Continuous Sensor Placement

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***Abstract*—Existing solutions to the sensor placement problem are based on sensor selection, in which the best subset of available sampling locations is chosen such that a desired estimation accuracy is achieved. However, the achievable estimation accuracy of sensor placement via sensor selection is limited to the initial set of sampling locations, which are typically obtained by gridding the continuous sampling domain. To circumvent this issue, we propose a framework of continuous sensor placement. A continuous variable is augmented to the grid-based model, which allows for off-the-grid sensor placement. The proposed offline design problem can be solved using readily available convex optimization solvers.**

***Index Terms*—Convex optimization, joint sparsity, sensor placement, sensor selection, sparse sensing, sparsity.**

I. Problem Statement

SENSOR networks are widely used in a variety of applications like environmental monitoring, security and safety, to list a few. Sensors are devices capable of sensing a certain physical phenomenon, processing data, and communicating information to a central unit. Sensors are geographically deployed, and the data acquired by such distributed sensors are often used to solve statistical inference problems like field (e.g., heat, sound) estimation, target localization, and so on.

The number of sensors available are often limited due to various factors like availability of physical or data storage space, economical constraints, or due to energy-efficiency reasons. Such a restriction on the number of sensors naturally limits the estimation accuracy. Moreover, selecting different sampling locations for the sensors generally leads to different values of the mean-squared-error. In this article, we are interested in finding the best placement of the sensors such that a desired estimation accuracy is ensured and the number of sensors are as low as possible. In other words, the focus will be on designing a sparse sensing technique to capture only informative data, thereby reducing the costs associated with sensing, data storage, and communication overheads.

Let $y(x)$ denote the observation signal with a continuous-domain argument, where without loss of generality (w.l.o.g.) $x \in \mathbb{R}$ denotes the sampling domain. We will restrict ourselves to the one-dimensional spatial domain, but the ideas presented can be applied directly to higher dimensions and even to temporal or spatio-temporal domains[1]. Assume that $y(x)$ represents the measured physical field over a continuous one-dimensional space x, and it satisfies the linear model

$$y(x) = \mathbf{a}^H(x)\boldsymbol{\theta} + n(x) \tag{1}$$

where $\boldsymbol{\theta} \in \mathbb{R}^N$ collects the parameters to be estimated, $\mathbf{a}(x) \in \mathbb{C}^N$ is the known linear model representing the mapping between the parameters and the measurements, and $n(x)$ is the noise. For example, in field estimation, $\boldsymbol{\theta}$ might contain the source location and its field intensity. Furthermore, we assume $\mathbf{a}(x) = 0$ for $x < 0$ and $x > x_{\max}$. In other words, $\mathbf{a}(x)$ is completely described by its variation in the interval $x \in [0, x_{\max}]$ where we can place the sensors.

The fundamental question of interest is—where to place the sensors such that the estimation error is as low as possible? We next state the problem more precisely.

Continuous sensor placement problem. *Given the model $\mathbf{a}(x)$ and a desired mean-squared-error of the estimate, find the placement of the sensors in the range $[0, x_{\max}]$ such that the number of sampling locations is minimum and the desired mean-squared-error is achieved.*

II. State of the Art

A. Sensor Selection

The traditional solution to the sensor placement problem is to grid the interval $[0, x_{\max}]$ and to choose the best spatial locations among them [1]–[7]. The field is measured at $M = \lfloor x_{\max}/\delta \rfloor$ discrete locations $\{x_m = (m-1)\delta | m \in \{1, 2, \ldots, M\}\}$ obtained by regularly sampling the continuous space, where δ denotes the sampling interval; and it determines the resolution of the discrete field.

Let $\{y_m = y(x_m)\}$ be the discrete-domain measurements, $\{\mathbf{a}_m = \mathbf{a}(x_m)\}$ denote the discretized model, and $\{n_m = n(x_m)\}$ represent the error. Using the above notations, we can write the discrete-domain version of (1) as

$$y_m = \mathbf{a}_m^H\boldsymbol{\theta} + n_m, m = 1, 2, \ldots, M. \tag{2}$$

We need to design how to choose the minimum number of sampling locations out of M initial ones such that a desired estimation performance of the inverse problem (2) can be guaranteed. This problem is referred to as *sensor selection*.

[1]More generally, x could lie in a p-dimensional space, e.g., $p = 4$ represents a (three-dimensional) spatio-temporal observation domain.

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For the linear model (2) with spatially white noise of variance σ^2, it is straightforward to compute the mean-squared-error of the least-squares estimate as

$$\mathrm{MSE}(\widehat{\boldsymbol{\theta}}) = \sigma^2 \mathrm{tr}\left\{\left(\sum_{m=1}^{M} \mathbf{a}_m \mathbf{a}_m^H\right)^{-1}\right\}, \quad (3)$$

where the notation $\mathrm{tr}\{\cdot\}$ denotes the trace operator.

The problem of choosing the best subset of sampling locations (or sensors) out of M given locations is combinatorial in nature. To simplify this problem, solutions based on greedy methods [1], [2] and convex optimization [4], [5], [7]–[9] are proposed. Sensor selection can be formulated as the design of a Boolean selection vector. More specifically, the problem is to design a selection vector

$$\mathbf{w} = [w_1, w_2, \ldots, w_M]^T \in \{0,1\}^M,$$

where $w_m = (0)1$ indicates whether the sensor is (not) selected. Introducing the selection variable in (3), sensor selection can be expressed as the following cardinality minimization problem

$$\underset{\mathbf{w}\in\{0,1\}^M}{\arg\min}\ \|\mathbf{w}\|_0 \quad \text{s.to} \quad \sigma^2 \mathrm{tr}\left\{\left(\sum_{m=1}^{M} w_m \mathbf{a}_m \mathbf{a}_m^H\right)^{-1}\right\} \le \lambda, \quad (4)$$

where the ℓ_0-(quasi) norm counts the number of non-zero entries of $\mathbf{w}$, and the threshold λ specifies the desired mean-squared-error. The above optimization problem is a non-convex Boolean program. To further simplify this problem, standard convex relaxations are used. For example, the ℓ_0-(quasi) norm is relaxed with a convex ℓ_1-norm or the sum-of-logs [4], and the Boolean constraint is relaxed to the convex set $[0,1]^M$.

B. Contribution

The main contribution of this article is a framework of continuous sensor placement, which allows for off-the-grid sensor placement. This is fundamentally different from the state-of-the-art sensor placement solutions based on sensor selection.

III. Sensitivity to Gridding

Existing solutions to the sensor placement problem are based on sensor selection. The performance of sensor placement via sensor selection is highly limited by the choice of the initial grid as the resolution might be too low, especially when $\mathbf{a}(x)$ is fast varying compared to the chosen grid. We illustrate this effect with the following numerical example.

Consider the linear model in (1) with the following specifications. Let the parameter vector $\boldsymbol{\theta}$ be of length 2×1. Consider a sum of sinusoids model for $\mathbf{a}(x) = [a_1(x), a_2(x)]^T$ with $a_i(x) = \sum_{p=1}^{P_i} \alpha_{p,i} \sin(2\pi f_{p,i} x)$ for $i = 1,2$. Let $\mathbf{f}_i = [f_{1,i}, \ldots, f_{P_i,i}]^T$ and $\alpha_i = [\alpha_{1,i}, \ldots, \alpha_{P_i,i}]^T$ for $i = 1,2$. We use the following parameters: $P_1 = 5$, $P_2 = 5$, $x_{\max} = 10$ m,

$$\mathbf{f}_1 = [0.1, 0.33, 0.67, 0.78, 0.95]^T,$$
$$\mathbf{f}_2 = [0.15, 0.7, 0.4, 0.58, 0.85]^T,$$
$$\alpha_1 = [0.5, 0.65, 0.3, -0.15, 0.45]^T,$$
$$\alpha_2 = [-0.25, -0.33, -0.6, 0.95, -0.25]^T$$

The noise variance is $\sigma^2 = 1$.

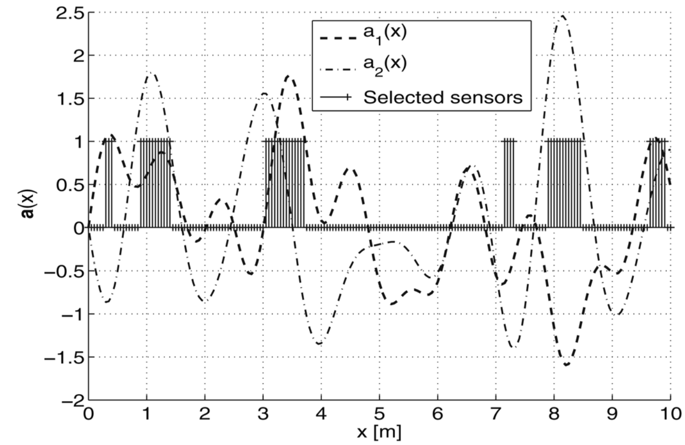

Fig. 1. The field is measured at $M = 201$ discrete locations with $\delta = 0.05$ m and $x_{\max} = 10$ m. The mean-squared-error bound is set to 0.026, leading to 50 selected sensors.

A. Coarse Gridding

Assume that the field can be measured at $M = 5$ potential locations. Let us consider the following case of candidate sampling locations $x_m := \{1,3,5,7,9\}$ m. Using (3), we can compute the mean-squared-error and it is 1.47. The optimal mean-squared-error using 5 sensors, on the other hand, is around 0.16.

We remark the following two observations. Firstly, the mean-squared-error resulting from the sensors at the above sampling locations is much lower than the optimal mean-squared-error. Secondly, any subset of these 5 sensors will naturally also result in a mean-squared-error lower than 1.47. Hence, due to the involved discretization, coarse gridding does not necessarily lead to the best placement in terms of the mean-squared-error.

B. Fine Gridding

Alternatively, the initial grid can be very dense with the candidate sampling locations at infinitesimal distance apart. This is obtained by choosing a very small δ. The sensor selection solver would then choose many sensors within one or more virtual spatial bins.

For the setup discussed in the previous subsection, the sensor selection for a fine grid with $M = 201$ potential sensors is illustrated in Fig. 1. The sensors within the most informative bin are selected first before going to the next informative bin, and so on, till the desired mean-squared-error is achieved.

The fine gridding has two main drawbacks. Firstly, it might not be practically feasible to place the sensors so close to each other. In addition, the reason why more sensors are selected within a certain bin is to improve the signal-to-noise ratio (SNR). However, instead of placing additional (expensive) sensors within a certain bin, the SNR can be improved by other (cheaper) means, e.g., temporal averaging using a single sensor. Finally, note that the solvers based on convex optimization techniques incur a cubic complexity making fine gridding also computationally less viable.

IV. Proposed Continuous Sensor Placement

The motivation behind a coarse discretization for sensor placement was computational tractability, but its performance is limited by the choice of the initial grid. On the contrary, fine gridding suffers from a high computational complexity and multiple closely spaced sensors for SNR improvement.

In this section, we present the proposed continuous sensor placement framework. To realize this, we augment the discrete model by including additional variables that account for the continuous nature of the sampling domain. The convex optimization based sensor selection solver is then adapted to solve this augmented formulation.

A. Taylor Interpolation

If $\mathbf{a}(x)$ is sufficiently smooth (i.e., its first-order derivative exists and is continuous), then local shifts of $\mathbf{a}(x)$ can be approximated using its derivative based on a first-order Taylor expansion:

$$\mathbf{a}(x_m + \Delta x_m) \approx \mathbf{a}_m + \Delta x_m \mathbf{a}'_m, m = 1, 2, \ldots, M, \tag{5}$$

where $\{x_m\}$ are the points on the initial grid, Δx_m represents the continuous variable around x_m with $|\Delta x_m| < 0.5\delta$, and $\mathbf{a}'_m = \partial \mathbf{a}_m / \partial x_m$ is the derivative. Such first-order interpolations have also been used in the context of sparse signal recovery to overcome problems due to gridding (see [10], [11]). Note that by using this idea, gridding actually results in binning, where we can place at most one sensor per bin. As a remark, other alternative interpolation techniques (e.g., polar interpolation [10]) can be considered.

B. Performance Measure

Before presenting the optimization problem and its solver for continuous sensor placement, we first derive a performance measure that determines the sensor placement. As in (3), we can compute the mean-squared-error based on the approximation (5) by completing the squares, i.e.,

$$\begin{aligned}\mathrm{MSE}(\widehat{\boldsymbol{\theta}}) &= \sigma^2 \mathrm{tr}\left\{\left(\sum_{m=1}^{M} \mathbf{a}(x_m + \Delta x_m)\mathbf{a}^H(x_m + \Delta x_m)\right)^{-1}\right\} \\ &\approx \sigma^2 \mathrm{tr}\left\{\left(\sum\nolimits_{m=1}^{M} \mathbf{a}_m\mathbf{a}_m^H + \Delta x_m^2 \mathbf{a}'_m \mathbf{a}'^H_m \right.\right. \\ &\qquad \left.\left. + \Delta x_m(\mathbf{a}'_m\mathbf{a}_m^H + \mathbf{a}_m\mathbf{a}'^H_m)\right)^{-1}\right\}. \end{aligned} \tag{6}$$

Introducing the Boolean selection vector $\mathbf{w}$ in (6) we obtain the performance function

$$\begin{aligned} f(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sigma^2 \mathrm{tr}&\left\{\left(\sum\nolimits_{m=1}^{M} w_m \mathbf{a}_m\mathbf{a}_m^H + u_m \mathbf{a}'_m\mathbf{a}'^H_m\right.\right. \\ &\left.\left. + v_m(\mathbf{a}'_m\mathbf{a}_m^H + \mathbf{a}_m\mathbf{a}'^H_m)\right)^{-1}\right\}, \end{aligned} \tag{7}$$

where $\mathbf{u} = [u_1, u_2, \ldots, u_M]^T$ with $u_m = w_m \Delta x_m^2$, and $\mathbf{v} = [v_1, v_2, \ldots, v_M]^T$ with $v_m = w_m \Delta x_m$. Note that $\mathbf{u} = \mathbf{v}^{\odot 2}$ (the notation $(\cdot)^{\odot 2}$ denotes element-wise squaring).

C. Optimization Problem

The optimization variables $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ in (7) are related through a structure. More specifically, the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ all share the same support set. Hence, instead of simply minimizing the cardinality of $\mathbf{w}$ as in (4), we can exploit the structure and jointly optimize their cardinality to minimize the number of sensors, and thus promote sparse sensing. Defining the matrix $\mathbf{Z} = [\mathbf{u}, \mathbf{v}, \mathbf{w}] \in \mathbb{R}^{M \times 3}$, the proposed continuous sensor placement problem can be formulated as

$$\begin{aligned} \arg\min_{\mathbf{Z}} \quad & \|\mathbf{Z}\|_{2,0} && \text{(8a)} \\ \text{s.to} \quad & f(\mathbf{u}, \mathbf{v}, \mathbf{w}) \le \lambda_{\mathrm{c}}, && \text{(8b)} \\ & \mathbf{Z} = [\mathbf{u}, \mathbf{v}, \mathbf{w}], \\ & \mathbf{u} = \mathbf{v}^{\odot 2}, && \text{(8c)} \\ & w_m \in \{0, 1\}, m = 1, 2, \ldots, M, && \text{(8d)} \\ & -0.5\delta \le v_m \le 0.5\delta, m = 1, 2, \ldots, M, && \text{(8e)} \\ & 0 \le u_m \le 0.25\delta^2, m = 1, 2, \ldots, M, && \text{(8f)} \end{aligned}$$

where the ℓ_2/ℓ_0-(quasi) norm counts the number of non-zero rows of $\mathbf{Z}$ as follows $\|\mathbf{Z}\|_{2,0} := |\{m : \sqrt{u_m^2 + v_m^2 + w_m^2} \neq 0\}|$, and the convex constraint (8b) specifies the estimation accuracy through a threshold λ_{c} (c denotes continuous). Since the continuous variable Δx_m takes values in the range $[-0.5\delta, 0.5\delta]$, we obtain the convex box constraints (8e) and (8f). The optimization problem (8) is non-convex due to: (a) cardinality cost, (b) Boolean constraint (8d), and (c) quadratic equality (8c). Therefore, it is (in general) difficult to solve (8) optimally.

We now use some standard convex relaxation techniques to simplify (8) and solve it sub-optimally. The ℓ_2/ℓ_0-(quasi) norm is relaxed with its best convex approximation, i.e., an ℓ_2/ℓ_1-mixed norm defined as $\|\mathbf{Z}\|_{2,1} := \sum_{m=1}^{M} \sqrt{u_m^2 + v_m^2 + w_m^2}$. The Boolean $w_m \in \{0, 1\}$ constraint is replaced with a convex set $w_m \in [0, 1]$. The constraint (8c) is equivalently expressed as $\mathbf{u} = \mathrm{diag}(\mathbf{U})$, where $\mathbf{U} = \mathbf{v}\mathbf{v}^H$ is a rank-1 matrix with $[\mathbf{U}]_{i,j \neq i} = 0, \ \forall i, j$ (the notation $\mathrm{diag}(\cdot)$ collects the diagonal elements of its argument in a vector, and $[\cdot]_{i,j}$ denotes the (i, j) th entry of the argument). Dropping the rank constraint on $\mathbf{U}$ and replacing the equality with an inequality as

$$\mathbf{U} \succeq \mathbf{v}\mathbf{v}^H \Leftrightarrow \begin{bmatrix} \mathbf{U} & \mathbf{v} \\ \mathbf{v}^H & 1 \end{bmatrix} \succeq 0,$$

we arrive at the relaxed continuous sensor placement problem:

$$\begin{aligned} \arg\min_{\mathbf{Z}, \mathbf{U}} \quad & \|\mathbf{Z}\|_{2,1} \\ \text{s.to} \quad & f(\mathbf{u}, \mathbf{v}, \mathbf{w}) \le \lambda_{\mathrm{c}}, \\ & \mathbf{Z} = [\mathbf{u}, \mathbf{v}, \mathbf{w}], \\ & \begin{bmatrix} \mathbf{U} & \mathbf{v} \\ \mathbf{v}^H & 1 \end{bmatrix} \succeq 0, \\ & \mathrm{diag}(\mathbf{U}) = \mathbf{u}, [\mathbf{U}]_{i,j \neq i} = 0, \forall i, j, \\ & 0 \le w_m \le 1, m = 1, 2, \ldots, M, \\ & -0.5\delta \le v_m \le 0.5\delta, m = 1, 2, \ldots, M, \\ & 0 \le u_m \le 0.25\delta^2, m = 1, 2, \ldots, M. \end{aligned} \tag{9}$$

Subsequently, an approximate Boolean solution for $\mathbf{w}$ has to be recovered either by deterministic or randomized rounding [4]. Finally, the sensor placements are given by shifting the locations of the selected sensors according to $\mathbf{v}$. The relaxed optimization problem can be solved using off-the-shelf solvers like CVX [12] or SeDuMi [13]. We underline here that the proposed sensor placement is not limited to the initial chosen grid points, and we basically replace grid points with bins allowing one sensor per bin. However, this feature comes at an additional complexity compared to that of solving the sensor selection problem with a fixed discrete grid. The increase in complexity is due to the additional variables like the continuous perturbation parameter and the associated box constraints.

D. Simulation Results

To validate the proposed sensor placement approach, we refer to the sum of sinusoids example introduced earlier in Section III. Let the initial coarse grid include $M = 11$ sampling locations $\{x_m = (m-1)\delta | m \in \{1, 2, \ldots, 11\}\}$ with $\delta = 1$. Note that the proposed continuous sensor placement framework is not limited to the sum of sinusoids model, but is valid for any general known model.

Fig. 2(a) illustrates the sensor placement via sensor selection. The best subset of sensors is computed by solving the relaxed version of (4). For sensor selection, we choose $\lambda := 0.47$ such that 5 sensors are selected out of 11 available sensors. For the considered scenario, the mean-squared-error achieved with such a sensor placement is ≈ 0.47.

Fig. 2(b) illustrates the results from the proposed continuous sensor placement obtained by solving the relaxed optimization problem (9). We use $\lambda_c = 0.064$, which has also been chosen such that 5 sensors are selected. The mean-squared-error achieved with the proposed placement is ≈ 0.36, which is lower than the mean-squared-error obtained by the sensor placement through sensor selection. The threshold λ_c is an underestimate of the mean-squared-error (unlike λ), and this is due to the approximation in (6). The threshold corresponding to a certain mean-squared-error can be chosen by computing the solution path for different λ_c values (see Section VI of [4] for more details).

Finally, in Table I we evaluate the mean-squared-error with 5 selected sensors obtained by solving the sensor selection problem and the proposed continuous placement for different grid densities. The optimization problems are solved in MATLAB using SeDuMi [13]. We consider different grid densities $M = \{5, 11, 21, 41, 81\}$ and in each case we use a threshold that selects 5 sensors. The continuous sensor placement offers better mean-squared-error with a reasonable increase in complexity.

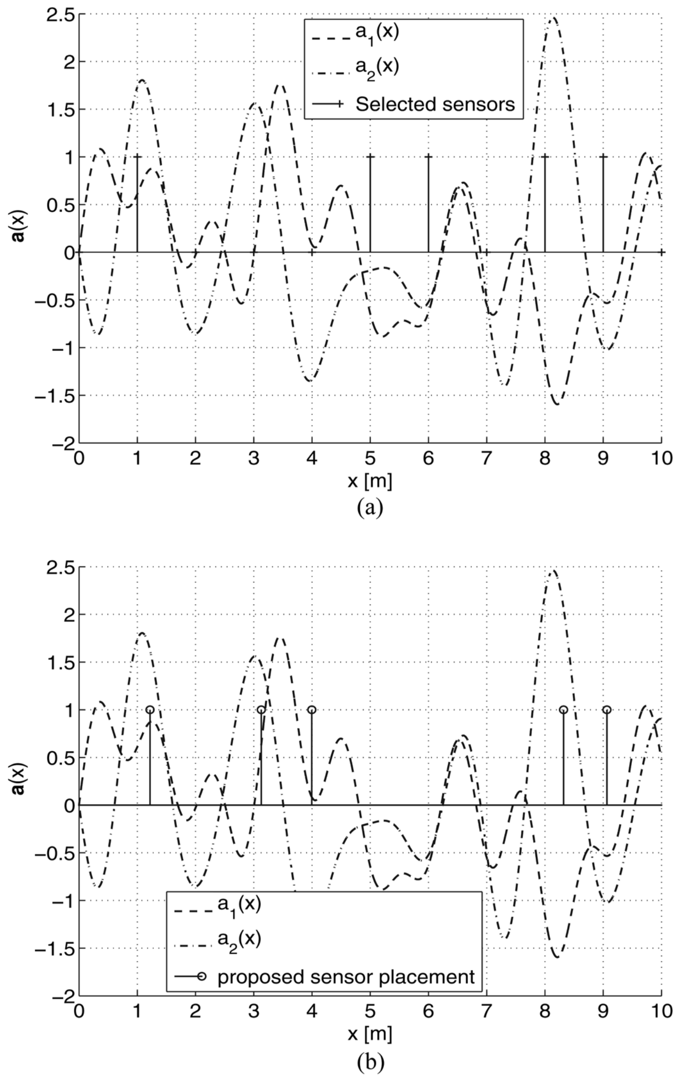

Fig. 2. The field is measured at $M = 11$ discrete locations with $\delta = 1$ m and $x_{\max} = 10$ m. A different threshold λ is used for (a) and (b), such that 5 sensors are selected (a) Sensor placement via sensor selection (b) Proposed continuous placement.

TABLE I
MEAN-SQUARED-ERROR WITH 5 SELECTED SENSORS

	$M=5$	$M=11$	$M=21$	$M=41$	$M=81$
Sensor selection	1.47	0.47	0.28	0.20	0.18
Continuous placement (proposed)	1.32	0.36	0.22	0.18	0.17

V. CONCLUSIONS

Sensor placement is a sampling design problem. Classic solutions to the sensor placement problem are based on sensor selection. In sensor selection, the continuous domain is first discretized leading to initial locations for placing the sensors. Sensor selection then solves for the best subset of sampling locations out of these potential locations subject to a performance constraint. As a consequence, the estimation accuracy is limited to the initial grid points. In this work, we have proposed a continuous sensor placement framework that allows for off-the-grid sensor placement to further improve the performance. To this end, we have supplemented the discrete model with additional variables that account for the continuous nature of the domain. The proposed solver promotes sparse sensing, and thus enables energy-efficient sensing schemes.

REFERENCES

[1] A. Krause, A. Singh, and C. Guestrin, “Near-optimal sensor placements in gaussian processes: Theory, efficient algorithms and empirical studies,” *J. Mach. Learn. Res.*, vol. 9, pp. 235–284, 2008.

[2] J. Ranieri, A. Chebira, and M. Vetterli, “Near-optimal sensor placement for linear inverse problems,” *IEEE Trans. Signal Process.*, vol. 62, no. 5, pp. 1135–1146, Mar. 2014.

[3] S. P. Chepuri, G. Leus, and A.-J. van der Veen, “Sparsity-exploiting anchor placement for localization in sensor networks,” in *Proc. 21st Eur. Signal Processing Conf. (EUSIPCO)*, Sep. 2013, pp. 1–5.

[4] S. P. Chepuri and G. Leus, “Sparsity-promoting sensor selection for non-linear measurement models,” *IEEE Trans. Signal Process.*, Sep. 2013, submitted for publication.

[5] S. P. Chepuri and G. Leus, “Sparsity-promoting adaptive sensor selection for non-linear filtering,” in *Proc. IEEE Int. Conf. Acoustics, Speech and Signal Processing (ICASSP)*, May 2014, pp. 5100–5104.

[6] H. Zhang, J. Moura, and B. Krogh, “Dynamic field estimation using wireless sensor networks: Tradeoffs between estimation error and communication cost,” *IEEE Trans. Signal Process.*, vol. 57, no. 6, pp. 2383–2395, 2009.

[7] S. Liu, E. Masazade, M. Fardad, and P. Varshney, “Sparsity-aware field estimation via ordinary kriging,” in *Proc. IEEE Int. Conf. Acoustics, Speech and Signal Processing (ICASSP)*, May 2014, pp. 3948–3952.

[8] S. Joshi and S. Boyd, “Sensor selection via convex optimization,” *IEEE Trans. Signal Process.*, vol. 57, no. 2, pp. 451–462, Feb. 2009.

[9] H. Rad, A. Simonetto, and G. Leus, “Sparsity-aware sensor selection: Centralized and distributed algorithms,” *IEEE Signal Process. Lett.*, vol. 21, no. 2, pp. 217–220, Feb. 2014.

[10] C. Ekanadham, D. Tranchina, and E. Simoncelli, “Recovery of sparse translation-invariant signals with continuous basis pursuit,” *IEEE Trans. Signal Process.*, vol. 59, no. 10, pp. 4735–4744, Oct. 2011.

[11] H. Zhu, G. Leus, and G. Giannakis, “Sparsity-cognizant total least-squares for perturbed compressive sampling,” *IEEE Trans. Signal Process.*, vol. 59, no. 5, pp. 2002–2016, May 2011.

[12] M. Grant and S. Boyd, *CVX: Matlab Software for Disciplined Convex Programming, version 2.0 beta*. Austin, TX, USA: CVX, Sep. 2012 [Online]. Available: http://cvxr.com/cvx

[13] J. F. Sturm, *Using SeDuMi 1.02, a Matlab Toolbox for Optimization over Symmetric Cones*, 1998 [Online]. Available: http://www.tandfonline.com/doi/abs/10.1080/10556789908805766#.VEZ2pyLQWSo