Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

exam EE2S31 SIGNAL PROCESSING
July 30th 2018, 13:30–16:30

Closed book; two sides A4 of handwritten notes permitted

This exam consists of five questions (38 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 1 (10 points)

The random variable X and Y have the joint probability density function (pdf)

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{for} \quad -1 \le x \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) Show that the marginal pdfs $f_X(x)$ and $f_Y(y)$ are given by

$$f_X(x) = \frac{1}{2} - \frac{x}{2} \quad \text{for} \quad -1 \le x \le 1.$$

and

$$f_Y(y) = \frac{y}{2} + \frac{1}{2} \quad \text{for} \quad -1 \le y \le 1.$$

b) Calculate $E[X|X \ge 0]$.

c) Calculate $E[X|Y]$.

d) Determine the correlation $E[XY]$.

e) Argue whether or not X and Y are: 1) orthogonal, 2) correlated and 3) independent.

Answer:

(2 p) (a)

$$f_X(x) = \int_{y=x}^{y=1} \frac{1}{2} dy = \left[\frac{y}{2}\right]_{y=x}^{y=1} = \frac{1}{2} - \frac{x}{2} \quad \text{for} \quad -1 \le x \le 1.$$

$$f_Y(y) = \int_{x=-1}^{x=y} \frac{1}{2} dx = \left[\frac{x}{2}\right]_{x=-1}^{x=y} = \frac{y}{2} + \frac{1}{2} \quad \text{for} \quad -1 \le y \le 1.$$

(2 p) (b) $P(X \ge 0) = \int_{x=0}^{x=1} (\frac{1}{2} - \frac{x}{2}) dx = \left[\frac{x}{2} - \frac{x^2}{4}\right]_{x=0}^{x=1} = \frac{1}{4}$

$$f_{X,X\ge 0}(x) = \begin{cases} \frac{1}{\frac{1}{4}}\left(\frac{1}{2} - \frac{x}{2}\right) = 2(1-x) & \text{for} \quad 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$E[X|X \ge 0] = \int_0^1 x 2(1-x) dx = \left[x^2 - \frac{2}{3}x^3\right]_0^1 = \frac{1}{3}$.

(2 p) (c) $E[X|Y] = \int_{-1}^{y} x \frac{1}{y+1} dx = \left[\frac{x^2}{2}\frac{1}{y+1}\right]_{-1}^{y} = \frac{y^2-1}{2(y+1)} = \frac{1}{2}(y-1)$

(2 p) (d) $E[XY] = \int_{-1}^{1}\int_{x}^{1} \frac{xy}{2} dydx = \int_{-1}^{1} \left[\frac{xy^2}{4}\right]_x^1 dx = \int_{-1}^{1} \left[\frac{x}{4} - \frac{x^3}{4}\right] dx = \left[\frac{x^2}{8} - \frac{x^4}{16}\right]_{-1}^{1} = 0.$

(2 p) (e) 1) X and Y are not independent, as $f_X f_Y \neq f_{X,Y}$. 2) X and Y are orthogonal, as the correlation $E[XY] = 0$. Finally, $E[X] = -1/3$ and $E[Y] = 1/3$. Therefore, the covariance is $cov(X,Y) = E[XY] - E[X]E[Y] = 1/9$, and thus, are X and Y correlated. Furthermore, knowing that X and Y are correlated, implies that they are also dependent.

Question 2 (8 points)

We consider the non-stationary stochastic process $X(t)$ with pdf

$$f_{X(t)}(x) = \begin{cases} \frac{1}{3t} & \text{for} \quad 3t \leq x \leq 6t \quad \text{and} \quad t > 0 \\ \delta(x) & \text{for} \quad t \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

a) Calculate $E[X(t)]$ for $t > 0$, and show that it equals $E[X(t)] = 4.5t$.

Given is a linear time invariant system with impulse response

$$h(t) = \begin{cases} 1 & \text{for} \quad 0 \leq t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The input to this system is process $X(t)$, and the output is $Y(t)$.

b) Calculate $E[Y(t)]$ for $t > 0$.

Now we consider a different system with input process $U(t)$ and output process $V(t)$, and impulse response $g(t)$ given by

$$g(t) = \begin{cases} 1 & \text{for} \quad 1 \leq t \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

The autocorrelation function of the input $U(t)$ is given by $R_U(\tau) = \sigma^2\delta(\tau)$.

c) Calculate the cross-correlation $R_{UV}(\tau)$ between input and output.

d) Draw the autocorrelation $R_V(\tau)$ of output $V(t)$.

Answer:

(a) For $t > 0$ we have

$$E[X(t)] = \int_{3t}^{6t} \frac{x}{3t} dx = \left[\frac{x^2}{6t}\right]_{3t}^{6t} = \frac{36t^2 - 9t^2}{6t} = 4.5t$$

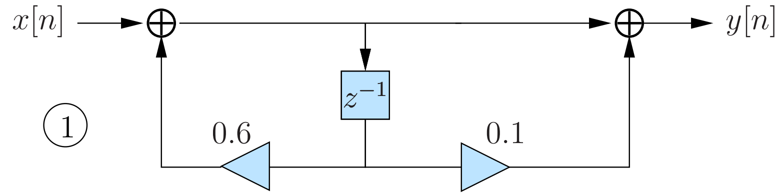
(b) For $t > 0$: $E[Y(t)] = \int_{-\infty}^{\infty} h(u)E\left[X(t-u)\right] du = \int_0^1 4.5(t-u)du = \frac{9}{2}t - \frac{9}{4}$.

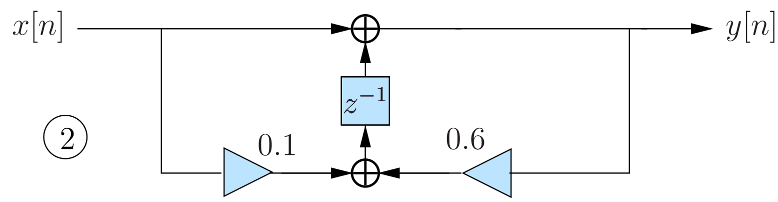
(c) $R_{UV}(\tau) = \int_1^2 \sigma^2\delta(\tau - u)du = \sigma^2$ for $1 \leq \tau \leq 2$. Elsewhere, $R_{UV}(\tau) = 0$.

(d) $R_V(\tau) = \sigma^2(1 - |\tau|)$ for $-1 \leq \tau \leq 1$, elsewhere, $R_V(\tau) = 0$. We can look at $g(\tau)$ as a time-shited block shaped impulse reponse around zero, where the time shift does not influence the final correlation. Convolving $g(\tau)$ and $g(-\tau)$ gives a triangular function of hight σ^2 and baseline from $\tau = -1$ up to $\tau = +1$.

Question 3 (8 points)

Given two realizations:





a) What is the transfer function $H_1(z)$ of the first realization?

b) Prove or motivate that the second realization has the same transfer function.

In a practical scenario, the output of each multiplier is quantized. We use a uniform midtread quantiser with stepsize Δ. For small Δ, the quantization error is modeled as additive noise $e[n]$, which is a realization of an uncorrelated wide-sense stationary process with a uniform distribution over the interval $[-\frac{\Delta}{2}, \frac{\Delta}{2})$, uncorrelated with the input signal.

c) Compute the mean and variance of the quantization noise process.

d) For each of the two realizations shown before, all multipliers are quantized. Compute the variance of the total quantization noise at the output of each digital filter. Which filter has less quantization noise at the output?

Answer:

a) $H(z) = \dfrac{1 + 0.1z^{-1}}{1 - 0.6z^{-1}}$.

b) The second realization is the transpose of the first realization. Therefore they have the same transfer function.

You could also explicitly compute the transfer function of the second filter.

c) $e[n]$ is zero mean and has variance $\sigma_e^2 = \dfrac{\Delta^2}{12}$ (as shown in the course slides).

d) Define equivalent noise sources $e_1[n]$ en $e_2[n]$ at the output of each multiplier. Compute the transfer function $K_1(z)$ and $K_2(z)$ of each noise source to the output of the filter.

For realization 1:

$$\begin{aligned} e_1[n] \to y[n]: &\quad K_1(z) = \tfrac{1+0.1z^{-1}}{1-0.6z^{-1}} = 1 + \tfrac{0.7z^{-1}}{1-0.6z^{-1}} \\ e_2[n] \to y[n]: &\quad K_2(z) = 1 \end{aligned}$$

$$k_1 = [\boxed{1},\ 0.7,\ 0.7(0.6),\ 0.7(0.6)^2,\ \cdots]$$

$$\sum_{n=0}^{\infty}|k_1[n]|^2 = 1+(0.7)^2\sum_{n=0}^{\infty}(0.6)^{2n} = 1+(0.7)^2\frac{1}{1-(0.6)^2} = 1.7656$$

The noise power at the output of realization 1 is

$$\sigma_1^2 = \frac{\Delta^2}{12}\left(\sum_{n=0}^{\infty}|k_1[n]|^2 + \sum_{n=0}^{\infty}|k_2[n]|^2\right) = \frac{\Delta^2}{12}\cdot 2.76$$

For realization 2:

$$K_1(z) = K_2(z) = \frac{z^{-1}}{1-0.6z^{-1}}$$

$$\sum_{n=0}^{\infty}|k_1[n]|^2 = \sum_{n=0}^{\infty}|k_2[n]|^2 = \frac{1}{1-(0.6)^2} = 1.85$$

The noise power at the output of realization 2 is

$$\sigma_2^2 = \frac{\Delta^2}{12}\left(\sum_{n=0}^{\infty}|k_1[n]|^2 + \sum_{n=0}^{\infty}|k_2[n]|^2\right) = \frac{\Delta^2}{12}\cdot 3.12$$

Therefore, the realization with the smallest noise power at the output is realization 1.

Question 4 (4 points)

Given the signals $x[n] = [\boxed{1}, 1, 0, 0, 0]$, $h[n] = [\boxed{0}, 1, 2, 3, 4]$, $s[n] = [\boxed{1}, 0, 0, 0, 0]$.

Denote the corresponding DFTs (based on $N = 5$) by $X[k]$, $H[k]$, $S[k]$.

a) Compute the signal $y[n]$ such that $Y[k] = H[k]\,X[k]$.

b) Is there a signal $g[n]$ such that $S[k] = H[k]\,G[k]$? Give equations that explicitly show how you could compute that signal.

Answer:

a) Multiplication of DFTs corresponds to *circular* convolution. The result has length $N = 5$. Hence

$$y = h \circledast x = [\boxed{0}, 1, 2, 3, 4] + [\boxed{4}, 0, 1, 2, 3] = [\boxed{4}, 1, 3, 5, 7]$$

b) We require $s = h \circledast g$, with $g = [g[0], g[1], \cdots, g[4]]$.

Circular convolution corresponds to the following matrix multiplication:

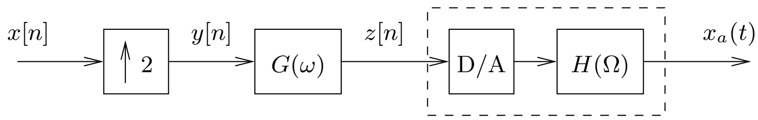
$$\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0&4&3&2&1\\1&0&4&3&2\\2&1&0&4&3\\3&2&1&0&4\\4&3&2&1&0\end{bmatrix}\begin{bmatrix}g[0]\\g[1]\\g[2]\\g[3]\\g[4]\end{bmatrix}$$

We can solve this as

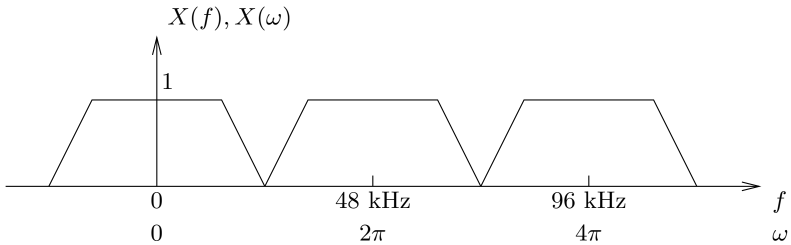
$$\begin{bmatrix}g[0]\\g[1]\\g[2]\\g[3]\\g[4]\end{bmatrix} = \begin{bmatrix}0&4&3&2&1\\1&0&4&3&2\\2&1&0&4&3\\3&2&1&0&4\\4&3&2&1&0\end{bmatrix}^{-1}\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} = \cdots = \begin{bmatrix}-0.18\\0.22\\0.02\\0.02\\0.02\end{bmatrix}$$

Alternatively, let $G[k] = \dfrac{S[k]}{H[k]}$ and compute $g[n]$ as the IFFT of $G[k]$.

Question 5 (8 points)



The figure above shows a block diagram of a two-times oversampled D/A convertor. The spectrum of the input signal $x[n]$ is as follows:



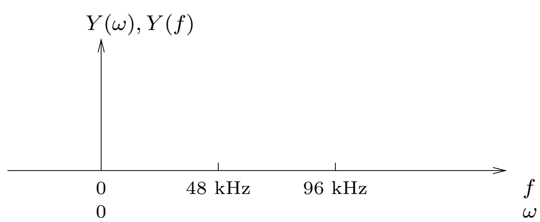
The spectrum is shown both as a function of the angular frequency ω (dimensionless) and of the corresponding frequency f expressed in Hertz (Hz).

a) What is the sampling frequency f_s at which the input signal $x[n]$ has been sampled? Motivate your answer.

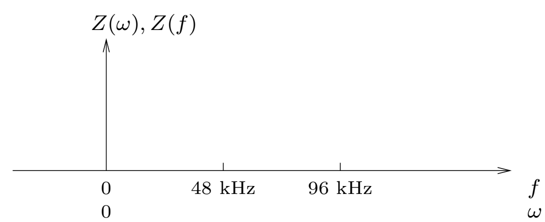
b) Draw the spectrum of the analog signal $x_a(t)$ that we wish to recover.

c) Explain in words what the purpose of the different blocks in the block diagram is, and what the advantage is of such an oversampled D/A convertor over a standard (non-oversampled) D/A convertor.

d) Assume that the digital filter $G(\omega)$ is a "perfect"brick-wall filter with cut-off frequency $f_c = 24$ kHz. Copy the figures below and sketch the spectrum of $y[n]$ and $z[n]$ both as a function of f and ω.



a)



b)

e) Suppose we would omit the expander (upsampler). Why can't we directly filter out the frequencies above 24 kHz?

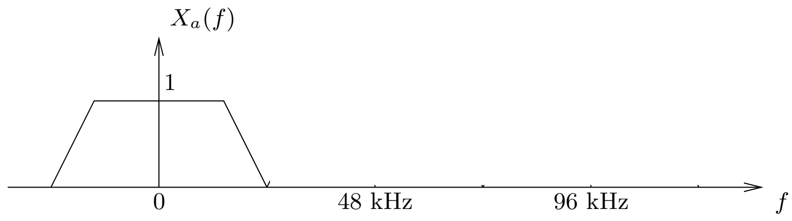
The actual digital-to-analog conversion takes place in the dashed box in the block diagram.

f) What is the maximum transition bandwidth (frequency band between the pass- and stop-band) of the analog interpolation filter $H(\Omega)$ such that we can perfectly reconstruct our audio signal?

Answer:

a) The sampling frequency is 48 kHz. This is the frequency that corresponds to the normalized angular frequency $\omega = 2\pi$.

b)



c) The first block (expander) increases the sampling frequency by a factor 2 without effecting the spectrum (only the periodicity of the spectrum changes) by inserting a zero in between every sample of $x[n]$. The digital (low-pass) filter interpolates the values in between the non-zero values of $y[n]$. The block D/A generates a continuous-time pulse train of which the height of the pulses correspond to the values of the samples of $z[n]$. The analog low-pass filter finally interpolates the values in between the pulses.

d)

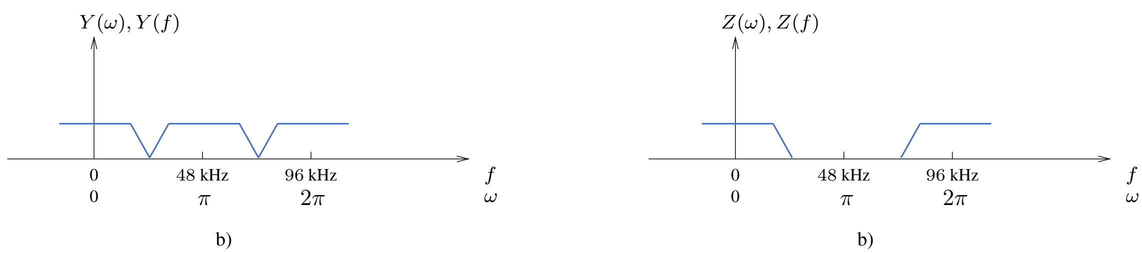


Figure 3: Sketch of frequency domain representation of y and z, both as a function of f and ω.

e) Because in a digital filter the maximal frequency is $\omega = \pi$, which before upsampling corresponds to $f = f_s/2 = 24$ kHz.

Even a perfect brickwall filter is useless.

f) Passband until 24 kHz, stopband at 72 kHz, hence the transition bandwidth is 48 kHz.