Answers Exam EE2S31 Signaalbewerking

July 29th, 2016

Answer each question on a **separate sheet**. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet. It is allowed to answer in Dutch or English.

Question 1 (9 points)

(2 p) (a) $f_X(x) = \int_{y=0}^{x^2} \frac{x^4}{2} dy = \left[\frac{x^4 y}{2}\right]_{y=0}^{y=x^2} = \frac{x^6}{2}$

$f_Y(y) = \int_{x=\sqrt{y}}^{1} \frac{x^4}{2} dx = \left[\frac{x^5}{10}\right]_{x=\sqrt{y}}^{x=1} = \frac{1}{10}(1 - y^{5/2})$.

(1 p) (b) $f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(y) = \frac{5x^4}{1-y^{5/2}}$ for $0 \leq x \leq 1$ and $0 \leq y \leq x^2$

(2 p) (c) Calculate $\hat{X} = E[X] = \int_0^1 \frac{5x^5}{2} dx = \frac{5}{12}$.

(2 p) (d) Calculate $\hat{X} = E[X|y] = \int_0^1 f_{X|Y}(x|y) dx = \int_0^1 \frac{x^3}{2(1-y^{3/2})} dx = \frac{1}{8(1-y^{3/2})}$.

(1 p) (e) If X and Y are dependent, knowing something about the realizatio of Y tells something about the realization of x. The conditional information on y gives thus additional information on x.

(1 p) (f) When X and Y are independent.

Question 2 (11 points)

(1 p) (a)

(1 p) (b) The distribution of $X(t)$ will be time dependent and not shift invariant. Process $X(t)$ is therefore thus non-stationary.

(2 p) (c) The moment generating function for A is $E[e^{sA}] = \frac{\lambda}{\lambda - s}$. $E[A] = \left.\frac{d\phi_A(s)}{ds}\right|_{s=0} = \frac{1}{\lambda}$. $E[A^2] = \left.\frac{d^2\phi_A(s)}{ds^2}\right|_{s=0} = \frac{2}{\lambda^2}$.

(1 p) (d) $E[X(t)] = t + \frac{1}{\lambda}$.

(1 p) (e) $R_X(t,\tau) = E[(t+A)(t+\tau+A)] = t^2 + t\tau + (2t+\tau)E[A] + E[A^2]$.

(3 f) (h)

$$\begin{aligned} f(\tau) &= h(\tau) * h(-\tau) \\ &= \int_{-\infty}^{\infty} 4e^{-2t}u(t)4e^{-2t+2\tau}u(-\tau+t)dt \\ &= \begin{cases} 16e^{2\tau}\int_{\tau}^{\infty} e^{-4t}dt = 4e^{-2\tau} & \text{if } \tau \geq 0 \\ 16e^{2\tau}\int_{0}^{\infty} e^{-4t}dt = 4e^{2\tau} & \text{if } \tau < 0 \end{cases} \end{aligned}$$

$$R_Y(\tau) = R_X(\tau) * f(\tau) = \sigma_X^2 4e^{2|\tau|}.$$

Consider now the case where the input to the system in Fig. 1 has changed into $X'(t) = X(t) + 1$

(2 p) (g) $E[Y(t)] = E[X'(t)]\int_0^{\infty} h(t)dt = \int_0^{\infty} 4e^{-2t}dt = 2$

Question 3 (9 points)

(2 p) (a) The input-output relation is given by $Y(z) = bX(z) + az^{-1}Y(z)$ so that

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{1 - az^{-1}} = \frac{bz}{z - a},$$

with region of convergence $|z| > |a|$ since the filter is causal (ROC is the exterior of a circle). The transfer function has a zero at $z = 0$ and a pole at $z = a$.

(1 p) (b) The impulse response is given by $h(n) = ba^n u(n)$.

(1 p) (c) The system is BIBO stable if all poles lie inside the unit circle. Hence $|a| < 1$ and no restrictions on b.

(2 p) (d) The distribution of the quantisation noise is given by $p_E(e) = \Delta^{-1}$ for $e \in [-\frac{\Delta}{2}, \frac{\Delta}{2})$. Hence, the mean is zero due to the symmetric. The variance is given by

$$\sigma_E^2 = \int_{-\Delta/2}^{\Delta/2} e^2 \Delta^{-1} de = \Delta^{-1} \left. \frac{1}{3} e^3 \right|_{-\Delta/2}^{\Delta/2} = \frac{\Delta^2}{12}.$$

(3 p) (e) The transfer function from both noise source to the output is given by $\frac{z}{z-a}$, which corresponds to an impulse response $a^n u(n)$. The total variance seen at the output of the filter is thus given by

$$\sigma_{\text{tot}}^2 = 2\frac{\Delta^2}{12} \sum_n |a^n u(n)|^2 = 2\frac{\Delta^2}{12} \frac{1}{1 - a^2}.$$

Question 4 (11 points)

(1 p) (a) The signal is continuous-time since the spectrum is non-periodic.

(1 p) (b) The signal is non-periodic since the spectrum is continuous.

(1 p) (c) $x_s(n) = x(nT_s)$ with $T_s = 1/f_s$.

(1 p) (d)

(2 p) (e) The first block (upsampler) increases the sampling frequency by a factor 4 without effecting the spectrum (only the periodicity of the spectrum changes) by inserting 3 zeros in between every sample of $x_s(n)$. The digital (low-pass) filter interpolates the values in between the non-zero values of $y(n)$. The block D/A generates a continuous-time pulse train of which the height of the pulses correspond to the values of the samples of $z(n)$. The analog low-pass filter finally interpolates the values in between the pulses.

(1 p) (f) Since the sampling frequency is 44 kHz and the signal has a bandwidth of 20 kHz, the passband should be 20 kHz, followed by a transition bandwidth of 4 kHz. The stopband starts at 24 kHz.

(3 p) (g)

$$Y(\omega) = X_s(4\omega), \quad Z(\omega) = Y(\omega)H(\omega) = X_s(4\omega)H(\omega).$$

(1 p) (h) Transition bandwidth is 20 kHz, transition bandwidth from $20 - 156$ kHz, and stopband from 156 kHz on.