Multichannel detection of Gaussian signals with uncalibrated receivers

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Abstract— **We consider the detection of unknown Gaussian signals received by an array of uncalibrated non-identical sensors, which is a problem that appears in radio astronomy. The problem is formulated as a test on the covariance structure, the GLRT for this problem is stated and related to a simpler ad-hoc detector. We compare the method to the conventional multichannel subspace detector and show its robustness to non-identical channels on data collected with the Westerbork radio telescope.**

I. Introduction

In this letter we study the detection of spatially correlated signals impinging on an array of uncalibrated non-identical sensors, in the presence of spatially uncorrelated noise. The noise covariance matrix is diagonal but otherwise unknown.

The motivation for this study comes from an application in radio astronomy, where we wish to detect and suppress man-made interfering sources impinging on an array of telescopes. The output of the receiver after processing is essentially a sequence of short-term (∼10 second) sample correlation matrices, composed of the contributions of astronomical sources in the pointing direction, the additive receiver noise, and the interference. The receiver noise is largely independent among the sensors, but the receiver gains are not identical, with differences of up to a few dB. Calibration of this is done separately and taken into account off-line. An interfering source is usually in the near field and received through the side-lobes of the parabolic dishes, hence the received signals are correlated but with arbitrary unknown gains. In many cases the interference is intermittent in nature (e.g., TDMA signals as in the GSM system) and our objective is to detect its presence on-line on milli-second periods, and discard those periods which are deemed contaminated (temporal excision) [1]. Note that the astronomical signals of interest are much weaker than the receiver noise and hence it is necessary to detect interference even if it is much below the noise power. The astronomical signals themselves are too weak to be detected at these short time scales.

When the interferers are much weaker than the system noise and the receivers are non-identical, the change in eigenstructure of the sample covariance matrix is not detectable unless one of two steps is taken. The first is pre-calibration and whitening. The second which is easier to implement on-line is to use a different model where the noise covariance matrix is assumed diagonal but not necessarily equal to $\sigma^2\mathbf{I}$, and to detect deviation from this nominal model. This is the approach taken here. The Generalized Likelihood Ratio Test (GLRT) for this problem turns out to be the determinant of the sample correlation matrix, a fact which is not very well known in signal processing but has been used for a long time in certain other disciplines. After formulating the problem and stating the GLRT, we show that in first order approximation it is equivalent to a simpler detector. We then demonstrate the results of the excision using the GLRT detector and compare it to a detector which assumes identical receivers.

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II. Problem formulation

Assume that we have a set of q narrow-band Gaussian signals impinging on an array of p sensors. The received signal can be described in complex envelope form by

$$\mathbf{x}(k) \;=\; \sum_{i=1}^{q} \mathbf{a}_i s_i(k) + \mathbf{n}(k) \;=\; \mathbf{A}\mathbf{s}(k) + \mathbf{n}(k) \qquad (1)$$

where $\mathbf{x}(k) = [x_1(k), \cdots, x_p(k)]^T$ is a $p\times 1$ vector of received signals at sample times k, $\mathbf{A} = [\mathbf{a}_1, \cdots, \mathbf{a}_q]$, where $\mathbf{a}_i$ is the array response vector for the i'th signal, $\mathbf{s}(k) = [s_1(k), \cdots, s_q(k)]^T$ is a $q \times 1$ vector of gaussian source signals at sample times k with covariance matrix $\mathbf{R}_s = E(\mathbf{s}\mathbf{s}^{\mathrm{H}})$, $\mathbf{n}(k)$ is the $p \times 1$ additive noise vector, which is assumed to have independent gaussian entries with unknown diagonal covariance matrix $\mathbf{R}_n = \mathrm{diag}\{\nu_1, \ldots, \nu_{\mathrm{p}}\}$.

We would like to detect the presence of signals satisfying the above model, i.e., given data vectors $\mathbf{x}(1), \ldots, \mathbf{x}(N)$ decide whether $q = 0$ or $q > 0$. We do not assume parametric knowledge of the array manifold $\mathbf{a}(\theta)$ or a calibration of the noise power in each channel. Under these assumptions the only way to distinguish between signal and noise is to use the fact that the noise is spatially uncorrelated. Since parametrizing the signal plus noise case by a (possibly) low rank matrix plus diagonal would lead to an untractable ML problem, we try to distinguish between a diagonal and an arbitrary Hermitian matrix. To summarize, the detection problem is given by the hypotheses

$$\begin{array}{lll} \mathcal{H}_0: & \mathbf{x}(k) \sim \mathcal{CN}(0, \mathbf{R}_0) & \\ \mathcal{H}_1: & \mathbf{x}(k) \sim \mathcal{CN}(0, \mathbf{R}_1), & k = 1, \cdots, N. \end{array} \qquad (2)$$

where $\mathcal{CN}(0, \mathbf{R}_\ell)$ denotes the zero-mean complex normal distribution with covariance $\mathbf{R}_\ell$,

$$\mathbf{R}_0 := \begin{bmatrix} \nu_1 & & 0 \\ & \ddots & \\ 0 & & \nu_p \end{bmatrix}, \quad \mathbf{R}_1 \text{ any positive definite matrix}$$

Generalizations of this problem have been studied in the psychometrics, biometrics and statistics literature since the 1930s under the heading of *factor analysis*, which is concerned with detecting the rank q and estimating the factors of a sample covariance matrix with model $\mathbf{R} = \mathbf{A}\mathbf{A}^{\mathrm{H}} + \mathbf{D}$, where $\mathbf{A}:\ p\times q$ and $\mathbf{D} \geq 0$ diagonal (but usually for real-valued matrices) [2]. The problem has received much less attention in the signal processing literature. Related recent work includes e.g. direction estimation using two subarrays with mutually uncorrelated noise [3], [4].

III. The GLRT detector

In this section we first give a short derivation of the GLRT for the detection problem (2). Note that both hypotheses are composite and we have to derive maximum likelihood estimates of the parameters for each of the hypotheses. Under the assumptions for either hypothesis the likelihood function assuming $\mathcal{H}_\ell$ is given by

$$L(\mathbf{X}|\mathcal{H}_\ell) \;\equiv\; L(\mathbf{X}|\mathbf{R}_\ell) \;=\; \left(\frac{1}{|\mathbf{R}_\ell|} e^{-\mathrm{tr}\left(\mathbf{R}_\ell^{-1}\hat{\mathbf{R}}\right)}\right)^N, \quad \ell = 0, 1,$$

where $\mathbf{X} = [\mathbf{x}(1), \ldots, \mathbf{x}(N)]$ and $\hat{\mathbf{R}} = \frac{1}{N}\sum_{k=1}^{N} \mathbf{x}(k)\mathbf{x}(k)^H$ is the sample covariance matrix, $|\cdot|$ denotes the determinant and $\mathrm{tr}(\cdot)$ the trace operator. The ML estimate of $\mathbf{R}_\ell$ is found by maximizing $L(\mathbf{X}|\mathbf{R}_\ell)$ over $\mathbf{R}_\ell$, or equivalently the log-likelihood function

$$\mathcal{L}(\mathbf{X}|\mathbf{R}_\ell) = N\left(-\log(|\mathbf{R}_\ell|) - \mathrm{tr}(\mathbf{R}_\ell^{-1}\hat{\mathbf{R}})\right).$$

Under $\mathcal{H}_0$ we have to estimate $\nu_1, \ldots, \nu_p$. To that end we set the derivative of $\mathcal{L}$ with respect to ν_i to zero, giving $-\frac{1}{\nu_i} + \frac{\hat{\mathbf{R}}_{ii}}{\nu_i^2} = 0$. This yields $\hat{\nu}_i = \hat{\mathbf{R}}_{ii}$. Under $\mathcal{H}_1$ we obtain similarly that the ML estimate of $\mathbf{R}_1$ is given by $\hat{\mathbf{R}}$. Therefore the GLRT test statistic is given by the "Hadamard ratio"

$$\frac{L(\mathbf{X}|\mathcal{H}_0)}{L(\mathbf{X}|\mathcal{H}_1)} = \frac{|\hat{\mathbf{R}}|^N}{\prod_{i=1}^{p} \hat{\mathbf{R}}_{ii}^N} = |\hat{\mathbf{C}}|^N, \qquad (3)$$

where $\hat{\mathbf{C}}$ is the sample correlation matrix given by $\hat{\mathbf{C}} = \mathbf{W}\hat{\mathbf{R}}\mathbf{W}$ and $\mathbf{W} = \mathrm{diag}\{\frac{1}{\sqrt{\hat{\mathbf{R}}_{11}}}, \ldots, \frac{1}{\sqrt{\hat{\mathbf{R}}_{pp}}}\}$. Note that $0 \le |\hat{\mathbf{C}}| \le 1$, where equality to 1 is obtained asymptotically for $N \to \infty$ if $q = 0$. Thus, for a certain threshold $\gamma = \gamma(N)$ between 0 and 1, the GLRT is

$$T_1 \equiv |\hat{\mathbf{C}}| \underset{\mathcal{H}_1}{\overset{\mathcal{H}_0}{\gtrless}} \gamma \qquad (4)$$

This result is identical to that in the real-valued case (see [2, p.137]). The expression is rather satisfactory since in the absence of sensor calibration data all the spatial information exists in the spatial correlation coefficients between the different sensors, and the GLRT suggests a proper way of combining these different correlations. It is also quite easy to implement and does not involve any eigenstructure computations.

Furthermore, extrapolating the results from the real-valued case [2], it is known that under $\mathcal{H}_0$, $-2N\log|\hat{\mathbf{C}}|$ has asymptotically a chi-square distribution with $p^2 - p$ degrees of freedom: this can be used to determine a threshold γ corresponding to a desired probability of false alarm P_{FA}. According to Box [5], at least $N > 50$ samples are needed, and a better asymptotic fit is obtained by replacing N by $N - \frac{1}{6}(2p + 11)$.

A related ad-hoc detector to which we can compare is based on the Frobenius-norm of the off-diagonal entries of $\hat{\mathbf{C}}$. Since the diagonal entries are equal to 1, it is equivalent to take the norm of $\hat{\mathbf{C}}$ itself, i.e.,

$$T_2 \equiv \|\hat{\mathbf{C}}\|_F \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma' \qquad (5)$$

In fact, it is straightforward to prove that, for weak signals, the performance of this detector must be approximately equal to that of the GLRT. Indeed, for weak signals, the eigenvalues of $\hat{\mathbf{C}}$ are equal to $\lambda_i = 1 + \epsilon_i$, for small ϵ_i. Note that $\mathrm{tr}(\hat{\mathbf{C}}) = \mathrm{p} \Rightarrow \sum_{\mathrm{i}} \lambda_{\mathrm{i}} = \mathrm{p} \Rightarrow \sum_{\mathrm{i}} \epsilon_{\mathrm{i}} = 0$. We can write

$$\begin{aligned} T_1 &= \textstyle\prod_i \lambda_i = e^{\sum_i \log\lambda_i} \\ \Rightarrow \quad \log(T_1) &= \textstyle\sum_i \log\lambda_i = \sum_i \epsilon_i - \frac{1}{2}\epsilon_i^2 + \mathcal{O}(\epsilon_i^3) \\ &= -\textstyle\sum_i \frac{1}{2}\epsilon_i^2 + \mathcal{O}(\epsilon_i^3) \end{aligned}$$

whereas

$$\begin{aligned} & T_2^2 = \|\hat{\mathbf{C}}\|_F^2 = \textstyle\sum_i \lambda_i^2 = \sum_i 1 + 2\epsilon_i + \epsilon_i^2 \\ \Rightarrow \quad & -\tfrac{1}{2}(T_2^2 - p) = -\textstyle\sum_i \frac{1}{2}\epsilon_i^2 \end{aligned}$$

Since a monotonic transformation of a test statistic does not change the outcome of the test if the threshold is modified accordingly,[1] the two detectors are equivalent up to third order. Computing the Frobenius-norm requires only $\mathcal{O}(p^2)$ operations, versus $\mathcal{O}(p^3)$ for the determinant test (implemented via a Cholesky factorization of $\hat{\mathbf{C}}$).

[1] Note that the decisions in (4) and (5) are opposite, hence the change of sign in the second transformation.

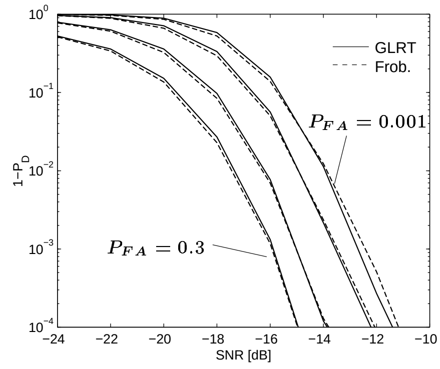

Fig. 1. Probability of missed detection vs. SNR. P_{FA} = 0.3, 0.1, 0.01, 0.001.

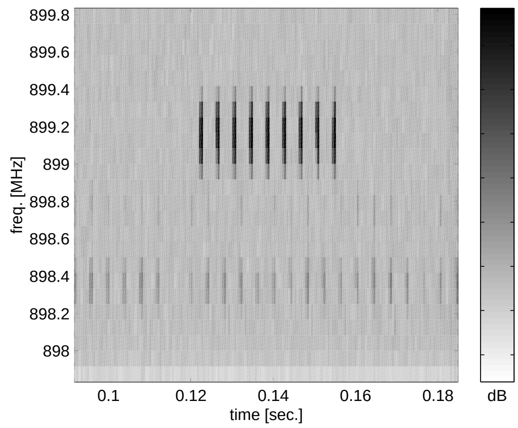

Fig. 2. Time-frequency spectrum of channel 1, showing GSM interference

IV. Simulation results

To test the performance of the detector in a simulation, we have used an array with 8 elements, with receiver noise covariance $\mathbf{R}_n = \mathrm{diag}[0,\ 0,\ 0,\ -3,\ -4,\ -5,\ -6,\ -7]$ dB. We consider a single signal, with signal to noise ratio (SNR) at the input of the first sensor varied from -24 dB to -10 dB. Both signal and noise are Gaussian. Each experiment is based on $N = 50$ samples of the array output. Figure 1 shows the probability of detection vs. SNR for various probabilities of false alarm P_{FA}. The graph is based on 100 000 experiments per value of the SNR. We see that indeed the performance of both detectors is similar.

V. Application

The main motivation for the detection problem above stems from an application to interference mitigation in radio astronomy. We apply the detector to sample data collected with the Westerbork radio telescope. The data was recorded using the 8-channel NOEMI project data recorder [1]. We selected a bandwidth of 2 MHz, around 899 MHz, with a duration of 3 seconds. This band is contaminated with various GSM mobile telephony signals. Such signals are intermittent, occupying time slots of length 0.577 ms in frames of 4.6 ms. A segment of the data is shown in figure 2. The received data channels were split into subbands of 83 kHz by means of windowing and short-term FFTs, and subsequently correlated per frequency bin. Each covariance matrix is an average based on 21 samples and covers a

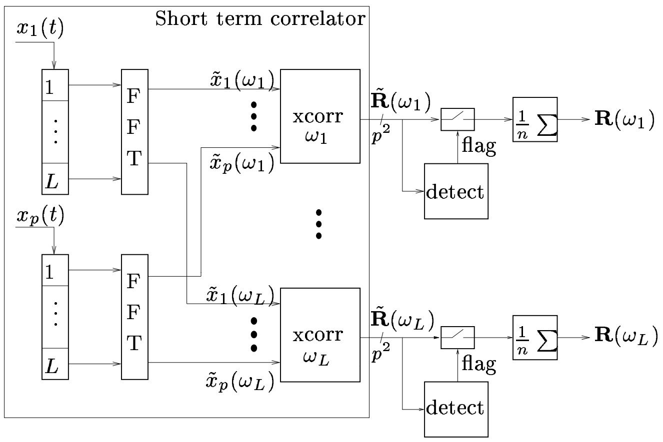

Fig. 3. Computational structure of the blanking process

period of 0.24 ms. Two detectors have been applied. The first is the detector of (3), and the other one is given by

$$T_3 \equiv \frac{|\hat{\mathbf{R}}|}{[\frac{1}{p}\mathrm{tr}(\hat{\mathbf{R}})]^{\mathrm{p}}} \tag{6}$$

where $\hat{\mathbf{R}}$ is the sample covariance matrix. This detector is a GLRT assuming identical channels (or $\mathbf{R}_n = \sigma^2\mathbf{I}$) [2].

Since $N = 21$ is small, we have not used the theoretical thresholds. Instead, we have excised the worst 10 percent of the data at each frequency channel and generated spectral estimates by further averaging the covariance matrices of the remaining 90 percent of the data. The processing structure is shown in figure 3.

Figure 4 shows the power spectrum of channel 1 and the cross-spectrum of channels 1 and 3, respectively, before and after blanking. Without excision, we can see that several interfering signals are present, most weak but one rather strong. We can clearly see that while both detectors excised properly the strong interference, the detector based on the $\mathbf{R}_n = \sigma^2\mathbf{I}$ assumption failed to excise the weak features of the interference.

References

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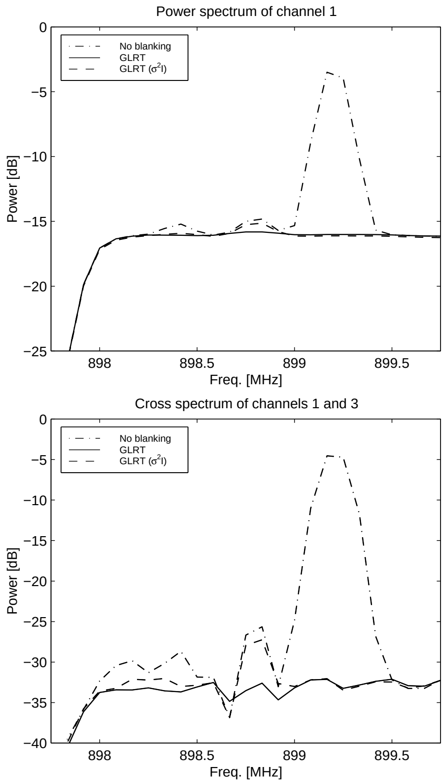

Fig. 4. Power spectra and cross-spectra of channels 1 and 3, before and after interference excision