Answers Exam EE2S31 Signaalbewerking

June 24th, 2016

Answer each question on a **separate sheet**. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet. It is allowed to answer in Dutch or English.

Question 1 (8 points)

(2 p) (a) $\Phi_X(s) = E[e^{sX}] = \lambda \int_0^\infty e^{x(s-\lambda)}dx = \frac{\lambda}{\lambda - s}$

(2 p) (b) $E[X] = \frac{d\Phi_X(s)}{ds}\,|_{s=0} = \frac{\lambda}{(\lambda-s)^2}\,|_{s=0} = \frac{1}{\lambda}$ and

$E[X^2] = \frac{d^2\Phi_X(s)}{ds^2}\,|_{s=0} = \frac{d\frac{\lambda}{(\lambda-s)^2})}{ds}\,|_{s=0} = \frac{2\lambda}{(\lambda-s)^3}\,|_{s=0} = \frac{2}{\lambda^2}$

(1 p) (c) For the sum of two independent RVs it holds that the MGF is the product of the two MGFs, that is, $\Phi_Y(s) = \left(\frac{\lambda}{\lambda-s}\right)^2$. The pdf is then given by

$$f_Y(y) = \begin{cases} \lambda^2 y e^{-\lambda y} & \text{for} \quad y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The joint pdf $f_{X,Y}(x,y)$ between processes X and Y is given by

$$f_{X,Y}(x,y) = \lambda^2 e^{-\lambda y} \quad \text{for} \quad 0 \leq x \leq y \leq \infty.$$

(3 p) (d) The conditional pdf $f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(x) = \frac{1}{y}$ for $0 \leq x \leq y \leq \infty$. $E[X|y]$ is then given by

$$E[X|y] = \int_0^y \frac{x}{y} dx = \frac{1}{2}y.$$

Question 2 (7 points)

(2 p) (a) Decorrelated implies $R_Y[k] = \sigma_X^2\delta[k]$. Thus, $|H(f)| = \sqrt{S_Y(f)/S_X(f)} = \sqrt{\frac{1}{3+e^{-j2\pi f}+e^{j2\pi f}}}$.

(2 p) (b) $R_{XY}[k] = \sigma_X^2\delta[k] + 3\sigma_X^2\delta[k-1] + \sigma_X^2\delta[k-2]$ (just a delay). and $R_Y[k] = R_X[k]$ as a delay does not change the autocorrelation function. Hence $|H(f)|^2 = 1$.

(1 p) (c) $R_{XY}[k] = R_X[k] \star h[k] = \sigma_X^2 a^n u[n]$.

(2 p) (d) $R_Y[k] = R_{XY}[k] \star h[-k]$

$$R_Y[k] = \sigma_X^2 \sum_m a^m u[m] a^{-(k-m)} u[-(k-m)] = \sigma_X^2 a^{-k} \sum_m a^{2m} u[m] u[m-k].$$

for $k \geq 0$ we get

$$R_Y[k] = \sigma_X^2 a^{-k} \sum_{m=k}^{\infty} a^{2m} = \frac{\sigma_X^2 a^k}{1-a^2}$$

and for $k < 0$ we get

$$R_Y[k] = \sigma_X^2 a^{-k} \sum_{m=0}^{\infty} a^{2m} = \frac{\sigma_X^2 a^{-k}}{1-a^2}.$$

Altogether we get

$$R_Y[k] = \frac{\sigma_X^2 a^{|k|}}{1-a^2}$$

Question 3 (9 points)

(1 p) (a) The variance of the difference signal $X(z) - R(z)$ is in general smaller than the variance of $X(z)$ itself, and therefore requires less bits to obtain a certain SNR.

(1 p) (b) If we oversample by a factor 2 and quantise the oversampled data, the quantisation error (white) will be spread out over the complete spectrum, of which only half of it contains signal information. We can, therefore, filter out half of the spectrum without distorting the signal but halving the variance of the noise signal. This results in a 3 dB gain in SNR.

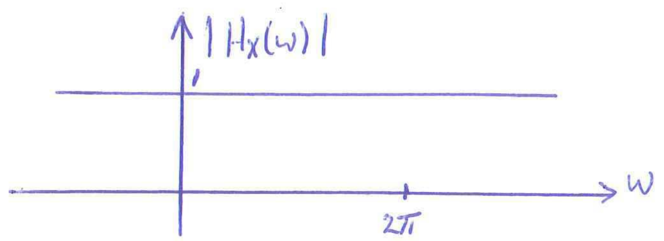
(2 p) (c) The total transfer function of the DM can be expressed as

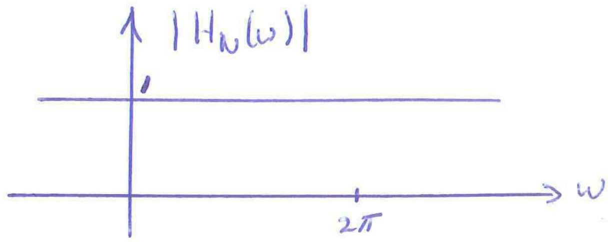
$$R(z) = H_{\text{acc}}(z)C(z) = H_{\text{acc}}(z)\left(\frac{z^{-1}}{1+z^{-1}H_{\text{acc}}(z)}X(z) + \frac{1}{1+z^{-1}H_{\text{acc}}(z)}N(z)\right),$$

so that

$$H_X(z) = z^{-1} \quad \text{and} \quad H_N(z) = 1.$$

(1 p) (d) $|H_X(\omega)| = |H_N(\omega)| = 1$ for all $\omega \in [0, 2\pi)$.





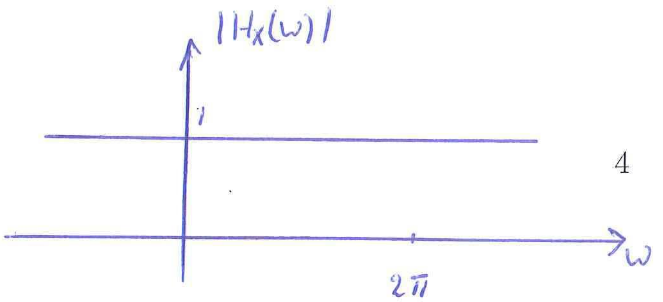
(2 p) (e) The total transfer function of the SDM can be expressed as

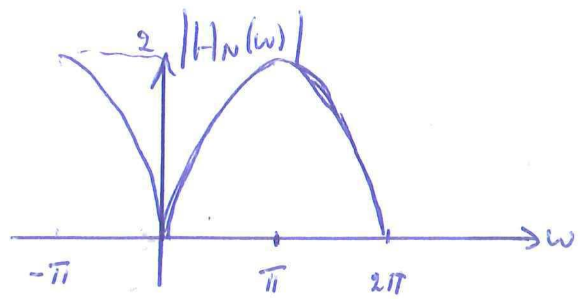
$$C(z) = \frac{z^{-1}H_{\text{acc}}(z)}{1+z^{-1}H_{\text{acc}}(z)}X(z) + \frac{1}{1+z^{-1}H_{\text{acc}}(z)}N(z),$$

so that

$$H_X(z) = z^{-1} \quad \text{and} \quad H_N(z) = 1 - z^{-1}.$$

(1 p) (f) $|H_X(\omega)| = 1$ and $|H_N(\omega)| = 2|\sin(\omega/2)|$ for all $\omega \in [0, 2\pi)$.



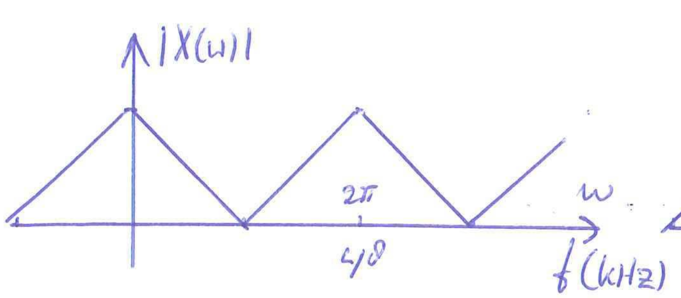


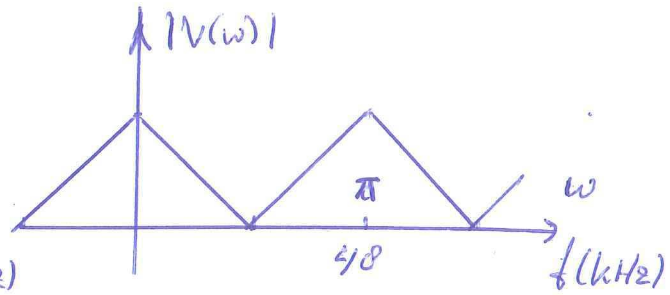
Question 4 (7 points)

(1 p) (a) The expander block inserts $L-1$ zeros in between every sample of $x(n)$, thereby increasing the sampling rate by a factor L. The filter $H(z)$ band limits the signal to a bandwidth of π/M so that no aliasing will occur after decimation by a factor M done in the last block (keeps 1 out of M samples, thereby decreasing the sampling rate).

(1 p) (b) $L = 2, M = 3$, since $48L/M = 32$.

(2 p) (c) We have $V(\omega) = X(\omega L)$.

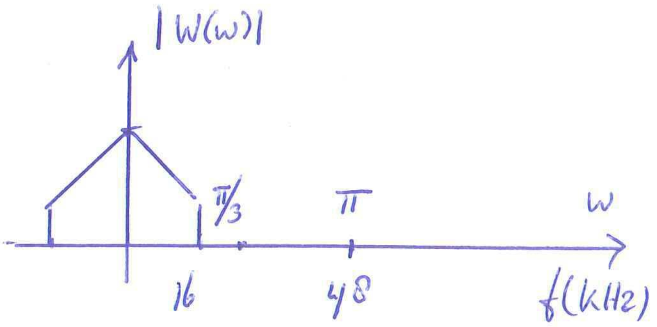


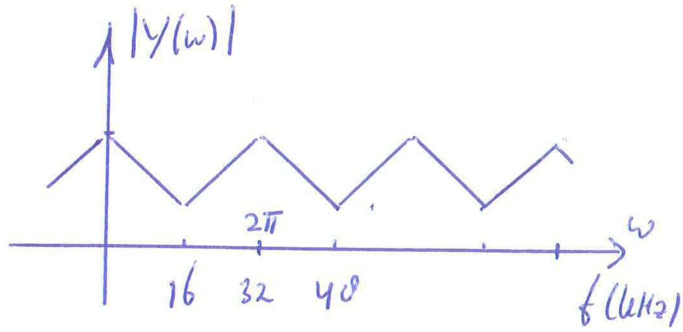


(1 p) (d) The filter should be an ideal (brick-wall) filter with cut-off frequency $\pi/M = \pi/3$ which corresponds to an unnormalised cut-off frequency of 16 kHz.

(2 p) (e)

$$y(\omega) = \frac{1}{3}\sum_{k=0}^{M-1} W\left((\omega - 2\pi k)/M\right) = \frac{1}{3}\sum_{k=0}^{M-1} X\left((\omega L - 2\pi k)/M\right) H\left((\omega - 2\pi k)/M\right)$$





(1 p) (g) The 1-bit quantiser gives rise to approximately 6 dB SNR. The remaining 90 dB should be obtained by oversampling and noise shaping. Since we get an SNR improvement of 9 dB per doubling of the sampling frequency, we conclude that we need an oversampling rate of $2^{10} = 1024$.