Projection-Based Model-Order Reduction of Large-Scale Maxwell Systems

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Abstract — **In this paper we present a structure-preserving model-order reduction technique to efficiently compute electromagnetic wave fields on unbounded domains. As an approximation space, we take the span of the real and imaginary parts of frequency-domain solutions of Maxwell's equations. Reduced-order models for the electromagnetic field belong to this space and the expansion coefficients of these models are determined from a Galerkin condition. We show that the models constructed in this manner are structure-preserving and interpolate the electromagnetic field responses at the expansion frequencies. Moreover, for monostatic field responses (coinciding sources and receivers), the first-order derivative of a reduced-order model with respect to frequency interpolates this first-order derivative of the unreduced monostatic field response as well. A two-dimensional numerical example illustrates the performance of the proposed reduction method.**

1 INTRODUCTION

We are interested in the time- or frequency-domain electromagnetic field in configurations that consist of penetrable objects of bounded extent embedded in a homogeneous background medium of unbounded extent. The external electric and magnetic current densities that excite the electromagnetic field have a bounded support as well and are located in the homogeneous background medium.

To numerically compute the electromagnetic field in such a configuration, we take the rectangular domain $\Omega = \Omega_x \times \Omega_y \times \Omega_z$ with $\Omega_i = \{i \in \mathbb{R}; |i| \leq \ell_i, i, 0 < \ell_i < \infty\}$, $i = x, y, z$, as our domain of interest, where the side lengths ℓ_i are chosen such that only the homogeneous background medium is present in its complement $\bar{\Omega}$, that is, the support of all sources and objects are proper subsets of Ω. Subsequently, we consider Maxwell's equations in the Laplace or s-domain with $s \in \mathbb{C}^+$ (right-half of the complex s-plane) and stretch the spatial coordinates in each Cartesian direction using the stretching functions

$$\hat{\chi}_i(i, s) = a_i(i) + s^{-1} b_i(i), \qquad i = x, y, z, \tag{1}$$

where a_i and b_i are real-valued and s-independent functions that satisfy $a_i(i) \geq 1$, $b_i(i) \geq 0$, and

$$a_i(i) = 1 \quad \text{and} \quad b_i(i) = 0 \quad \text{for } i \in \Omega_i, \tag{2}$$

with $i = x, y, z$. Note that the stretching functions satisfy

$$\hat{\chi}_i^*(i, s) = \hat{\chi}_i(i, s^*), \quad i = x, y, z, \tag{3}$$

where the asterisk denotes complex conjugation.

With complex stretching included, the stretched Maxwell equations are given by [1] –[5]

$$-\boldsymbol{\nabla}_{\rm s} \times \hat{\mathbf{H}} + s\varepsilon\hat{\mathbf{E}} = -\hat{\mathbf{J}}^{\rm ext} \tag{4}$$

and

$$\boldsymbol{\nabla}_{\rm s} \times \hat{\mathbf{E}} + s\mu\hat{\mathbf{H}} = -\hat{\mathbf{K}}^{\rm ext}, \tag{5}$$

where

$$\boldsymbol{\nabla}_{\rm s} = \mathbf{i}_x \hat{\chi}_x^{-1} \partial_x + \mathbf{i}_y \hat{\chi}_y^{-1} \partial_y + \mathbf{i}_z \hat{\chi}_z^{-1} \partial_z \tag{6}$$

is the stretched nabla operator.

It is well known that for properly chosen stretching functions, electromagnetic waves propagate without any reflection into the complement of Ω and strongly decay as the waves propagate away from the boundary $\partial\Omega$ of Ω into the complement $\bar{\Omega}$. We may therefore safely impose a perfectly conducting material boundary condition at a rectangular surface $\partial\Omega_{\rm out}$ that is completely located within the complement of Ω and sufficiently far away from the boundary $\partial\Omega$ of our domain of interest essentially without affecting the electromagnetic field in Ω. We refer to the domain that is enclosed by the closed boundary surfaces $\partial\Omega_{\rm out}$ and $\partial\Omega$ as a Perfectly Matched Layer (PML) and denote it by $\Omega_{\rm PML}$. Our total computational domain consists of the union of Ω and $\Omega_{\rm PML}$, while our domain of interest is Ω, since the PML is only used to simulate the extension to infinity.

2 THE SEMIDISCRETE FREQUENCY-DEPENDENT MAXWELL SYSTEM

Discretizing the stretched s-domain Maxwell equations in space on $\Omega \cup \Omega_{\rm PML}$ using two-point finite differences for the first-order spatial derivatives (see

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[6], for example) and taking a temporal Dirac distribution operative at $t = 0$ as a source wavelet, we end up with the large-scale semidiscrete Maxwell system

$$\left[\hat{\mathbf{D}}(s) + s\mathbf{M}\right] \hat{\mathbf{f}}(s) = -\mathbf{q}', \tag{7}$$

with $s \in \mathbb{C}^+$ and where $\mathbf{q}'$ a finite-difference approximation of the external electric- or magnetic-current density, $\hat{\mathbf{f}}(s)$ is the electromagnetic field vector, $\mathbf{M}$ a diagonal and positive definite medium matrix, and $\hat{\mathbf{D}}(s)$ a spatial discretization matrix containing the discretized stretched curl-operators. Note that the latter matrix is frequency-dependent due to the application of the coordinate stretching technique. Furthermore, there exists a diagonal and nonsingular step size matrix $\hat{\mathbf{W}}(s)$ such that

$$\hat{\mathbf{D}}^T(s)\hat{\mathbf{W}}(s) = \hat{\mathbf{W}}(s)\hat{\mathbf{D}}(s). \tag{8}$$

Since only the diagonal elements of $\hat{\mathbf{W}}(s)$ that correspond to step sizes located within the PML are s-dependent, we have that

$$\mathbf{W}_\Omega := \mathbf{S}\hat{\mathbf{W}} \tag{9}$$

is an s-independent step size matrix, where $\mathbf{S}$ is the support matrix of the domain of interest Ω and

$$\mathcal{L}_{\rm fd} = \frac{1}{2}\mathbf{f}^T(t)\mathbf{W}_\Omega\mathbf{M}\,\mathbf{f}(t) \tag{10}$$

is a finite-difference approximation of the free-field Lagrangian

$$\mathcal{L} = \frac{1}{2}\iiint_\Omega \varepsilon|\mathbf{E}|^2\,\mathrm{d}V - \frac{1}{2}\iiint_\Omega \mu|\mathbf{H}|^2\,\mathrm{d}V. \tag{11}$$

Finally, we note that since the stretching functions satisfy Eq. (3), we also have

$$\hat{\mathbf{D}}^*(s) = \hat{\mathbf{D}}(s^*) \quad \text{and} \quad \hat{\mathbf{W}}^*(s) = \hat{\mathbf{W}}(s^*) \tag{12}$$

for $s \in \mathbb{C}^+$.

If we now multiply Eq. (7) by the step size matrix $\hat{\mathbf{W}}(s)$, we obtain

$$\hat{\mathbf{A}}(s)\hat{\mathbf{f}}(s) = \mathbf{q}, \tag{13}$$

where the source vector $\mathbf{q} = -\hat{\mathbf{W}}(s)\mathbf{q}'$ is s-independent, since the external current densities are located within the domain of interest and not inside the PML. Moreover, the Maxwell system matrix in the above equation is given by

$$\hat{\mathbf{A}}(s) = \hat{\mathbf{W}}(s)\left[\hat{\mathbf{D}}(s) + s\mathbf{M}\right] \tag{14}$$

and using Eqs. (8) and (12), it is easily verified that this matrix satisfies

$$\hat{\mathbf{A}}^T(s) = \hat{\mathbf{A}}(s) \quad \text{and} \quad \hat{\mathbf{A}}^*(s) = \hat{\mathbf{A}}(s^*) \tag{15}$$

for $s \in \mathbb{C}^+$.

3 REDUCED-ORDER ELECTROMAGNETIC WAVE FIELD MODELS

To construct the reduced-order models for the electromagnetic field, we first solve Eq. (13) for $m \geq 1$ different frequenties and construct the subspace

$$\mathcal{K}_m = \mathrm{span}\{\hat{\mathbf{f}}(s_1), \hat{\mathbf{f}}(s_2), ..., \hat{\mathbf{f}}(s_m)\}.$$

Having this subspace available, we take

$$\mathcal{K}_m^{\rm R} = \mathrm{span}\{\mathrm{Re}\,\mathcal{K}_m, \mathrm{Im}\,\mathcal{K}_m\}$$

as an expansion and projection space to construct the reduced order models [7]. Observe that $\hat{\mathbf{f}}(s_i) \in \mathcal{K}_m^{\rm R}$ and $\hat{\mathbf{f}}(s_i^*) \in \mathcal{K}_m^{\rm R}$ for $i = 1, 2, ..., m$, since $\hat{\mathbf{f}}(s^*) = \hat{\mathbf{f}}^*(s)$.

Now let $\mathbf{V}_m$ be a basis matrix of $\mathcal{K}_m^{\rm R}$. We approximate the field vector $\hat{\mathbf{f}}(s)$ by an element from this subspace and write the approximation as

$$\hat{\mathbf{f}}_m(s) = \mathbf{V}_m\hat{\mathbf{a}}_m(s), \tag{16}$$

where $\hat{\mathbf{a}}_m(s)$ is the $2m$-by-1 vector of expansion coefficients. The residual that corresponds to this approximation is given by

$$\hat{\mathbf{r}}_m(s) = \mathbf{q} - \hat{\mathbf{A}}(s)\hat{\mathbf{f}}_m(s), \tag{17}$$

and we determine the expansion coefficients by requiring that the residual is orthogonal to the subspace $\mathcal{K}_m^{\rm R}$, that is, the expansion coefficients follow from the Galerkin condition

$$\mathbf{V}_m^T\hat{\mathbf{r}}_m(s) = \mathbf{0}. \tag{18}$$

Assuming that the reduced-order Maxwell system matrix

$$\hat{\mathbf{R}}_m(s) = \mathbf{V}_m^T\hat{\mathbf{A}}(s)\mathbf{V}_m \tag{19}$$

is nonsingular for $s \in \mathbb{C}^+$, the expansion coefficients follow as

$$\hat{\mathbf{a}}_m(s) = \hat{\mathbf{R}}_m^{-1}(s)\mathbf{V}_m^T\mathbf{q} \tag{20}$$

and we arrive at the reduced-order model

$$\hat{\mathbf{f}}_m(s) = \mathbf{V}_m\hat{\mathbf{R}}_m^{-1}(s)\mathbf{V}_m^T\mathbf{q}. \tag{21}$$

We note that from the definition of the reduced Maxwell system matrix (Eq. (19)) and Eq. (15) it follows that $\hat{\mathbf{R}}_m(s)$ satisfies

$$\hat{\mathbf{R}}_m^T(s) = \hat{\mathbf{R}}_m(s) \quad \text{and} \quad \hat{\mathbf{R}}_m^*(s) = \hat{\mathbf{R}}_m(s^*) \tag{22}$$

for $s \in \mathbb{C}^+$. In addition, we have

$$\hat{\mathbf{f}}_m(s_i) = \hat{\mathbf{f}}(s_i) \quad \text{and} \quad \hat{\mathbf{f}}_m(s_i^*) = \hat{\mathbf{f}}(s_i^*), \tag{23}$$

that is, the reduced-order model interpolates the exact field vector at the frequencies s_i, $i =$

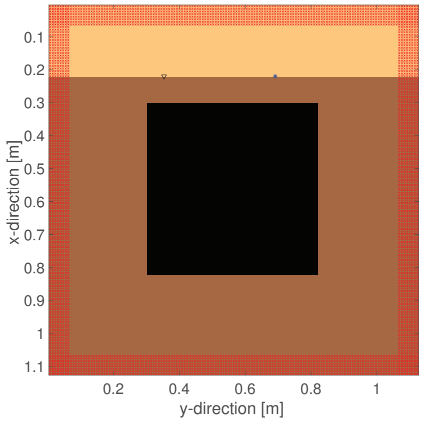

Figure 1: A z-directed external electric current source (star) and a receiver (triangle) located at the interface between two halfspaces. The upper halfspace consists of air, while the lower halfspace has a relative permittivity $\varepsilon_{\mathrm{r}} = 1$ and a conductivity $\sigma = 5 \cdot 10^{-4}$ S/m. The anomaly has a relative permittivity $\varepsilon_{\mathrm{r}} = 4$ and a conductivity $\sigma = 10^{-2}$ S/m.

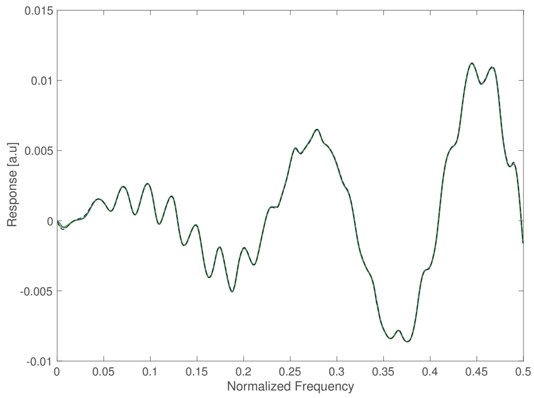

Figure 2: Electric field strength response at the receiver location over the complete frequency band of interest. Green solid line: FDFD comparison solution, black solid line: polynomial reduced-order model of order $m = 3500$, black dashed line: reduced-order model presented in this paper of order $m = 71$. All lines essentially overlap.

$1, 2, ..., m$ and their complex conjugates. Moreover, for monostatic field responses $\mathbf{q}^T\hat{\mathbf{f}}(s)$ we also have

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{q}^T\hat{\mathbf{f}}_m(s)\Big|_{s=s_i,s_i^*} = \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{q}^T\hat{\mathbf{f}}(s)\Big|_{s=s_i,s_i^*} \qquad (24)$$

for $i = 1, 2, ..., m$. In other words, for monostatic field responses the first-order derivative of the monostatic reduced-order model with respect to frequency interpolates the first-order derivative of the exact monostatic field response at $s = s_i$ and $s = s_i^*$ for $i = 1, 2, ..., m$.

4 NUMERICAL RESULTS

To illustrate the performance of our reduction technique, we compute the electromagnetic wave field response for the simple two-dimensional ground penetrating radar setup shown in Figure 1. This configuration consists of two halfspaces and a box shaped anomaly located in the lower halfspace. Air is present in the upper halfspace, while the lower halfspace is lossy with a relative permittivity $\varepsilon_{\mathrm{r}} = 1$ and a conductivity $\sigma = 5 \cdot 10^{-4}$ S/m. The anomaly has a relative permittivity $\varepsilon_{\mathrm{r}} = 4$ and a conductivity $\sigma = 10^{-2}$ S/m. Both the z-directed electric current source (star in Figure 1 and the receiver (triangle in Figure 1 are located at the interface between the two halfspaces and the receiver measures the z-component of the electric field strength over a frequency interval running from 50 MHz to 2.4 GHz.

Furthermore, Maxwell's equations are discretized in space using second-order finite-differences such that we have eight points per smallest wavelength ar 2.4 GHz. This leads to a discretized Maxwell system with $N = 62 \cdot 10^3$ unknowns.

Using the full unreduced operator, we first compute the transfer function for this configuration using an FDFD method to obtain a comparison solution for our reduced order models. In addition, we also compute reduced-order models over the frequency band of interest using the polynomial Krylov reduction technique presented in [4]. The full unreduced and frequency-independent Maxwell operator in this technique is only needed to form matrix-vector products and no systems of equations need to be solved. Using the polynomial reduction technique, a reduced-order model of order $m = 3500$ is required to accurately model the electric field response at the receiver location over the frequency band of interest (see Figure 2). To construct a reduced-order model based on the technique presented in this paper, we choose $m = 71$ equidistant frequencies on the imaginary axis with an imaginary part ranging between 50 MHz and 2.4 GHz. The resulting reduced-order response is shown in Figure 2 and this response essentially coincides with the FDFD comparison solution over the complete frequency band of interest. Clearly, the technique presented in this paper produces a model with an order that is much smaller than the model obtained via the polynomial Krylov technique. However, the cost of each iteration is larger when using the present technique, since it requires the solution of a system of equations for each interpolation frequency. This leads us to conclude that if the online costs of constructing a reduced-order model are the dominant factor in an electromagnetic wave field computation, then the polynomial technique may be preferred, while if offline costs are to be minimized (in an inversion algorithm, for example) then the reduced-order model with the smallest order should be used, which is provided by the technique described in this paper.

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