Localization with TOA as a Constrained Robust Stochastic Least Squares Problem

Sayit Korkmaz[1] and Alle-Jan van der Veen[1]
Circuits and Systems, TU Delft
{sayit,allejan}@cas.et.tudelft.nl

***Abstract*— Localizing a source given range measurements from base stations with fixed locations is a nonlinear optimization problem. A typical solution to such a problem is to run iterative algorithms which do not guarantee convergence or use maximum likelihood with grid searches. An alternative is then to transform the problem to a new one for which we know the solution is simple and computationally stable. We formulate the time-of-arrival base localization problem as a constrained robust stochastic least squares problem. While we can solve such problems efficiently the price paid for computational stability is positioning accuracy. Hence there is a trade-off between two desirable properties, accuracy and computational stability.**

I. Introduction

Localization with time-of-arrival (TOA) usually consists of two steps. In the first step the TOA's are obtained from channel measurements. In the second step these parameters are used to obtain the final estimate of location. In the second step the problem is still a non-linear optimization problem. Although a maximum likelihood approach can be used, it is computationally costly to implement with grid searches. Closed form algorithms meet the need for a rough initial estimate. When we have an initial estimate from closed form solutions this can be used as a starting point for an iterative or grid-search based optimization. Furthermore closed form algorithms constitute a good trade-off between computational complexity and positioning accuracy. Indeed there is a lot of options in the trade-offs between computational stability and positioning accuracy.

Due to its importance, many closed form solutions exist for localization in the literature in the context of aerospace and acoustic applications [1]–[6]. The algorithms are usually derived for far field scenarios.

Common to all of the closed form solutions is the linearization of the quadratic equations by some form of Gaussian elimination. After this linearization, we obtain a linear set of equations in the unknown coordinates of the mobile station. Usually a variation of least squares is used to obtain the unknown coordinates from the linear set of equations. Nevertheless when we perform Gaussian elimination the noise terms enter the equations in a squared manner. Indeed squaring noise only causes error. This also makes exact analytical analysis difficult and in the literature there is only approximate analysis of noise [1], [5]. In this paper, we show that it is possible also to make the noise terms appear linear at the price of increasing the number of unknowns. What we mean is that when we write down the equations there are no noise terms that are squared.

[1]The authors are with Delft University of Technology, Dept. Electrical Eng. (EEMCS), 2628 CD Delft, The Netherlands. This research was supported in part by NWO-STW under the VICI programme (DTC.5893).

II. Review of classical TOA closed form localization algorithms

In this section we will review the classical methods of TOA closed form localization. In the following equations, x, y denote the Cartesian coordinates of the mobile station and x_l, y_l, x_k, y_k denote the coordinates of the base station labeled with l and k respectively and $x_{kl} = x_k - x_l$ and $y_{kl} = y_k - y_l$. r_l and r_k are the distances of the mobile station to the base stations labeled with l and k and $r_{kl} = r_k - r_l$. We consider a 2D scenario for ease of development. Extention to 3D is straightforward. For simplicity we consider the case of 4 base stations. It assumed that exact knowledge of the locations of base stations is available. First we introduce the least squares closed for localization for TOA from [6]. The distance equations are:

$$r_l^2 = (x - x_l)^2 + (y - y_l)^2 \tag{1}$$
$$r_k^2 = (x - x_k)^2 + (y - y_k)^2 \tag{2}$$

By subtracting the pair of equations from eachother we obtain the following set of equations:

$$\mathbf{b}_{TOA} = \mathbf{H}_{TOA}\boldsymbol{\theta}_{TOA} \tag{3}$$

$$\mathbf{H}_{TOA} = 2\begin{bmatrix} x_{21} & y_{21} \\ x_{31} & y_{31} \\ x_{41} & y_{41} \end{bmatrix} \qquad \boldsymbol{\theta}_{TOA} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{4}$$

$$\mathbf{b}_{TOA} = \begin{bmatrix} x_2^2 + y_2^2 - x_1^2 - y_1^2 - r_2^2 + r_1^2 \\ x_3^2 + y_3^2 - x_1^2 - y_1^2 - r_3^2 + r_1^2 \\ x_4^2 + y_4^2 - x_1^2 - y_1^2 - r_4^2 + r_1^2 \end{bmatrix} \tag{5}$$

In a practical setting, the ranges in the vector $\mathbf{b}_{TOA}$ are replaced with their estimates and the noisy vector becomes $\hat{\mathbf{b}}_{TOA}$. Note that since the ranges appear quadratically in the vector $\mathbf{b}_{TOA}$, the noise on the vector becomes non-Gaussian and biased. The initial estimate of the location is obtained via simple least squares as follows:

$$\hat{\boldsymbol{\theta}}_{TOA} = (\mathbf{H}_{TOA}^T\mathbf{H}_{TOA})^{-1}\mathbf{H}_{TOA}^T\hat{\mathbf{b}}_{TOA} \tag{6}$$

There are many other possibilities yet this approach will be our benchmark. Indeed some authors prefer to make first order approximations to the quadratic terms and then proceed with these simplified models. Nevertheless the algorithms always become dirty in the sense that we can not get rid of nonlinearities.

In the next section we will formulate a novel way of Gaussian elimination applied to TOA measurements.

III. The proposed algorithm

Consider the following set of equations for ranges where we do not introduce noise yet:

$$r_l^2 = (x - x_l)^2 + (y - y_l)^2 \tag{7}$$
$$r_k^2 = (x - x_k)^2 + (y - y_k)^2 \tag{8}$$

By subtracting the second equation from the first we obtain

$$\begin{aligned}(r_k - r_l)(r_k + r_l) = (x - x_k)^2 + (y - y_k)^2 \\ - (x - x_l)^2 - (y - y_l)^2\end{aligned} \tag{9}$$

With simple algebraic manipulations, we obtain the following set of equations

$$r_l^2 = (x - x_l)^2 + (y - y_l)^2 \tag{10}$$
$$r_k^2 = (x - x_k)^2 + (y - y_k)^2 \tag{11}$$
$$2x_{kl}x + 2y_{kl}y + r_{kl}r_k + r_{kl}r_l = x_k^2 + y_k^2 - x_l^2 - y_l^2 \tag{12}$$

The unknowns in these equations are x, y, r_k, r_l. Now this may sound strange. Why should we call ranges as unknowns? Indeed we never measure the exact ranges. We always estimate a parameter which has also some uncertainty. Of course we may give a certain region of values where we are almost sure that the true range is. In other words, measured ranges obtained from the channel are random variables with some mean value and variance. So we must never treat them as exact deterministic parameters.

Compared to the previous case we have a small increase in the number of unknowns. This is the price we paid. In matrix form the equations are written as follows:

$$\mathbf{b} = \mathbf{H}\boldsymbol{\theta} \tag{13}$$

$$\mathbf{H} = \begin{bmatrix} 2x_{21} & 2y_{21} & r_{21} & r_{21} & & \\ 2x_{31} & 2y_{31} & r_{31} & & r_{31} & \\ 2x_{41} & 2y_{41} & r_{41} & & & r_{41} \\ 2x_{32} & 2y_{32} & & r_{32} & r_{32} & \\ 2x_{42} & 2y_{42} & & r_{42} & & r_{42} \\ 2x_{43} & 2y_{43} & & & r_{43} & r_{43} \end{bmatrix} \tag{14}$$

$$\boldsymbol{\theta} = \begin{bmatrix} x \\ y \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \mathbf{b} = \begin{bmatrix} x_2^2 + y_2^2 - x_1^2 - y_1^2 \\ x_3^2 + y_3^2 - x_1^2 - y_1^2 \\ x_4^2 + y_4^2 - x_1^2 - y_1^2 \\ x_3^2 + y_3^2 - x_2^2 - y_2^2 \\ x_4^2 + y_4^2 - x_2^2 - y_2^2 \\ x_4^2 + y_4^2 - x_3^2 - y_3^2 \end{bmatrix} \tag{15}$$

In this formulation the vector $\mathbf{b}$ is exact. It consists only from the coordinates of the base stations. But the matrix $\mathbf{H}$ must be replaced with its estimated version as $\hat{\mathbf{H}}$. Because it consists of the range differences. As we said earlier ranges are random variables hence the matrix $\hat{\mathbf{H}}$ partially consists of random variables. We again do not know the exact matrix but nevertheless we can describe its statistical behavior since it consists of range differences and we assume that we know the statistical behavior of ranges. And note that ranges enter the matrix in an additive manner. There are no squaring operations. This is what we mean when we say that noise terms are made linear, although some may vies them as multiplicative. In the end this does not make any difference in the development of the optimal algorithms.

Until now we treated the vector $\boldsymbol{\theta}$ as comletely unknown. But this is not true again. Indeed we know the statistical behavior of some of its parameters. The ranges are described with some mean value and variance. In addition to this fact the coordinates x, y and the ranges are not independent. We must use all of these information if we aim to solve the problem completely. Indeed we will use the statistical knowledge about ranges but we will ignore the dependence between the unknown coordinates and the ranges.

In the next section we provide a very simple algorithm which will later be improved.

A. *Simple Closed Form Localization*

In this case part of $\boldsymbol{\theta}$ is actually observed. To exploit this, we can pose the following optimization problem:

$$\text{minimize } \|\mathbf{b} - \hat{\mathbf{H}}\boldsymbol{\theta}\|_2^2 \tag{16}$$
$$\text{subject to } \mathbf{A}\boldsymbol{\theta} = \hat{\mathbf{r}} \tag{17}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \hat{\mathbf{r}} = \begin{bmatrix} \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \\ \hat{r}_4 \end{bmatrix} \tag{18}$$

For the ℓ_2 norm the problem has an explicit solution given as

$$\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}_1 - (\hat{\mathbf{H}}^T\hat{\mathbf{H}})^{-1}\mathbf{A}^T[\mathbf{A}(\hat{\mathbf{H}}^T\hat{\mathbf{H}})^{-1}\mathbf{A}^T]^{-1}(\mathbf{A}\boldsymbol{\theta}_1 - \hat{\mathbf{r}}) \tag{19}$$

where

$$\boldsymbol{\theta}_1 = (\hat{\mathbf{H}}^T\hat{\mathbf{H}})^{-1}\hat{\mathbf{H}}^T\mathbf{b} \tag{20}$$

The problem with this approach is we do not exploit the uncertainty in the matrix $\hat{\mathbf{H}}$ and the constraints do not account for the uncertainties in the ranges. Nevertheless it is a good trade-off between computational complexity and positioning accuracy. It is just the case when we have only point estimates of the parameters but not variances. Then what is the algorithm to handle the case where we know also the variance of range estimates. This will be answered in the next section.

B. Localization with TOA as a robust stochastic least squares problem

In this section we assume that we have estimates of ranges and their variances. In other words we have second order description of the uncertainties in the ranges. How should we use that information in an efficient way. As remarked earlier ranges appear both in the matrix $\mathbf{H}$ and the unknowns $\boldsymbol{\theta}$. The good thing is that now we can describe the second order statistical behavior of the matrix $\mathbf{H}$ and the unknowns $\boldsymbol{\theta}$.

$$\hat{r}_i = \bar{r}_i + n_i \tag{21}$$

So we know $\bar{r}_i$ and we know $E\{n_i^2\}$ and $E\{.\}$ denotes expectation operation.

The question now is to find an algorithm that can handle uncertainties in the matrix $\mathbf{H}$. It turns out that the proposed set of equations can be solved via robust stochastic least squares method. The basic idea is to minimize the expected value of the least squares cost function.

$$\text{minimize } E\big\{\|\mathbf{b} - \hat{\mathbf{H}}\boldsymbol{\theta}\|_2^2\big\} \tag{22}$$
$$\text{subject to } l_k \leq r_k \leq u_k \qquad k = 1,2,3,4 \tag{23}$$

This is a constrained robust stochastic least squares problem. Note that we have handled the uncertainties in the matrix $\mathbf{H}$ via expectation operation, and uncertainties in the unknowns via convex constraints. This reduces the problem to a convex optimization problem which can be efficiently and reliably solved.

By using the well known results from robust stochastic optimization [7], our optimization problem takes the following form:

$$\text{minimize } \|\mathbf{b} - \bar{\mathbf{H}}\boldsymbol{\theta}\|_2^2 + \boldsymbol{\theta}^T\mathbf{P}\boldsymbol{\theta} \tag{24}$$
$$\text{subject to } l_k \leq r_k \leq u_k \qquad k = 1,2,3,4 \tag{25}$$

where $\bar{\mathbf{H}} = E\{\hat{\mathbf{H}}\}$ and $\mathbf{P} = E\{\mathbf{U}^T\mathbf{U}\}$.

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & n_{21} & n_{21} & & \\ 0 & 0 & n_{31} & & n_{31} & \\ 0 & 0 & n_{41} & & & n_{41} \\ 0 & 0 & & n_{32} & n_{32} & \\ 0 & 0 & & n_{42} & & n_{42} \\ 0 & 0 & & & n_{43} & n_{43} \end{bmatrix} \tag{26}$$

and $n_{ij} = n_i - n_j$.

The upper and lower bounds u_k, l_k are obtained from the standard deviation of noise on the range measurement r_k.

IV. CRLB Analysis

In this section we provide the CRLB without derivation. Since it follows directly from [8]. The elements of the Fisher information matrix are given as follows:

$$[\mathcal{I}(\boldsymbol{\theta})]_{ij} = \Big[\frac{\partial\boldsymbol{\mu}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}_i}\Big]^T \mathbf{C}^{-1}\Big[\frac{\partial\boldsymbol{\mu}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}_j}\Big] \tag{27}$$

The parameters in the expression are given as follows

$$\boldsymbol{\theta} = [x\ y]^T \tag{28}$$

$$\boldsymbol{\mu}(\boldsymbol{\theta}) = \begin{bmatrix} \sqrt{(x-x_1)^2 + (y-y_1)^2} \\ \sqrt{(x-x_2)^2 + (y-y_2)^2} \\ . \\ . \\ . \\ \sqrt{(x-x_M)^2 + (y-y_M)^2} \end{bmatrix} \tag{29}$$

$$\mathbf{C} = \begin{bmatrix} \sigma_1^2 & & & & \\ & \sigma_2^2 & & & \\ & & . & & \\ & & & . & \\ & & & & \sigma_M^2 \end{bmatrix} \tag{30}$$

$$\frac{\partial\boldsymbol{\mu}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}_1} = \begin{bmatrix} \frac{x-x_1}{\sqrt{(x-x_1)^2+(y-y_1)^2}} \\ \frac{x-x_2}{\sqrt{(x-x_2)^2+(y-y_2)^2}} \\ . \\ . \\ \frac{x-x_M}{\sqrt{(x-x_M)^2+(y-y_M)^2}} \end{bmatrix} \tag{31}$$

$$\frac{\partial\boldsymbol{\mu}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}_2} = \begin{bmatrix} \frac{y-y_1}{\sqrt{(x-x_1)^2+(y-y_1)^2}} \\ \frac{y-y_2}{\sqrt{(x-x_2)^2+(y-y_2)^2}} \\ . \\ . \\ \frac{y-y_M}{\sqrt{(x-x_M)^2+(y-y_M)^2}} \end{bmatrix} \tag{32}$$

Now we can fix the variances on the x and y coordinates as follows

$$var\{\hat{\theta}_i\} \geq [\mathcal{I}^{-1}(\boldsymbol{\theta})]_{ii} \tag{33}$$

V. Simulations

In order to test the performance of the proposed algorithms we performed extensive computer simulations. In the first set of simulations we tested the performance of the TOA based algorithms in the near field and in the far field. The coordinates of the base stations were fixed throughout the simulations to $(0,0),(0,10),(10,0),(5,5),(10,10)$. We change the location of the mobile station from $(15,15)$ to $(300,300)$. In all the simulations we change the noise variance and use the RMS as the measure of performance. In the below notation x, y denotes the coordinates of the mobile station and $x(i), y(i)$ denote the estimate at the i^{th} simulation.

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x(i)-x)^2 + (y(i)-y)^2} \tag{34}$$

We repeated the simulation 10000 times for every noise variance. Figure 1 and Figure 2 show the performance for the near filed case.

In order to test the performance of the constrained robust stochastic least squares algorithm we performed extensive computer simulations. The coordinates of the base stations were fixed throughout the simulations to $(0,0),(0,10),(10,0),(5,5),(10,10)$. The location of the mobile station was fixed to $(15,15)$. In all the simulations the

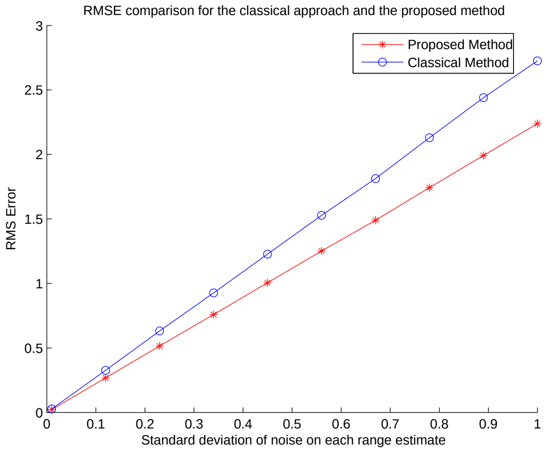

Fig. 1. RMSE for localizations with TOA for the case where mobile station is at $(15, 15)$

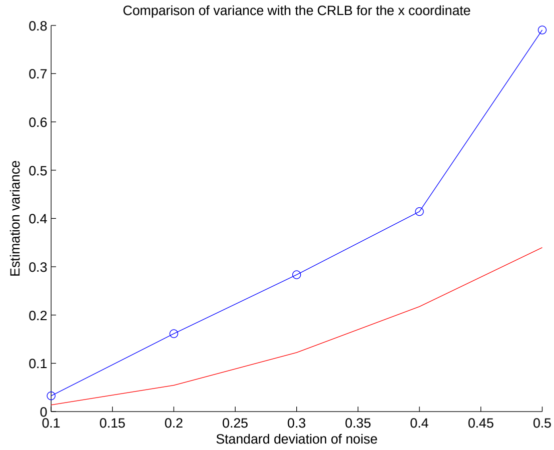

Fig. 3. Comparison of estimation variance for the x coordinate with CRLB

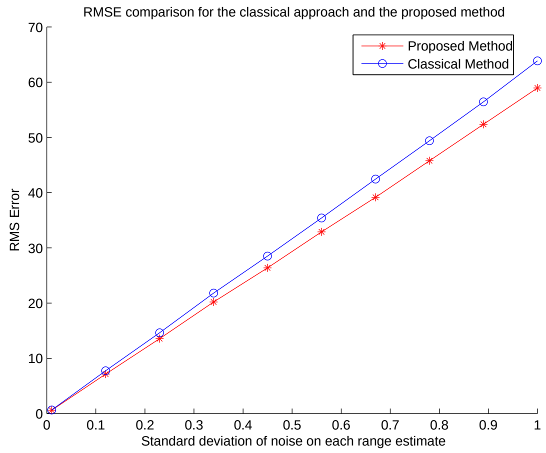

Fig. 2. RMSE for localization with TOA for the case where mobile station is at $(300, 300)$

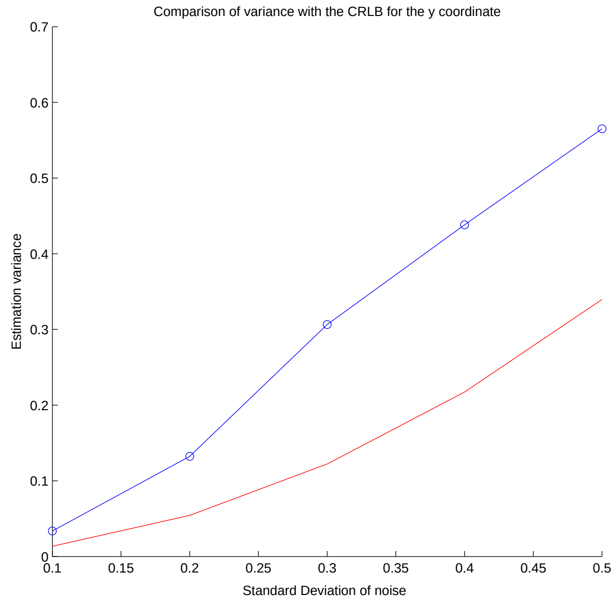

Fig. 4. Comparison of estimation variance for the y coordinate with CRLB

noise variance on each range measurement was fixed for all base stations. We performed 100 Monte-Carlo simulations for each noise variance. Fig. 3 and Fig. 4 show the comparison with CRLB. We see from the simulations that the proposed algorithm has a guarantee of convergence but does not achieve CRLB.

VI. Acknowledgemets

We have used the cvx toolbox for implementing the constrained robust stochastic least squares optimization problem.

References

[1] Y. T. Chan and K. C. Ho, "A simple and efficient estimator for hyperbolic location," *IEEE Transactions on Signal Processing*, vol. 42, no. 8, pp. 1905–1915, Aug. 1994.

[2] J. O. Smith and J. S. Abel, "Closed-form least-squares source location estimation from range-difference measurments," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 35, no. 12, pp. 1661–1669, Dec. 1987.

[3] R. Schmidt, "Least squares range difference location," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 32, no. 1, pp. 234–242, Jan. 1996.

[4] Y. Huang, J. Benesty, G. W. Elko, and R. M. Mersereau, "Real-time passive source localization: A practical linear-correction least-squares approach," *IEEE Transactions on Speech and Audio Processing*, vol. 9, no. 8, pp. 943–956, Nov. 2001.

[5] K. W. Cheung, H. C. So, W. K. Ma, and Y. T. Chan, "Least squares algorithms for time-of-arrival-based mobile location," *IEEE Transactions on Signal Processing*, vol. 52, no. 4, pp. 1121–1128, Apr. 2004.

[6] A. H. Sayed, A. Tarighat, and N. Khajehnouri, "Network-based wireless location," *IEEE Signal Processing Magazine*, pp. 24–40, July 2005.

[7] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge: Cambridge University Press, 2004.

[8] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*. New Jersey: Prentice Hall, 1993.