Exam EE2S31 Signaalbewerking

July 29th, 2016

Answer each question on a **separate sheet**. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet. It is allowed to answer in Dutch or English.

Question 1 (9 points)

A joint probability density function of the random variables X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{x^4}{2} & \text{for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq x^2 \\ 0 & \text{elsewhere.} \end{cases}$$

(2 p) (a) Compute the marginal pdfs of the random variables X and Y.

(1 p) (b) Compute the conditional pdf $f_{X|Y}(x|y)$.

For the remainder of this question, assume that $f_{X|Y}(x|y)$ is given by

$$f_{X|Y}(x|y) = \begin{cases} \frac{3x^2}{1-y^{3/2}} & \text{for } 0 \leq x \leq 1, \text{ and } 0 \leq y \leq x^2 \\ 0 & \text{elsewhere,} \end{cases}$$

and $f_X(x)$ is given by

$$f_X(x) = \begin{cases} \frac{5x^4}{2} & \text{for } 0 \leq x \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

In an experiment we observe realizations of random variable Y, while we want to make an estimate of X. To do so, we determine two different estimators for X.

(2 p) (c) Calculate $\hat{X} = E[X]$.

(2 p) (d) Calculate $\hat{X} = E[X|y]$.

(1 p) (e) Explain in words which of the two estimators from Question 1(c) and 1(d) is better.

(1 p) (f) Under which conditions are the two estimators in Question 1(c) and 1(d) equal?

Question 2 (11 points)

Given is the following stochastic process $x(t)$

$$X(t) = t + A,$$

with time t, $t \geq 0$ and random variable A with probability density function

$$f_A(a) = \begin{cases} \lambda e^{-\lambda a} & \text{for } a \geq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

(1 p) (a) Plot three different realizations of process $X(t)$.

(1 p) (b) Argue whether or not this is a stationary process.

(2 p) (c) Calculate the first two moments of A, i.e., $E[A]$ and $E[A^2]$.
Hint: The moment generating function for a random variable A is defined as $E[e^{sA}]$.

(1 p) (d) Determine the expected value of process $X(t)$.

(1 p) (e) Determine the autocorrelation function $R_X(t, \tau)$ of process $X(t)$.

For the following questions, we consider a different process $X(t)$ which is used as an input to the following linear time invariant system:

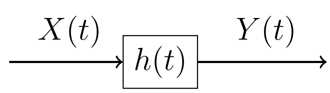

Figure 1: Given LTI system.

The impulse reponse is given by

$$h(t) = \begin{cases} 4e^{-2t} & \text{for } t \geq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

The input process is zero-mean, uncorrelated with variance σ_X^2.

(3 f) (f) Calculate the autocorrelation function $R_Y(t)$.

Consider now the case where the input to the system in Fig. 1 has changed into $X'(t) = X(t) + 1$

(2 p) (g) Calculate the expected value of the output for input $X(t)'$.

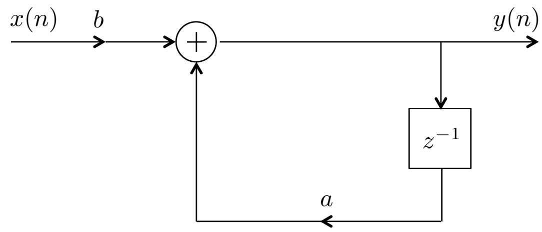

Figure 2: Realisation of a digital filter.

Question 3 (9 points)

Consider the causal digital filter of which the realisation is depicted in Figure 2.

(2 p) (a) Show that the transfer function of the filter is given by

$$H(z) = \frac{bz}{z-a}, \quad |z| < |a|,$$

and give the corresponding pole-zero plot.

(1 p) (b) Compute the impulse response of the filter.

(1 p) (c) For what values of a and b is the filter BIBO stable? Motivate your answer.

In a practical scenario we have to quantise the two outputs of the multipliers in Figure 2. Assume that the quantiser we use is a uniform midtread quantiser with stepsize Δ. Assuming Δ is small enough, the quantisation error can be modelled as an additive noise signal $e(n)$, as depicted in Figure 3, which is a realisation of an uncorrelated wide-sense stationary process, having a uniform distribution over the interval $[-\frac{\Delta}{2}, \frac{\Delta}{2})$, and is uncorrelated with the input signal.

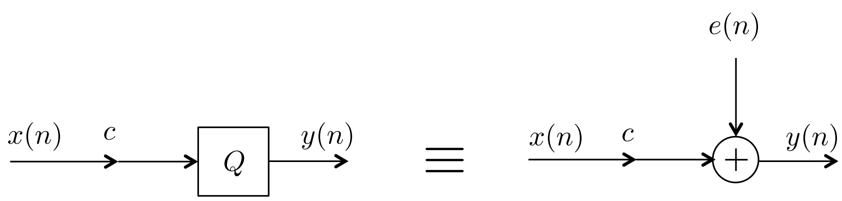

Figure 3: Additive noise model for quantisation noise.

(2 p) (d) Compute the mean and variance of the quantisation noise process.

(3 p) (e) Compute the variance of the total quantisation noise at the output of the digital filter depicted in Figure 2.

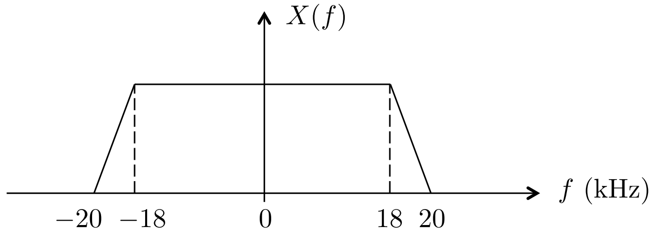

Figure 4: Spectrum $X(f)$.

Question 4 (11 points)

Consider an audio signal x of which its spectrum is given as depicted in Figure 4.

(1 p) (a) Is this audio signal a discrete-time or continuous-time signal? Motivate your answer.

(1 p) (b) Is this audio signal a periodic or a non-periodic signal? Motivate your answer.

Suppose we sample the audio signal $x(t)$ with sampling frequency $f_s = 44$ kHz.

(1 p) (c) What is the relation between $x(t)$ and $x_s(n)$?

(1 p) (d) Sketch the spectrum of $x_s(n)$, both as a function of the normalised angular frequency ω (dimensionless) and the frequency f expressed in cycles/sec (or, equivalently, Hertz (Hz)).

Suppose we use a 4-times oversampled D/A convertor to reconstruct the original audio signal $x(t)$ out of its samples $x_s(n)$, of which the block diagram is depicted in Figure 5.

(2 p) (e) Explain in words what the purpose of the different blocks in Figure 5 is and what the advantage is of such an oversampled D/A convertor over a standard (non-oversampled) D/A convertor.

(1 p) (f) What are the specifications of the filter $H(\omega)$ in terms of pass, stop, and transition band?

(3 p) (g) Give an expression for the signals $y(n)$ and $z(n)$ as a function of the input signal $x_s(n)$ in the frequency domain and give a sketch of the spectra of y and z both as a function of the normalised angular frequency ω (dimensionless) and the frequency f expressed in cycles/sec (or, equivalently, Hertz (Hz)).

(1 p) (h) What are the specifications of the analog low-pass filter $H_I(\Omega)$ in terms of pass, stop, and transition band?

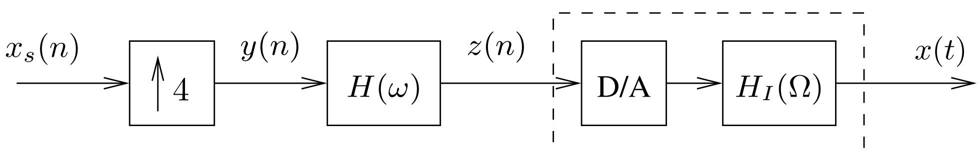

Figure 5: Block diagram of the oversampled D/A convertor.