \\
&= \sigma_X^2 \frac{(1/3)^{k}}{1-(1/3)^2}\,.
\end{aligned}$$

Altogether, we get $R_Y[k] = \frac{9}{8}\sigma_X^2 (1/3)^{|k|}$.

Alternatively, via the Fourier domain:

$$
\begin{aligned}
P_Y &= = |H(\phi)|^2 P_X(\phi) \\
&= \frac{\sigma_X^2}{|1 - 1/3e^{-j2\pi\phi}|} \\
&= \frac{\sigma_X^2}{1 + 1/9 - \frac{2}{3}\cos(2\pi\phi)} ,
\end{aligned}
$$

which leads to

$$R_Y[k] = \frac{8}{9}\sigma_X^2 (\frac{1}{3})^{|k|} .$$

Question 4 (10 points)

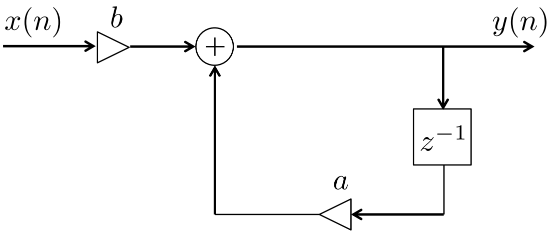
Consider a first-order IIR filter of which the transfer function is given by

$$H(z) = \frac{b}{1 - az^{-1}} ,$$

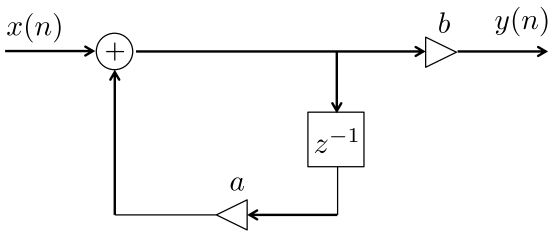
where $b, a \in \mathbb{R}$.

(1 p) (a) Give an expression for the impulse response $h(n)$ of $H(z)$.

This first-order IIR filter can be implemented by each of the following realisations; they only differ in the location of the multiplier b:

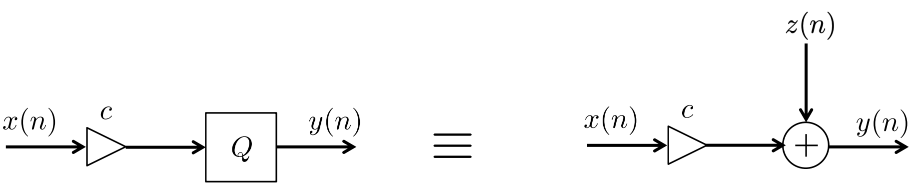


Realization A



Realization B

In a practical scenario we have to quantise the outputs of the multipliers. Assume that the quantisers we use are uniform midtread quantisers with stepsize Δ. Assuming Δ is small enough, the quantisation error can be modelled as an additive noise signal $z(n)$, which is a realisation of an uncorrelated wide-sense stationary process, having a uniform distribution over the interval $[-\frac{\Delta}{2}, \frac{\Delta}{2})$, and uncorrelated with the input signal, schematically shown as follows:



(2 p) (b) Compute the mean and variance of the quantisation noise process.

(3 p) (c) Compute the variance of the total quantisation noise at the output of the digital filter with realization A.

(3 p) (d) Compute the variance of the total quantisation noise at the output of the digital filter with realization B.

(1 p) (e) Given a value of the multiplier a, for what values of b is realisation A favourable over realization B? Motivate your answer.

Solution

a)

$$h(n) = ba^n u(n).$$

b) The distribution of the quantisation noise is given by $p_Z(z) = \Delta^{-1}$ for $z \in [-\frac{\Delta}{2}, \frac{\Delta}{2})$. Hence, the mean is zero due to symmetry. The variance is given by

$$\sigma_Z^2 = \int_{-\Delta/2}^{\Delta/2} z^2 \Delta^{-1} dz = \Delta^{-1} \frac{1}{3} z^3 \Big|_{-\Delta/2}^{\Delta/2} = \frac{\Delta^2}{12}.$$

c) The two noise sources can be moved to the input of the feedback loop which has an impulse response $g(n) = a^n u(n)$. Hence, the total quantisation noise variance at the output of the filter is given by

$$\sigma_{q_a}^2 = 2\frac{\Delta^2}{12} \sum_{n=-\infty}^{\infty} |g(n)|^2 = 2\frac{\Delta^2}{12} \sum_{n=0}^{\infty} (a^n)^2 = 2\frac{\Delta^2}{12} \frac{1}{1-a^2}.$$

d) The noise source due to the multipliers b appear directly at the output of the filter and gives, therefore, a contribution of $\Delta^2/12$. The noise source due to the multiplier a can be moved to the input of the filter, which now includes the multiplier b. With this, the total quantisation noise variance at the output of the filter is given by

$$\begin{aligned} \sigma_{q_b}^2 &= \frac{\Delta^2}{12} \left(1 + \sum_{n=-\infty}^{\infty} |h(n)|^2\right) \\ &= \frac{\Delta^2}{12} \left(1 + \sum_{n=0}^{\infty} (ba^n)^2\right) \\ &= \frac{\Delta^2}{12} \left(1 + \frac{b^2}{1-a^2}\right). \end{aligned}$$

e) We have $\sigma_{q_a}^2 < \sigma_{q_b}^2$ if $2 < 1 + b^2 - a^2$. Hence, we have $\sigma_{q_a}^2 < \sigma_{q_b}^2$ if $b > \sqrt{1 + a^2}$.