Chapter 4 – Frequency Analysis: The Fourier Series

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Fourier series

Jean Baptiste Joseph Fourier (1768-1830)

Fourier’s idea:

*any* periodic function can be written as a weighted sum of sines and cosines of different frequencies.

(not exactly true)
Fourier series

\[ x(t) = c_0 + 2 \sum_{k=1}^{\infty} \left( c_k \cos(k\Omega_0 t) + d_k \sin(k\Omega_0 t) \right), \quad \Omega_0 = \frac{2\pi}{T_0} \]

When Fourier submitted his paper in 1807, the committee (which included Lagrange, Laplace, Malus and Legendre, among others) concluded:

... the manner in which the author arrives at these equations is not exempt of difficulties and [...] his analysis to integrate them still leaves something to be desired on the score of generality and even rigour.

In that book, Fourier extended his finding to non-periodic signals, stating that such a signal can be represented by a weighted integral of a series of sine and cosine functions. Such an integral is termed the *Fourier transform*. 
What in this chapter?

• Eigenfunctions and LTI systems
• Complex and trigonometric Fourier series
• Spectrum of periodic signals
• Fourier series and Laplace transform
• Properties of Fourier series
• Convergence of Fourier series
Consider a LTI system with input signal \( x(t) = e^{s_0 t}, t \in \mathbb{R} \):

\[
y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s_0 (t-\tau)} d\tau
\]

\[
= e^{s_0 t} \int_{-\infty}^{\infty} h(\tau) e^{-s_0 \tau} d\tau
= e^{s_0 t} H(s_0)
\]

assuming \( H(s_0) \) exists \( (s_0 \in \text{ROC}) \)

\* \( e^{s_0 t} \) is called an \textit{eigenfunction} of the LTI system
Eigenfunctions revisited

Special case:

- $e^{s_0 t} = e^{j\Omega_0 t}$: harmonic signal \quad \Rightarrow \quad y(t) = H(j\Omega_0)e^{j\Omega_0 t}$

The function $H(j\Omega)$ is called the frequency response of the LTI system:

$$y(t) = H(j\Omega_0)e^{j\Omega_0 t} = |H(j\Omega_0)|e^{j(\Omega_0 t + \angle H(j\Omega_0))}$$

- magnitude response $|H(j\Omega_0)|$ modifies the magnitude of $e^{j\Omega_0 t}$
- phase response $\angle H(j\Omega_0)$ modifies the phase of $e^{j\Omega_0 t}$
Express $x$ as a linear combination of harmonics $e^{j\Omega_k t}$:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\Omega_k t} \quad \Rightarrow \quad y(t) = \sum_{k=-\infty}^{\infty} H(j\Omega_k) X_k e^{j\Omega_k t}$$

Fourier series

or

Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega \quad \Rightarrow \quad y(t) = \int_{-\infty}^{\infty} H(j\Omega) X(\Omega) e^{j\Omega t} d\Omega$$
Why study this special case?

- Fourier transform is probably the most widely applied signal processing tool in science and engineering
- Ties together two of the most used phenomenas known to engineers: those of time and frequency
- Time and frequency are dual domains
- Many signal manipulations are done in the frequency domain including filtering, sampling, modulation, etc.
- Harmonic signals appear naturally in many applications
- Steady-state analysis
Fourier series

Can we represent/approximate any signal by sinusoids?

Approximation using 9 sinusoids

Error using 9 sinusoids

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Fourier series

Key questions:

• Does any signal has such a representation?

• How should we choose the frequencies of the constituent sinusoids?

• How do we find the weights?

• How many do we need, finite or infinite many?

• How does the sequence converge (in norm, pointwise, uniformly)?
Orthonormal system

\[ e_1, e_2 \]

\[ \mathbb{R}^2 \]

\[ (x, e_1) \]

\[ (x, e_2) \]

\[ (e_k, e_m) = \begin{cases} 
1, & k = m \\
0, & k \neq m 
\end{cases} \]

Pythagoras' theorem

\[ x = \sum_{k=1}^{2} (x, e_k) e_k, \quad \|x\|^2 = \sum_{k=1}^{2} |(x, e_k)|^2 \]
Orthonormal system

- The functions \( \{\psi_k(t), t \in [a, b]\} \) are called orthonormal (orthogonal and normalized) if

\[
(\psi_k, \psi_m) = \int_a^b \psi_k(t)\psi_m^*(t)dt = \begin{cases} 
0 & k \neq m \\
1 & k = m 
\end{cases}
\]

- To ensure the existence of the norm and the inner product, we assume the functions have finite energy. That is, \( \psi_k \in L^2([a, b]) \) where

\[
L^2(E) = \{f : \int_E |f(t)|^2dt < \infty\}
\]
**Orthonormal system**

**Theorem:** Let \( \{ \psi_k \}_{k=1}^{\infty} \) denote a complete orthonormal system in \( L^2(E) \) and let \( x \in L^2(E) \). Then

\[
x = \sum_{k=1}^{\infty} (x, \psi_k) \psi_k
\]

Moreover, we have (Parseval’s identity)

\[
\|x\|^2 = \sum_{k=1}^{\infty} |(x, \psi_k)|^2
\]
Complex Fourier series

The Fourier series representation of a periodic signal $x(t)$ of period $T_0$ is given by

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}, \quad \Omega_0 = \frac{2\pi}{T_0}$$

with Fourier coefficients $X_k$

$$X_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jk\Omega_0 t} dt, \quad k \in \mathbb{Z}$$

Moreover, we have (Parseval’s identity):

$$\|x\|^2 = T_0 \sum_{k=-\infty}^{\infty} |X_k|^2$$
Complex Fourier series

- Fourier series determine the frequency components of periodic signals and how the power is distributed over the frequency components, called the *spectrum*.
- The spectrum of periodic signals is *discrete* (line spectrum).
- For real signals, the spectrum is conjugate symmetric:

\[
X_{-k} = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{jk\Omega_0 t} dt = X_k^*.
\]

- As a consequence, \( |X_{-k}| = |X_k| \) and \( \angle X_{-k} = -\angle X_k \)
Complex Fourier series

Example (periodic pulse train):

- DC (average) value of 1
Complex Fourier series

- Complex Fourier series

\[ X_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\Omega_0 t} dt = \frac{2}{T_0} \int_{-T_0/4}^{T_0/4} e^{-jk\Omega_0 t} dt = \begin{cases} \frac{-1}{jk\pi} e^{-jk\Omega_0 t} \bigg|_{-T_0/4}^{T_0/4}, & k \neq 0 \\ 1, & k = 0 \end{cases} \]

\[ = \begin{cases} \frac{2}{k\pi} \sin(k\pi/2), & k \neq 0 \\ 1, & k = 0 \end{cases} \]
Complex Fourier series

Fourier series is given by

\[ x(t) = 1 + \frac{2}{\pi} e^{j\Omega_0 t} + \frac{2}{\pi} e^{-j\Omega_0 t} - \frac{2}{3\pi} e^{j3\Omega_0 t} - \frac{2}{3\pi} e^{-j3\Omega_0 t} \ldots \]

Period of train of rectangular pulses and its magnitude and phase line spectra.
Complex Fourier series

Note that $|X_k| = |X_{-k}|$ and that $\angle X_k = -\angle X_{-k}$.

Hence, we can rewrite the Fourier series as

$$x(t) = 1 + \frac{2}{\pi} e^{j\Omega_0 t} + \frac{2}{3\pi} e^{-j3\Omega_0 t} - \frac{2}{3\pi} e^{j3\Omega_0 t} - \frac{2}{3\pi} e^{-j3\Omega_0 t} \ldots$$

$$= 1 + \frac{4}{\pi} \cos(\Omega_0 t) - \frac{4}{3\pi} \cos(3\Omega_0 t) + \cdots$$
Trigonometric Fourier series

The **Fourier series representation** of a periodic signal $x(t)$ of period $T_0$ is given by

$$x(t) = c_0 + 2 \sum_{k=0}^{\infty} (c_k \cos(k\Omega_0 t) + d_k \sin(k\Omega_0 t)), \quad \Omega_0 = \frac{2\pi}{T_0}$$

with **Fourier coefficients** $c_k$ and $d_k$

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos(k\Omega_0 t) dt, \quad k = 0, 1, 2, \cdots$$

$$d_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin(k\Omega_0 t) dt, \quad k = 1, 2, \cdots$$
Trigonometric Fourier series

Observe that

- \( c_k = \frac{1}{2}(X_k + X_{-k}) \), \( d_k = \frac{j}{2}(X_k - X_{-k}) \)
- \( X_k = c_k - j d_k \), \( X_{-k} = c_k + j d_k \)
- \( |X_k| = \sqrt{c_k^2 + d_k^2} \), \( \angle X_k = -\tan^{-1}\left(\frac{d_k}{c_k}\right) \)
- if \( x \) even symmetric \( (x(t) = x(-t)) \), all \( d_k \)s are zero
- if \( x \) odd symmetric \( (x(t) = -x(-t)) \), all \( c_k \)s are zero
Trigonometric Fourier series

Example (periodic pulse train):

- DC (average) value of 1
- \( x(t) \) is even symmetric (\( d_k \)s are zero)
Trigonometric Fourier series

- Trigonometric Fourier series

\[
c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cos(k\Omega_0 t) dt = \frac{4}{T_0} \int_{0}^{T_0/4} \cos(k\Omega_0 t) dt
\]

\[
= \begin{cases} 
\frac{2}{k\pi} \sin(k\pi/2), & k \neq 0 \\
1, & k = 0
\end{cases}
\]

Hence, the Fourier series is given by

\[
x(t) = 1 + \frac{4}{\pi} \cos(\Omega_0 t) - \frac{4}{3\pi} \cos(3\Omega_0 t) + \cdots
\]
Fourier series

Some observations:

• Notice that

$$\lim_{k \to \infty} X_k = \lim_{k \to \infty} c_k = 0$$

Riemann-Lebesgue lemma: If $x \in L^2(E)$, then

$$\lim_{k \to \pm \infty} \int_E x(t) e^{-jk\Omega_0 t} dt = 0$$

$$\lim_{k \to \infty} \int_E x(t) \cos(k\Omega_0 t) dt = \lim_{k \to \infty} \int_E x(t) \sin(k\Omega_0 t) dt = 0$$
Fourier series

Some observations:

- Notice that
  \[ \lim_{k \to \infty} X_k = \lim_{k \to \infty} c_k = 0 \]
  and that the decay is of order \( O(k^{-1}) \)

- \( x(T_0/4) = 2 \) but the Fourier series yields 1 at \( t = T_0/4 \) ???

- The Fourier series seems to have convergence problems around discontinuities (Gibb’s phenomena)
Gibb’s Phenomena

\[ x(t) = 1 + \frac{4}{\pi} \cos(\Omega_0 t) - \frac{4}{3\pi} \cos(3\Omega_0 t) + \cdots \]
Convergence of Fourier series

Some (important) remarks:

- Let \( y(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} \) denotes the Fourier series representation of \( x(t) \). Then \( \|x - y\| = 0 \). That is

\[
\int_E |x(t) - y(t)|^2 dt = 0
\]

- We have convergence in norm, which does not imply that it converges pointwise to \( x(t) \)
Let \( x_1(t) \) be defined as

\[
x_1(t) = \begin{cases} 
x(t), & t \in [t_0, t_0 + T_0] \\
0, & \text{otherwise}
\end{cases}
\]

Then

\[
X_1(s) = \int_{t_0}^{t_0+T_0} x_1(t)e^{-st}dt.
\]

The Fourier coefficients \( X_k \) are given by

\[
X_k = \frac{1}{T_0} \left. X_1(s) \right|_{s = jk\Omega_0}
\]

\[
X_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x_1(t)e^{-jk\Omega_0 t}dt
\]
Properties of Fourier Series

- **Time-shifting:** \( y(t) = x(t + \tau) \quad \leftrightarrow \quad Y_k = e^{jk\Omega_0 \tau} X_k \)

**Direct approach:**

\[
Y_k = \frac{1}{T_0} \int_0^{T_0} x(t + \tau) e^{-jk\Omega_0 t} \, dt = \frac{1}{T_0} \int_0^{T_0} x(s) e^{-jk\Omega_0 (s-\tau)} \, ds
\]

\[
= \frac{e^{jk\Omega_0 \tau}}{T_0} \int_0^{T_0} x(s) e^{-jk\Omega_0 s} \, ds = e^{jk\Omega_0 \tau} X_k
\]

**Using Laplace (see Table 3.1):**

\[
Y_k = \frac{1}{T_0} Y_1(s) \bigg|_{s=jk\Omega_0} = \frac{1}{T_0} \left( e^{s\tau} X_1(s) \right) \bigg|_{s=jk\Omega_0} = e^{jk\Omega_0 \tau} X_k
\]
Properties of Fourier Series

- Frequency-shifting: \( y(t) = x(t)e^{-jm\Omega_0 t} \) \( \xrightarrow{\mathcal{F}} Y_k = X_{k+m} \)

Direct approach:

\[
Y_k = \frac{1}{T_0} \int_0^{T_0} x(t)e^{-jm\Omega_0 t}e^{-jk\Omega_0 t} dt = \frac{1}{T_0} \int_0^{T_0} x(t)e^{-j(k+m)\Omega_0 t} dt = X_{k+m}
\]

Using Laplace (see Table 3.1):

\[
Y_k = \frac{1}{T_0} Y_1(s) \bigg|_{s=jk\Omega_0} = \frac{1}{T_0} X_1(s+jm\Omega_0) \bigg|_{s=jk\Omega_0} = X_{k+m}
\]
Properties of Fourier Series

- Differentiation: 
  \[ y(t) = \frac{dx(t)}{dt} \quad \leftrightarrow \quad Y_k = jk\Omega_0 X_k \]

Direct approach:

\[
Y_k = \frac{1}{T_0} \int_{0}^{T_0} x'(t)e^{-jk\Omega_0 t} dt = \frac{1}{T_0} \int_{0}^{T_0} e^{-jk\Omega_0 t} d(x(t))
\]

\[
= \frac{1}{T_0} x(t)e^{-jk\Omega_0 t} \bigg|_{0}^{T_0} + \frac{jk\Omega_0}{T_0} \int_{0}^{T_0} x(t)e^{-jk\Omega_0 t} dt = jk\Omega_0 X_k
\]

Using Laplace (see Table 3.1):

\[
Y_k = \frac{1}{T_0} Y_1(s) \bigg|_{s=jk\Omega_0} = \frac{1}{T_0} \left( sX_1(s) \right) \bigg|_{s=jk\Omega_0} = jk\Omega_0 X_k
\]
Convergence of the Fourier Series

• Fourier series can be defined for functions \( x \in L^1(E) \supset L^2(E) \)

• The pointwise convergence of a Fourier series is a rather complicated problem

  – Dirichlet (1829) showed that if \( x \in L^1(E) \) and has a finite number of discontinuities and extrema, then the Fourier series converges everywhere to the local average

  – Kolmogorov (1926) has given an example of a function in \( L^1(E) \) in which the Fourier series diverges everywhere!

  – Carleson (1966) proved that if \( x \in L^2(E) \), then the Fourier series converges for almost all \( t \) to \( x(t) \) (Dirichlet conditions)
Convergence of the Fourier Series

Dirichlet’s theorem: if $x \in L^1([0, T_0])$, then

$$S_N(t) = \frac{1}{T_0} \int_0^{T_0} x(u) D_N(u - t) \, du$$

where

$$D_N(t) = \frac{\sin((N + \frac{1}{2})\Omega_0 t)}{\sin(\frac{1}{2}\Omega_0 t)}$$

is called Dirichlet’s kernel
Convergence of the Fourier Series

Conclusion:

• Convergence of Fourier series depends on *local* behaviour

• For large $N$, the Dirichlet kernel becomes a $\delta$-function:

$$\lim_{N \to \infty} S_N(t_0) = \frac{1}{2} \left( x(t_0^-) + x(t_0^+) \right)$$

If $x$ is continuous at $t = t_0$, then

$$\lim_{N \to \infty} S_N(t_0) = x(t_0)$$

In conclusion, the Fourier series converges in *norm* (we have equality in $L^2([0, T_0])$), but we only have *pointwise convergence* at points where $x$ is continuous!
Convergence of the Fourier Series

Example: \( x(t) = t, t \in [-\pi, \pi] \). Since \( x \) is odd, \( c_k = 0 \) for all \( k \) and

\[
d_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} t \sin(kt) dt
\]

\[
= \frac{-1}{2\pi k} t \cos(kt) \bigg|_{-\pi}^{\pi} + \frac{1}{2\pi k} \int_{-\pi}^{\pi} \cos(kt) dt = \frac{1}{k} (-1)^{k+1}
\]

Hence, the Fourier series becomes

\[
\frac{2}{1} \sin(t) - \frac{2}{2} \sin(2t) + \frac{2}{3} \sin(3t) - \frac{2}{4} \sin(4t) + \cdots
\]
Convergence of the Fourier Series

• Notice that \( \lim_{k \to \infty} d_k = 0 \) and that the decay is \( O(k^{-1}) \)

• \( x(\pi) = \pi \) but substituting \( t = \pi \) in the Fourier series yields 0
Convergence of the Fourier Series

approximation using 1 sinusoid

approximation using 2 sinusoids

approximation using 3 sinusoids

approximation using 10 sinusoids
Convergence of the Fourier Series

Example: \( x(t) = |t|, t \in [-\pi, \pi] \). Since \( x \) is even, \( d_k = 0 \) for all \( k \)

\[
c_k = \frac{1}{\pi} \int_{0}^{\pi} t \cos(kt) \, dt
\]

\[
\begin{align*}
(k \neq 0) & \quad \frac{1}{k \pi} t \sin(kt) \bigg|_{0}^{\pi} - \frac{1}{k \pi} \int_{0}^{\pi} \sin(kt) \, dt = \begin{cases} 
0, & k = 2, 4, \ldots \\
\frac{-2}{\pi k^2}, & k \text{ odd}
\end{cases} \\
& \quad = 0
\end{align*}
\]

and

\[
c_0 = \frac{1}{\pi} \int_{0}^{\pi} t \, dt = \frac{\pi}{2}
\]
Convergence of the Fourier Series

The Fourier series becomes

\[
\frac{\pi}{2} - \frac{4}{\pi} \left( \frac{\cos(t)}{1^2} + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots \right)
\]

Notice that \( \lim_{k \to \infty} c_k = 0 \) and that the decay is of order \( O(k^{-2}) \)
Convergence of the Fourier Series

approximation using 1 sinusoid

approximation using 2 sinusoids

approximation using 3 sinusoids

approximation using 10 sinusoids
Convergence of the Fourier Series

- If $x$ is $p$ times differentiable and all derivatives are in $L^1(E)$, then
  
  $$x^{(p)}(t) \xrightarrow{\mathcal{F}} (jk\Omega_0)^p X_k$$

- Applying the Riemann-Lebesgue lemma on $x^{(p)}$, we conclude that
  
  $$\lim_{k \to \pm \infty} (k\Omega_0)^p X_k = 0$$

  so that regularity of $x$ translates to rapid decay of $X_k$

This explains the faster decay of the Fourier coefficients of the function $x(t) = |t|$ as compared to those of $x(t) = t$. 
Convergence of the Fourier Series

\[ x(t) = \frac{dy(t)}{dt} \]

\[ y(t) \]

\[ (-2, -1, 0, 1, 2, \ldots) \]

\[ t \]

\[ (-2, -1, 0, 1, 2, \ldots) \]
What have we accomplished?

- Response of LTI systems to periodic signals (eigenfunction property)
- Harmonic (sinusoidal) representation of periodic/finite-length signals
- Spectrum of periodic/finite-length signals
- Connection between Fourier and Laplace
- Convergence properties of Fourier series
Where do we go?

- Extension of Fourier representation for aperiodic/infinite-length signals
- Unification of spectral theory for periodic and aperiodic signals
- Connection between Fourier and Laplace transforms
- Duality relation time and frequency domain
- Convolution and filtering
- Relation between pole/zero locations and frequency response