Emitter Localization Given Time Delay and Frequency Shift Measurements

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Abstract

Given time and frequency differences of arrival measurements, we estimate the position and velocity of an emitter by jointly eliminating non-linear nuisance parameters with an orthogonal projection matrix. Although simulation results show that this estimator does not always perform as well as the two-step estimator, the benefit is its computational simplicity. Whereas the complexity of the two-step estimator increases cubically with respect to the number of sensors, the complexity of the proposed estimator increases quadratically.

Index Terms

Source position estimation, time difference of arrival, frequency difference of arrival.

1. Introduction

Estimating the location of an emitter with a passive sensor array has been of considerable interest for many years, and has found many applications in several fields including radar, sonar, wireless communications, satellites, airborne systems, and acoustics [1]–[11]. With the common indirect estimation approach [1], [2], one or more parameters (e.g., angle or time of arrival) are measured, and the emitter

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parameters (position and/or velocity) are then determined. A different approach is to estimate the emitter parameters directly from the observations [10], [11]. Herein, we focus on the former approach assuming a stationary passive sensor array and a moving emitter.

Given the measurements of time differences of arrival (TDOAs) and frequency differences of arrival (FDOAs) between pairs of observed signals, the goal is to estimate the source position and velocity[1]. Weinstein proposed an estimation technique which is applicable for a linear array only and assumes a source in the far-field region [5]. The estimation procedure suggested by Ho and Xu [9] extended the two-step approach of Chan and Ho [8] by taking into account the FDOA measurements. The idea of Ho and Xu [9] is to obtain a set of linear equations by introducing two nuisance parameters (the range and range rate associated with the reference sensor and the source). In the first step, a weighted least squares (WLS) solution is proposed to estimate the position and velocity of the source together with these nuisance parameters, and in the second step, the relations between the nuisance parameters and the parameters of interest are used to solely estimate the position and velocity using another WLS minimization. The performance of this method was shown to be close to the Cramér-Rao lower bound (CRLB) [9, Appendix C]. Friedlander suggested to estimate the source position and velocity by extending his least squares (LS) method which was developed to locate a stationary source given TDOAs only [7]. The LS position estimate of a stationary source relies on an orthogonal projection matrix to eliminate the nuisance parameter (range between the reference sensor and the source). The notion of Friedlander's extension [7, Section V] was to use two similar orthogonal projections in a subsequent manner as follows: first obtain the LS source position as previously explained, and then eliminate the second nuisance parameter (range-rate between the reference sensor and the source) using the same orthogonal projection to get the LS velocity estimate. Our simulation results show that this subsequent projection approach has poor performance compared to the two-step approach [9] and the CRLB.

Herein, by exploiting the idea leading to Friedlander's TDOA-based positioning method [7], we propose

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[1]The TDOAs and FDOAs are obtained by maximizing the ambiguity function [12]. Their statistical properties are discussed in [13], [14], and [16], assuming a known, an unknown deterministic, and a random transmitted signal, respectively.

a LS estimator of the source position and velocity which is obtained from using a joint elimination (a single orthogonal projection) of the two nuisance parameters. It is noteworthy to mention that this LS estimate is closely related to the first step WLS estimate in [9] following the results in [15]. We show that the estimates are asymptotically unbiased, and also derive their covariance matrix. The performance of the proposed estimator is evaluated with simulations for a source in the near-field and far-field regions as a function of: i) the noise variance using a circular sensor array and a random sensor array, ii) the number of sensors, and iii) the ratio between the variances of the TDOA and the FDOA measurements. We show that there is a trade-off between performance and complexity. Although, the proposed algorithm does not always perform as well as the two-step approach, the main advantage is its computational complexity. Whereas the complexity of the previously suggested two-step estimator increases cubically with respect to the number of sensors, the complexity of the proposed estimator only increases quadratically.

Notation: uppercase and lowercase bold fonts denote matrices and vectors, respectively. $(\cdot)^T$, $(\cdot)^{-1}$ stand for transpose, and inverse, respectively. $\mathbf{I}_n$ is the $n \times n$ identity matrix, $\mathbf{0}_n$ is a $n \times 1$ vector with all elements equal to zero. $\mathrm{diag}(z_1, \ldots, z_N)$ is a diagonal matrix with $z_1, \ldots, z_N$ on the main diagonal. $E[\mathbf{x}]$ represents the expectation of the random vector $\mathbf{x}$. $\dot{\mathbf{x}}$ is the time derivative of $\mathbf{x}(t)$ with respect to t, i.e., $\dot{\mathbf{x}} = d\mathbf{x}(t)/dt$. $\|\mathbf{x}\|$ is the 2-norm of $\mathbf{x}$. $\otimes$ is the Kronecker product. $\mathbf{X}^{\perp}$ is the orthogonal projection matrix of $\mathbf{X}$, i.e., $\mathbf{X}^{\perp} = \mathbf{I} - \mathbf{X}(\mathbf{X}^H \mathbf{X})^{-1}\mathbf{X}^H$. $\bar{\mathbf{x}}$ is the concatenation of $\mathbf{x}$ and $\dot{\mathbf{x}}$, i.e., $\bar{\mathbf{x}} = [\mathbf{x}^T, \dot{\mathbf{x}}^T]^T$. $\hat{\mathbf{x}}$ is the estimate of $\mathbf{x}$ in the presence of Gaussian noise, i.e., $\hat{\mathbf{x}} = \mathbf{x} + \mathbf{e}$ where $\mathbf{e}$ is a zero mean Gaussian vector representing the estimation error. $\tilde{\mathbf{x}}$ represents the first order error of the estimate $\hat{\mathbf{x}}$, i.e., $\hat{\mathbf{x}} = \mathbf{x} + \tilde{\mathbf{x}}$.

2. PROBLEM FORMULATION

Consider M stationary sensors and a moving source distributed in a q-dimensional Cartesian coordinate system ($q = 2$ or $q = 3$). Let $\bar{\mathbf{p}}_s \triangleq [\mathbf{p}_s^T, \dot{\mathbf{p}}_s^T]^T$ be the $2q \times 1$ vector, where $\mathbf{p}_s$ and $\dot{\mathbf{p}}_s$ are the $q \times 1$ true unknown position and velocity vectors of coordinates of the source. Let $\mathbf{p}_m$, $m = 1, 2, ..., M$ denote the known $q \times 1$ vector of coordinates of the mth sensor (We note that the setup in [9] is developed for the case of moving sensors. The extension of the current problem formulation and the proposed method to

this case is straightforward). Let $\Delta t_{m,1}$ and $\Delta f_{m,1}$ be the true TDOA and FDOA between the signals received by the mth sensor and the first (reference) sensor. Denote by c the signal propagation speed and by f_c the carrier frequency of the signal. The true range $r_{m,1}$ and range-rate $\dot{r}_{m,1}$ differences are

$$r_{m,1} \quad \overset{\Delta}{=} \quad c\Delta t_{m,1} = d_{m,s} - d_{1,s} \tag{1}$$

$$\dot{r}_{m,1} \quad \overset{\Delta}{=} \quad \frac{c}{f_c}\Delta f_{m,1} = \dot{d}_{m,s} - \dot{d}_{1,s} \tag{2}$$

where the range $d_{m,s}$ and range-rate $\dot{d}_{m,s}$ between the mth sensor and the source are defined as,

$$d_{m,s} \quad \overset{\Delta}{=} \quad \|\mathbf{p}_s - \mathbf{p}_m\| \tag{3}$$

$$\dot{d}_{m,s} \quad \overset{\Delta}{=} \quad \frac{(\mathbf{p}_m - \mathbf{p}_s)^T \dot{\mathbf{p}}_s}{d_{m,s}} \tag{4}$$

We note that the TDOA and FDOA measurements are taken over a short interval and the assumption is that the source position and velocity to be estimated (assumed to be at some point in the interval) do not change much during the measurements.

Define the $2(M-1) \times 1$ vector $\bar{\mathbf{r}} \overset{\Delta}{=} [\mathbf{r}^T, \dot{\mathbf{r}}^T]^T$ where $\mathbf{r} \overset{\Delta}{=} [r_{2,1}, \ldots, r_{M,1}]^T$ and $\dot{\mathbf{r}} \overset{\Delta}{=} [\dot{r}_{2,1}, \ldots, \dot{r}_{M,1}]^T$ are $(M-1) \times 1$ vectors. In practice, we are given the noisy $2(M-1) \times 1$ vector,

$$\hat{\bar{\mathbf{r}}} = \bar{\mathbf{r}} + \boldsymbol{\delta} \tag{5}$$

where $\hat{\bar{\mathbf{r}}} \overset{\Delta}{=} [\hat{\mathbf{r}}^T, \hat{\dot{\mathbf{r}}}^T]^T$, and $\hat{\mathbf{r}} \overset{\Delta}{=} [\hat{r}_{2,1}, \ldots, \hat{r}_{M,1}]^T$, $\hat{\dot{\mathbf{r}}} \overset{\Delta}{=} [\hat{\dot{r}}_{2,1}, \ldots, \hat{\dot{r}}_{M,1}]^T$ are $(M-1) \times 1$ vectors containing the noisy measurements of the range and range-rate differences, respectively. The $2(M-1) \times 1$ vector $\boldsymbol{\delta} \overset{\Delta}{=} [\boldsymbol{\epsilon}^T, \boldsymbol{\xi}^T]^T$ is the additive noise where $\boldsymbol{\epsilon} \overset{\Delta}{=} [\epsilon_{2,1}, \ldots, \epsilon_{M,1}]^T$ and $\boldsymbol{\xi} \overset{\Delta}{=} [\xi_{2,1}, \ldots, \xi_{M,1}]^T$ are $(M-1) \times 1$ vectors. We assume that $\boldsymbol{\delta}$ is a zero mean Gaussian random vector with a covariance matrix $E[\boldsymbol{\delta}\boldsymbol{\delta}^T]$.

The problem we discuss is briefly expressed as follows: Given the vector of measurements $\hat{\bar{\mathbf{r}}}$, determine the vector of interest $\bar{\mathbf{p}}_s$.

3. THE PROPOSED LEAST-SQUARES ESTIMATOR

We start by developing a model which linearly depends on $\bar{\mathbf{p}}_s$ following the mathematical derivations introduced in [7]. Define the $(M-1) \times q$ matrix $\mathbf{S}$ and the $(M-1) \times 1$ vector $\mathbf{u}$ as,

$$\mathbf{S} \triangleq [\mathbf{p}_2 - \mathbf{p}_1, \cdots, \mathbf{p}_M - \mathbf{p}_1]^T \tag{6}$$

$$\mathbf{u} \triangleq \frac{1}{2} \left[\|\mathbf{p}_2\|^2 - \|\mathbf{p}_1\|^2 - r_{2,1}^2, \ldots, \|\mathbf{p}_M\|^2 - \|\mathbf{p}_1\|^2 - r_{M,1}^2 \right]^T \tag{7}$$

According to [7, Eq. (7a)] we have the following relation,

$$\mathbf{S}\mathbf{p}_s = \mathbf{u} - d_{1,s}\mathbf{r} \tag{8}$$

Next, define the $(M-1) \times 1$ time derivative vector of $\mathbf{u}$, denoted by $\dot{\mathbf{u}}$, and the 2×1 vector $\bar{\mathbf{d}}_{1,s}$ as

$$\dot{\mathbf{u}} \triangleq [-r_{2,1}\dot{r}_{2,1}, \ldots, -r_{M,1}\dot{r}_{M,1}]^T \tag{9}$$

$$\bar{\mathbf{d}}_{1,s} \triangleq [d_{1,s}, \dot{d}_{1,s}]^T \tag{10}$$

Then, according to [7, Eq. (60)] we get that,

$$\mathbf{S}\dot{\mathbf{p}}_s = \dot{\mathbf{u}} - [\ \dot{\mathbf{r}} \quad \mathbf{r}\ ]\bar{\mathbf{d}}_{1,s} \tag{11}$$

In [7] the two models in (8) and (11) were considered separately. Herein, we note that these two models contain the vectors of interest, i.e., the position and the velocity of the source. Hence, by combining (8) and (11) we get a linear model with respect to (w.r.t.) $\bar{\mathbf{p}}_s$ given as,

$$\mathbf{F}\bar{\mathbf{p}}_s + \mathbf{H}\bar{\mathbf{d}}_{1,s} = \bar{\mathbf{u}} \tag{12}$$

where the $2(M-1) \times 1$ vector $\bar{\mathbf{u}}$, the $2(M-1) \times 2q$ matrix $\mathbf{F}$, and the $2(M-1) \times 2$ matrix $\mathbf{H}$ are

$$\bar{\mathbf{u}} \triangleq [\mathbf{u}^T, \ \dot{\mathbf{u}}^T]^T \tag{13}$$

$$\mathbf{F} \triangleq \mathbf{I}_2 \otimes \mathbf{S} \tag{14}$$

$$\mathbf{H} \triangleq \begin{bmatrix} \mathbf{r} & \mathbf{0}_{M-1} \\ \dot{\mathbf{r}} & \mathbf{r} \end{bmatrix} \tag{15}$$

where $\mathbf{I}_n$ is an $n \times n$ identity matrix, $\otimes$ is a Kronecker product, and $\mathbf{0}_n$ is an $n \times 1$ vector of zeros.

The linear model in (12) contains both the unknown non-linear nuisance vector $\bar{\mathbf{d}}_{1,s}$ (range and range-rate of the source w.r.t. the reference sensor) and the unknown vector of interest $\bar{\mathbf{p}}_s$. In [9] the approach is to first estimate $\bar{\mathbf{d}}_{1,s}$ together with $\bar{\mathbf{p}}_s$, and then to use the relation between the two vectors to further refine the previous estimate of $\bar{\mathbf{p}}_s$. In [7] the estimation is based on: i) eliminating the term associated with $d_{1,s}$ in (8), with an orthogonal projection matrix [7, Eq. (8)], and obtaining the LS solution for $\mathbf{p}_s$; ii) eliminating the term associated with $\dot{d}_{1,s}$ in (11), using the same orthogonal projection matrix [7, Eq. (8)], and then obtaining the LS solution for $\dot{\mathbf{p}}_s$ (where $d_{1,s}$ involved in the latter solution is calculated using the estimate of $\mathbf{p}_s$ obtained after the first step).

We adopt a different approach. The idea is to jointly eliminate the unknown non-linear nuisance vector $\bar{\mathbf{d}}_{1,s}$ in (12) using an appropriate orthogonal projection matrix which leads to an equation that solely depends on the unknown vector of interest $\bar{\mathbf{p}}_s$. It is noteworthy to mention that this operation considers the two vectors $\bar{\mathbf{d}}_{1,s}$ and $\bar{\mathbf{p}}_s$ as independent, and ignores the fact that they are mathematically related.

We define the $2(M-1) \times 2(M-1)$ orthogonal projection matrix of $\mathbf{H}$ as,

$$\mathbf{P}^\perp = \mathbf{I}_{2(M-1)} - \mathbf{H} \left(\mathbf{H}^T \mathbf{H} \right)^{-1} \mathbf{H}^T \tag{16}$$

Pre-multiplying (12) with $\mathbf{P}^\perp$ yields a linear model which only depends on the vector of interest $\bar{\mathbf{p}}_s$,

$$\mathbf{P}^\perp \mathbf{F} \bar{\mathbf{p}}_s = \mathbf{P}^\perp \bar{\mathbf{u}} \tag{17}$$

In the presence of noise we replace the true vectors and matrices in (17) by their noisy versions (i.e., we write $\hat{\bar{\mathbf{u}}}$ instead of $\bar{\mathbf{u}}$), since we will adopt the noisy measurements vector $\hat{\bar{\mathbf{r}}}$ given in (5). This results in the error vector, denoted by $\boldsymbol{\eta}$, and (17) is then given by

$$\hat{\mathbf{P}}^\perp \hat{\bar{\mathbf{u}}} = \hat{\mathbf{P}}^\perp \mathbf{F} \hat{\bar{\mathbf{p}}}_s + \boldsymbol{\eta} \tag{18}$$

The LS estimate of $\bar{\mathbf{p}}_s$ is obtained by minimizing the square norm of $\boldsymbol{\eta}$, that is,

$$\hat{\bar{\mathbf{p}}}_s = \underset{\bar{\mathbf{p}}_s}{\arg\min} \left\| \hat{\mathbf{P}}^\perp (\mathbf{F} \bar{\mathbf{p}}_s - \hat{\bar{\mathbf{u}}}) \right\|^2 = \hat{\mathbf{Q}} \hat{\bar{\mathbf{u}}} \tag{19}$$

where $\hat{\mathbf{Q}}$ is a $2q \times 2(M-1)$ matrix defined as,

$$\hat{\mathbf{Q}} \triangleq (\mathbf{F}^T \hat{\mathbf{P}}^\perp \mathbf{F})^{-1} \mathbf{F}^T \hat{\mathbf{P}}^\perp \tag{20}$$

This concludes the derivation of the proposed estimator. Notice that following the results in [15], the LS estimator in (19) is related to the WLS estimator obtained in the first step in [9]. In the next sections we focus on the small error performance and the computational complexity of this LS estimator.

4. SMALL ERROR ANALYSIS

We examine the effect of noise on the position and the velocity estimates using small error analysis. The idea is to express the estimate $\hat{\bar{\mathbf{p}}}_s$ as $\hat{\bar{\mathbf{p}}}_s \cong \bar{\mathbf{p}}_s + \tilde{\bar{\mathbf{p}}}_s$ where $\tilde{\bar{\mathbf{p}}}_s$ is the first order error of $\hat{\bar{\mathbf{p}}}_s$ (higher order error terms of $\hat{\bar{\mathbf{p}}}_s$ depend on products involving both $\boldsymbol{\epsilon}$ and $\boldsymbol{\xi}$ and are therefore ignored). The approximated bias of the estimate $\hat{\bar{\mathbf{p}}}_s$ is then given by $E[\tilde{\bar{\mathbf{p}}}_s]$, and the approximated covariance of $\hat{\bar{\mathbf{p}}}_s$ is then given by $E[(\tilde{\bar{\mathbf{p}}}_s - E[\tilde{\bar{\mathbf{p}}}_s])(\tilde{\bar{\mathbf{p}}}_s - E[\tilde{\bar{\mathbf{p}}}_s])^T]$. We start by obtaining an explicit expression for $\tilde{\bar{\mathbf{p}}}_s$ and then analyze its two first moments.

Considering the estimate in (19), we express the noisy matrix $\hat{\mathbf{Q}}$ and the noisy vector $\hat{\bar{\mathbf{u}}}$ using first order approximations as $\hat{\mathbf{Q}} = \mathbf{Q} + \tilde{\mathbf{Q}}$ and $\hat{\bar{\mathbf{u}}} = \bar{\mathbf{u}} + \tilde{\bar{\mathbf{u}}}$, respectively (the explicit expressions for the first order error terms, $\tilde{\mathbf{Q}}$ and $\tilde{\bar{\mathbf{u}}}$ are given in Appendix A). We then get that

$$\begin{aligned}
\hat{\bar{\mathbf{p}}}_s &= \hat{\mathbf{Q}}(\bar{\mathbf{u}} + \tilde{\bar{\mathbf{u}}}) \\
&= \hat{\mathbf{Q}}(\mathbf{F}\bar{\mathbf{p}}_s + \mathbf{H}\bar{\mathbf{d}}_{1,s}) + \hat{\mathbf{Q}}\tilde{\bar{\mathbf{u}}} \\
&= \bar{\mathbf{p}}_s + \hat{\mathbf{Q}}\mathbf{H}\bar{\mathbf{d}}_{1,s} + \hat{\mathbf{Q}}\tilde{\bar{\mathbf{u}}} \\
&\cong \bar{\mathbf{p}}_s + (\mathbf{Q} + \tilde{\mathbf{Q}})\mathbf{H}\bar{\mathbf{d}}_{1,s} + \mathbf{Q}\tilde{\bar{\mathbf{u}}} \\
&= \bar{\mathbf{p}}_s + \tilde{\mathbf{Q}}\mathbf{H}\bar{\mathbf{d}}_{1,s} + \mathbf{Q}\tilde{\bar{\mathbf{u}}} \tag{21}
\end{aligned}$$

where in the second passing we substitute $\bar{\mathbf{u}}$ by $\mathbf{F}\bar{\mathbf{p}}_s + \mathbf{H}\bar{\mathbf{d}}_{1,s}$, in the third passing we used the result that $\hat{\mathbf{Q}}\mathbf{F} = \mathbf{I}$, in the forth passing we neglected the term $\tilde{\mathbf{Q}}\tilde{\bar{\mathbf{u}}}$ which involves products of errors, and finally

in the fifth passing we used the result that $\mathbf{Q}\mathbf{H} = \mathbf{0}$. The first order error of $\hat{\tilde{\mathbf{p}}}_s$ is thus

$$\tilde{\mathbf{p}}_s \overset{\triangle}{=} \tilde{\mathbf{Q}}\mathbf{H}\mathbf{d}_{1,s} + \mathbf{Q}\tilde{\tilde{\mathbf{u}}} \tag{22}$$

Substituting in (22) the expressions for $\tilde{\mathbf{Q}}$ and $\tilde{\tilde{\mathbf{u}}}$ (obtained in (32) and (35) in Appendix A) results in,

$$\tilde{\tilde{\mathbf{p}}}_s = \mathbf{Q}\mathbf{J}\boldsymbol{\delta} \tag{23}$$

where we define the $2(M-1) \times 2(M-1)$ matrix $\mathbf{J}$ as

$$\mathbf{J} \overset{\triangle}{=} - \begin{bmatrix} \text{diag}(\mathbf{r}) + d_{1,s}\mathbf{I}_{M-1} & \mathbf{0}_{M-1}\mathbf{0}_{M-1}^T \\ \text{diag}(\dot{\mathbf{r}}) + \dot{d}_{1,s}\mathbf{I}_{M-1} & \text{diag}(\mathbf{r}) + d_{1,s}\mathbf{I}_{M-1} \end{bmatrix} \tag{24}$$

Since $E[\boldsymbol{\delta}] = \mathbf{0}$, we conclude that the first order approximation of the bias of the estimate $\hat{\tilde{\mathbf{p}}}_s$ is zero, that is, $E[\tilde{\mathbf{p}}_s] = \mathbf{0}_{2q \times 1}$. The first order approximation of the covariance matrix of $\hat{\tilde{\mathbf{p}}}_s$ is

$$E[\tilde{\mathbf{p}}_s\tilde{\mathbf{p}}_s^T] = \mathbf{Q}\mathbf{J}E[\boldsymbol{\delta}\boldsymbol{\delta}^T]\mathbf{J}^T\mathbf{Q}^T \tag{25}$$

This concludes the derivation of the bias and the covariance matrix.

5. COMPUTATIONAL COMPLEXITY

We evaluate the computational complexity of the proposed LS positioning technique and compare it with the complexity of the two-step method. For simplicity we denote by $\text{RM}(X)$ the number of real multiplications (RMs) involved in calculating the parameter X.

A. Proposed estimator

The total number of RMs which are required to calculate $\hat{\tilde{\mathbf{p}}}_s$ with the proposed estimator (refer to Appendix B) is

$$\text{RM}(\hat{\tilde{\mathbf{p}}}_s) = \begin{cases} 32M^2 + 2M + 40, & q = 2 \,(\text{two-dimensional space}) \\ 48M^2 + 10M + 166, & q = 3 \,(\text{three-dimensional space}) \end{cases} \tag{26}$$

For a large number of sensors, the complexity of the proposed estimator increases quadratically w.r.t. M.

B. Two-step estimator

The two-step algorithm is detailed in [9, Section IV, p. 2458] and for exhibition simplicity is rewritten in Table II in Appendix B where we use the same notation as used in [9]. In Appendix B we detail the computational complexity of this method. According to this algorithm we need to refine the estimate by performing a few steps (at least two) if the source is in the near-field region. These steps (and their repetition) are neglected if the source is in the far-field region. However, in practice we cannot a-priori know whether the source is in the near-field region or the far-field region. Therefore, we need to consider the case of a source in the near-field only (worst case). Following the results in Appendix B, the total RMs which are required to calculate $\hat{\hat{\mathbf{p}}}_s$ with the two-step approach assuming a source in the near-field region is,

$$\text{RM}(\hat{\hat{\mathbf{p}}}_s) = \begin{cases} 48M^3 - 72M^2 + 468M + 2328, & q = 2 \,(\text{two-dimensional space}) \\ 48M^3 - 48M^2 + 768M + 8010, & q = 3 \,(\text{three-dimensional space}) \end{cases} \tag{27}$$

The main part of the calculation of this approach is calculating the weighting matrix required for the first estimation step, which involves the inversion of a $2(M-1) \times 2(M-1)$ matrix and requires $24(M-1)^3$ RMs. Therefore, the complexity of the two-step approach increases cubically with respect to the number of sensors in the array.

In Figure 1 we show the complexities of the proposed LS method and the two-step method for $q = 2$, and $q = 3$, and assuming a source in the near-field region, versus the number of sensors M where $M = 5, 6, \ldots, 20$. As can be seen, the ratio between the two complexities increases as the number of sensors in the array, M is increased.

6. NUMERICAL EXAMPLES

We present several simulation results that demonstrate the root mean square error (RMSE) of the position and velocity estimates using independent Monte-Carlo trials (we used 5000 trials). We compare the RMSE of the proposed LS estimator to those of the two-step method [9], and to the CRLB [9,
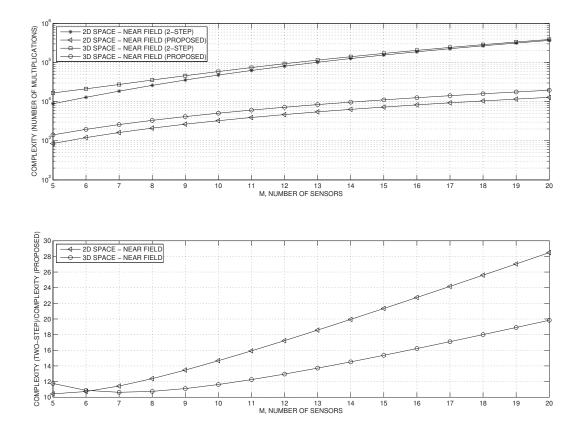
Fig. 1.   The number of real multiplications (upper plot) and the ratio between the number of real multiplications (lower plot) involved in the estimation of the two-step approach and the proposed approach for both a two-dimensional (2D) space and a three-dimensional (3D) space, and for a source in the near-field region.

Appendix C]. We also compute the theoretical RMSE of the proposed estimator according to (25), and the theoretical RMSE of the two-step estimator according to [9, Eq. (25)]. We assume that the transmitted signal is a white process with variance σ_s^2, independent of the noise processes which are all white, independent processes with variance σ_n^2. Also, the attenuations of the intercepted signal at all

sensors are assumed identical. We assume the covariance matrix of the noise vector $\boldsymbol{\delta}$ is [5, Section II]

$$\boldsymbol{\Lambda}_{\delta\delta} = \begin{bmatrix} E[\boldsymbol{\epsilon\epsilon}^T] & 0 \\ 0 & E[\boldsymbol{\xi\xi}^T] \end{bmatrix} = \begin{bmatrix} E[\boldsymbol{\epsilon\epsilon}^T] & 0 \\ 0 & \beta E[\boldsymbol{\epsilon\epsilon}^T] \end{bmatrix} \tag{28}$$

where $\beta \overset{\Delta}{=} \frac{12}{T^2}$ and T is the observation time, and [5, Eq. (10), Eq. (14)]

$$E[\boldsymbol{\epsilon\epsilon}^T] \overset{\Delta}{=} \gamma(\mathbf{I}_{M-1} + \mathbf{1}_{M-1}\mathbf{1}_{M-1}^T) \tag{29}$$

$$\gamma \overset{\Delta}{=} \frac{3\pi c^2}{TW^3} \frac{1 + M\text{SNR}}{M\text{SNR}^2} \tag{30}$$

$$\text{SNR} \overset{\Delta}{=} \sigma_s^2/\sigma_n^2 \tag{31}$$

where W is the signal bandwidth. This covariance matrix assumes that the transmitted signal is a Gaussian random process with a known power spectrum density. Other covariance matrices (obtained from a CRLB analysis) can be used also such as the covariance matrix given in [13] where it is assumed that the transmitted signal and the attenuations to the sensors are known, or the covariance matrix given in [14] where the signal is assumed to be deterministic but unknown and also the attenuations to the sensors are unknown.

In all the following plots we normalize the position RMSE by the distance between the source position and the origin, and normalize the velocity RMSE by the Euclidean norm of the source velocity vector.

In the first simulation we evaluate the RMSE versus the parameter γ for a sensor array with a circular configuration. We consider two cases for the source: far-field and near-field. In the far-field case the position and the velocity vectors of the source are $\mathbf{p}_s = [10000\cos(\pi/3), 10000\sin(\pi/3)]^T$ [meter] and $\dot{\mathbf{p}}_s = [30\sin(\pi/3), 30\cos(\pi/3)]^T$ [mester/sec], respectively. While in the near-field the position of the source is $\mathbf{p}_s = [1000\cos(\pi/3), 1000\sin(\pi/3)]^T$ [meter] with the same velocity vector. We consider eight sensors where $\mathbf{p}_m = 100 \cdot [\cos\left(\frac{2\pi m}{8}\right), \sin\left(\frac{2\pi m}{8}\right)]^T$ [meter], $m = 1, \dots, 8$. We vary the parameter $10\log_{10}(\gamma)$ from -50[dB meter^2] to -20[dB meter^2] (in case the source is in the near-field region) and from -80[dB meter^2] to -50[dB meter^2] (in case the source is in the far-field region). We assume that

$\beta = 0.1$ [Hz^2]. The normalized RMSE of the position and the velocity of the source using the proposed LS estimator, and the two step-approach are shown in Figure 2, where the CRLB is also plotted. We also add the RMSE of the subsequent orthogonal projection approach suggested in [7, Section V]. As can be seen, the RMSE of the LS solution is close to that of the two-step approach and the CRLB, while the RMSE of the subsequent orthogonal projection approach in [7, Section V] is inferior compared to the LS estimator and the two-step method. As a result we will not consider this approach in the following simulation results. We note that the theoretical performance of the two-step method is known to be close to the CRLB, and thus in this plot and in the subsequent plots the line representing the theoretical performance of the two-step method coincides with the CRLB.

In the second simulation we again evaluate the RMSE versus the parameter γ, but this time for a sensor array with a random configuration. We consider a source located in the the far-field region. The position and velocity vectors are $\mathbf{p}_s = [10000, 10000]^T$ [meter] and $\dot{\mathbf{p}}_s = [30, -20]^T$ [meter/sec], respectively. We consider eight sensors where $\mathbf{p}_m = r_m \cdot [\cos(\phi_m), \sin(\phi_m)]^T$ [meter], r_m is uniformly distributed on $[0, 100]$ [meter], and ϕ_m is uniformly distributed on $[-\pi, \pi]$. We perform 50 realizations of the sensor configuration, and then average the RMSEs. We vary $10\log_{10}(\gamma)$ as detailed in the previous simulation, and also assume that $\beta = 0.1$ [Hz^2]. The normalized RMSE of the position and the velocity of the source using the proposed LS solution, and the two step-approach are shown in Figure 3, where the CRLB is also plotted. In the left plot we show the result of one random configuration, while in the right plot we show the RMSE and the CRLB results are averaged over all the configurations. As can be observed, again the LS solution has a similar RMSE as that of the two-step approach for small values of γ (high SNR), and the two-step method achieves the CRLB for any SNR.

In the third simulation we evaluate the RMSE versus the number of sensors in the array. We consider a circular configuration as in the first example and a source in the far-field region. The position and the velocity vectors of the source are $\mathbf{p}_s = [10000\cos(\pi/3), 10000\sin(\pi/3)]^T$ [meter] and $\dot{\mathbf{p}}_s = [30\sin(\pi/3), 30\cos(\pi/3)]^T$ [meter/sec], respectively. We vary the number of sensors in the configuration from 8 to 32 with a step of 4. We consider a source in the far-field region, and set $10\log_{10}(\gamma) = -40$[dB
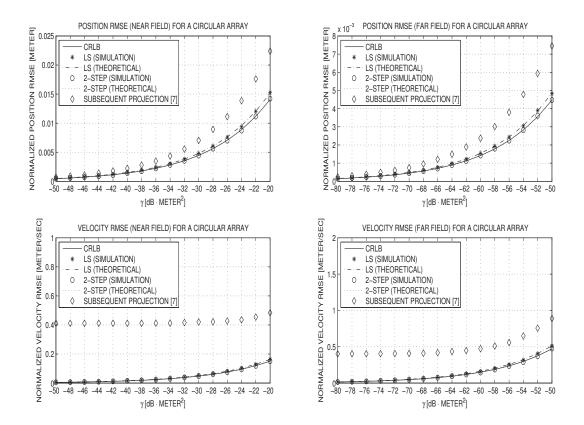
Fig. 2.   Normalized theoretical and simulated RMSE of the estimated position and velocity of the source in the far-field and near-field regions versus γ for an array with eight elements in a circular configuration, using the LS proposed method, the two-step approach, and the subsequent projection method [7], all compared with the CRLB.

meter^2], and $\beta = 0.1$ [Hz^2]. The normalized RMSE of the position and the velocity of the source using the proposed LS and the two step-approach is shown in Figure 4, where the CRLB is also plotted. Observe that compared to the two-step approach, the decrease of the RMSE of the LS method w.r.t. the number of sensors is smaller. In other words, the proposed approach provides increasingly worse accuracy (relative to the two-step approach) as the number of sensors in the array increases. On the other hand, as the number of sensors increases, the proposed approach becomes more computationally efficient.

In the fourth simulation we evaluate the RMSE versus the parameter β. We consider a circular configuration as in the first example and a source in the far-field region with the same position and
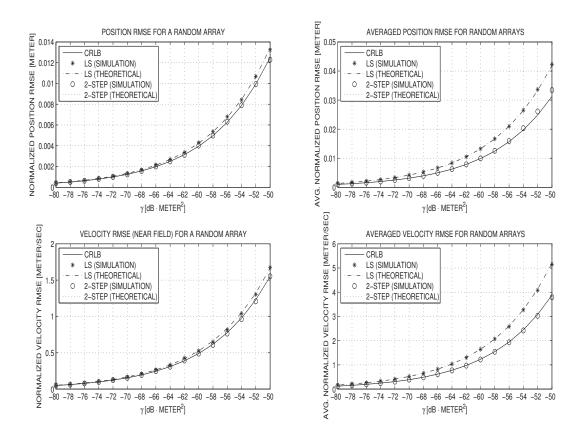
Fig. 3.   Normalized theoretical and simulated RMSE of the estimated position and velocity of the source in the far-field region versus γ for an array with eight elements in a random configuration, using the proposed LS method and the two-step approach, both compared with the CRLB (left plot - RMSE for one random configuration. right plot - RMSE averaged over 50 random configurations.)

velocity vectors as in the previous simulation. We vary the parameter β from 10^{-3} [Hz^2] to 10 [Hz^2]. We set $10\log_{10}(\gamma) = -40$[dB meter^2]. The normalized RMSE of the position and the velocity of the source using the proposed LS solution, and the two-step approach is shown in Figure 5, where the CRLB is also plotted. As can be seen, the LS and the two-step approach have similar velocity RMSE compared to the CRLB, while the position RMSE of the LS solution is poor. We note that the reason for the drop of some of the results of the two step method below the CRLB is due to the finite number of realizations that we simulated.
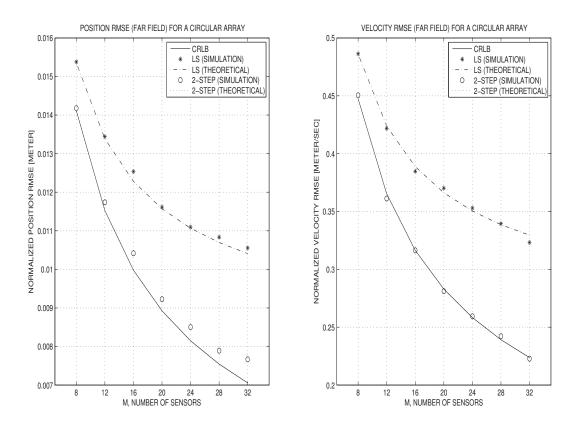
Fig. 4. Normalized theoretical and simulated RMSE of the estimated position and velocity of the source in the far-field region versus the number of sensors in an array with a circular configuration, using the LS proposed method and the two-step approach, both compared with the CRLB.

Finally, we compare the processing time (using MATLAB time commands), required for the proposed approach and the two-step approach to reach the estimate of the parameters of interest, as a function of the number of sensors in the array. We consider a circular array, with $10\log_{10}(\gamma) = -30$[dB meter^2], $\beta = 0.1$ [Hz^2], and a source in the near-field region. The position and the velocity vectors of the source are $\mathbf{p}_s = [1000\cos(60\pi/180), 1000\sin(60\pi/180)]^T$ [meter] and $\dot{\mathbf{p}}_s = [30, 15]^T$ [meter/sec], respectively. We vary the number of sensors from 5 to 20 with a step of 1. For each value of M we calculate the processing time of each method. In Figure 6 (upper subplot) we plot the absolute processing time of each method, and in Figure 6 (lower subplot) we plot the ratio between the processing time of the two-step
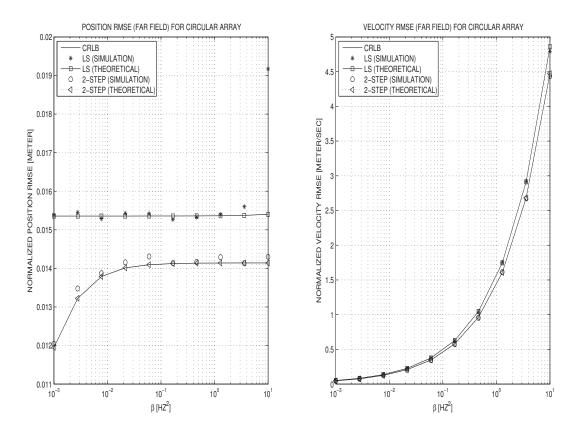
Fig. 5. Normalized theoretical and simulated RMSE of the estimated position and velocity of the source in the far-field region versus β for an array with eight elements in a circular configuration, using the proposed LS method and the two-step approach, both compared with the CRLB.

approach and the proposed approach. It can be seen that the complexity of the proposed approach is much smaller than the two-step approach especially for a large number of sensors.

7. CONCLUSIONS

We proposed a least squares method to estimate the position and velocity of an emitter given time and frequency differences of arrival measurements acquired by a sensor array. The idea is to obtain a linear model with respect to the parameters of interest by eliminating non-linear unknown nuisance parameters (range and range-rate differences between the reference sensor and the source) using an orthogonal
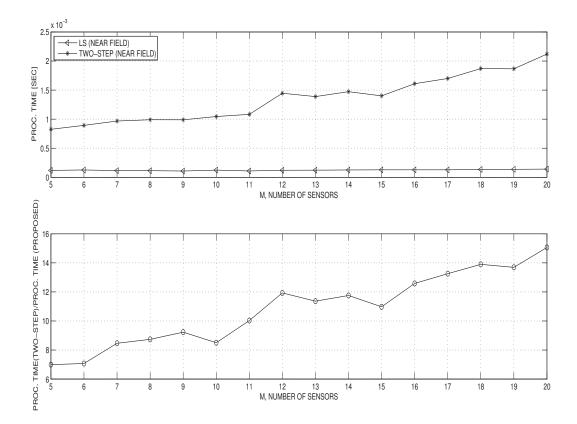
Fig. 6.   The total processing time of the proposed approach and the two-step approach (upper plot), and the ratio of the processing times (lower plot) versus the number of sensors in a circular array and a source in the near-field region.

projection matrix. Although the estimator does not always perform as well as the two-step estimator, the benefit is the reduction of the computational complexity by an order of the number of sensors.

APPENDIX A

EXPLICIT EXPRESSION OF $\tilde{\tilde{\mathbf{p}}}_s$

We derive the explicit expression of $\tilde{\tilde{\mathbf{p}}}_s$ as given in (23). We start by considering the first order approximation of $\hat{\tilde{\mathbf{u}}}$ and then the first order approximation of $\hat{\mathbf{Q}}$.

A. First order approximation of $\hat{\mathbf{u}}$

We approximate $\hat{\mathbf{u}}$ using a first order approximation, that is, $\hat{\mathbf{u}} = \bar{\mathbf{u}} + \tilde{\mathbf{u}}$. By substituting the noisy measurements vector $\hat{\mathbf{r}}$ given in (5) into (13), and neglecting terms that contain products of errors, we get that the first order error term of $\hat{\mathbf{u}}$ is given by,

$$\tilde{\mathbf{u}} \stackrel{\triangle}{=} \mathbf{R}\boldsymbol{\delta} \tag{32}$$

where we define the $2(M-1) \times 2(M-1)$ matrix,

$$\mathbf{R} \stackrel{\triangle}{=} - \begin{bmatrix} \mathrm{diag}(\mathbf{r}) & \mathbf{0}_{M-1}\mathbf{0}_{M-1}^T \\ \mathrm{diag}(\dot{\mathbf{r}}) & \mathrm{diag}(\mathbf{r}) \end{bmatrix} \tag{33}$$

and where $\mathrm{diag}(\mathbf{x})$ is a diagonal matrix with the elements of the vector $\mathbf{x}$ on the main diagonal.

B. First order approximation of $\hat{\mathbf{Q}}$

We approximate $\hat{\mathbf{Q}}$ using a first order approximation, that is, $\hat{\mathbf{Q}} = \mathbf{Q} + \tilde{\mathbf{Q}}$. We first start by expressing the noisy orthogonal projection matrix $\hat{\mathbf{P}}^\perp$ using a first order approximation, that is, $\hat{\mathbf{P}}^\perp = \mathbf{P}^\perp + \tilde{\mathbf{P}}^\perp$ (the explicit expression of the first order error term, $\tilde{\mathbf{P}}^\perp$, is given later). Substituting $\hat{\mathbf{P}}^\perp$ in (20) yields

$$\begin{aligned} \hat{\mathbf{Q}} &= (\mathbf{F}^T(\mathbf{P}^\perp + \tilde{\mathbf{P}}^\perp)\mathbf{F})^{-1}\mathbf{F}^T(\mathbf{P}^\perp + \tilde{\mathbf{P}}^\perp) \\ &= [(\mathbf{F}^T\mathbf{P}^\perp\mathbf{F})(\mathbf{I} + (\mathbf{F}^T\mathbf{P}^\perp\mathbf{F})^{-1}(\mathbf{F}^T\tilde{\mathbf{P}}^\perp\mathbf{F}))]^{-1}\mathbf{F}^T(\mathbf{P}^\perp + \tilde{\mathbf{P}}^\perp) \\ &\cong [\mathbf{I} - (\mathbf{F}^T\mathbf{P}^\perp\mathbf{F})^{-1}(\mathbf{F}^T\tilde{\mathbf{P}}^\perp\mathbf{F})](\mathbf{F}^T\mathbf{P}^\perp\mathbf{F})^{-1}\mathbf{F}^T(\mathbf{P}^\perp + \tilde{\mathbf{P}}^\perp) \\ &\cong \mathbf{Q} + (\mathbf{F}^T\mathbf{P}^\perp\mathbf{F})^{-1}\mathbf{F}^T\tilde{\mathbf{P}}^\perp - (\mathbf{F}^T\mathbf{P}^\perp\mathbf{F})^{-1}(\mathbf{F}^T\tilde{\mathbf{P}}^\perp\mathbf{F})(\mathbf{F}^T\mathbf{P}^\perp\mathbf{F})^{-1}\mathbf{F}^T\mathbf{P}^\perp \end{aligned} \tag{34}$$

where in the second passing we use the first order approximation $(\mathbf{I} + \mathbf{X})^{-1} \cong \mathbf{I} - \mathbf{X}$ given that $\mathbf{X} \ll \mathbf{I}$, and in the third passing we neglected terms that contain products of errors. Hence, the first order error term $\tilde{\mathbf{Q}}$ can be defined as

$$\tilde{\mathbf{Q}} \stackrel{\triangle}{=} (\mathbf{F}^T\mathbf{P}^\perp\mathbf{F})^{-1}\mathbf{F}^T\tilde{\mathbf{P}}^\perp - (\mathbf{F}^T\mathbf{P}^\perp\mathbf{F})^{-1}(\mathbf{F}^T\tilde{\mathbf{P}}^\perp\mathbf{F})(\mathbf{F}^T\mathbf{P}^\perp\mathbf{F})^{-1}\mathbf{F}^T\mathbf{P}^\perp \tag{35}$$

Notice that according to (22), in order to calculate $\tilde{\tilde{\mathbf{p}}}_s$ we need to multiply $\tilde{\mathbf{Q}}$ by $\mathbf{Hd}_{1,s}$. Recall that $\mathbf{P}^\perp \mathbf{H} = \mathbf{0}$. We thus conclude that we can neglect the second term in (35). By substituting (35) and (32) into (22) we get that

$$
\begin{aligned}
\tilde{\tilde{\mathbf{p}}}_s &= (\mathbf{F}^T\mathbf{P}^\perp\mathbf{F})^{-1}\mathbf{F}^T\tilde{\mathbf{P}}^\perp\mathbf{Hd}_{1,s} + \mathbf{QR}\boldsymbol{\delta} \\
&= (\mathbf{F}^T\mathbf{P}^\perp\mathbf{F})^{-1}\mathbf{F}^T(\tilde{\mathbf{P}}^\perp\mathbf{Hd}_{1,s} + \mathbf{P}^\perp\mathbf{R}\boldsymbol{\delta})
\end{aligned}
\tag{36}
$$

We now need to express the first order error term of $\mathbf{P}^\perp$, denoted by $\tilde{\mathbf{P}}^\perp$. By recalling the definition of $\mathbf{P}^\perp$ as given in (16), we start by expressing the matrix $\hat{\mathbf{H}}$ using a first order approximation, that is, $\hat{\mathbf{H}} = \mathbf{H} + \tilde{\mathbf{H}}$ (the explicit expression of $\tilde{\mathbf{H}}$ is presented later). Substituting this approximation in (16) (where we replace $\mathbf{H}$ by $\hat{\mathbf{H}}$) we get that

$$
\begin{aligned}
\hat{\mathbf{P}}^\perp &= \mathbf{I} - (\mathbf{H}+\tilde{\mathbf{H}})((\mathbf{H}+\tilde{\mathbf{H}})^T(\mathbf{H}+\tilde{\mathbf{H}}))^{-1}(\mathbf{H}+\tilde{\mathbf{H}})^T \\
&= \mathbf{I} - (\mathbf{H}+\tilde{\mathbf{H}})((\mathbf{H}^T\mathbf{H})(\mathbf{I}+(\mathbf{H}^T\mathbf{H})^{-1}(\tilde{\mathbf{H}}^T\mathbf{H}+\mathbf{H}^T\tilde{\mathbf{H}}))^{-1}(\mathbf{H}+\tilde{\mathbf{H}})^T \\
&\cong \mathbf{I} - [(\mathbf{H}+\tilde{\mathbf{H}})(\mathbf{I}-(\mathbf{H}^T\mathbf{H})^{-1}(\tilde{\mathbf{H}}^T\mathbf{H}+\mathbf{H}^T\tilde{\mathbf{H}}))](\mathbf{H}^T\mathbf{H})^{-1}(\mathbf{H}+\tilde{\mathbf{H}})^T \\
&\cong \mathbf{P}^\perp + \mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}(\tilde{\mathbf{H}}^T\mathbf{H}+\mathbf{H}^T\tilde{\mathbf{H}})(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T \\
&\quad -\mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\tilde{\mathbf{H}}^T - \tilde{\mathbf{H}}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T
\end{aligned}
\tag{37}
$$

where in the second passing we use the first order approximation $(\mathbf{I}+\mathbf{X})^{-1} \cong \mathbf{I}-\mathbf{X}$ given that $\mathbf{X} \ll \mathbf{I}$, and in the third passing we neglect terms that contain product of errors. Thus, we define the first order error term of $\tilde{\mathbf{P}}^\perp$ as

$$
\tilde{\mathbf{P}}^\perp \triangleq \mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}(\tilde{\mathbf{H}}^T\mathbf{H}+\mathbf{H}^T\tilde{\mathbf{H}})(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T - \mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\tilde{\mathbf{H}}^T - \tilde{\mathbf{H}}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T
\tag{38}
$$

Note that according to (38) we get that the product $\tilde{\mathbf{P}}^\perp\mathbf{Hd}_{1,s}$ which appears in (36) is given by

$$
\tilde{\mathbf{P}}^\perp\mathbf{Hd}_{1,s} = \mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\tilde{\mathbf{H}}\mathbf{d}_{1,s} - \tilde{\mathbf{H}}\mathbf{d}_{1,s} = -\mathbf{P}^\perp\tilde{\mathbf{H}}\mathbf{d}_{1,s}
\tag{39}
$$

Substituting (39) back into (36) results in

$$
\tilde{\tilde{\mathbf{p}}}_s = (\mathbf{F}^T\mathbf{P}^\perp\mathbf{F})^{-1}\mathbf{F}^T\mathbf{P}^\perp(\mathbf{R}\boldsymbol{\delta} - \tilde{\mathbf{H}}\mathbf{d}_{1,s})
\tag{40}
$$

Finally, we need to find an explicit expression for the first order error term $\tilde{\mathbf{H}}$. By substituting the noisy measurements vector $\hat{\tilde{\mathbf{r}}}$ given in (5) into (15) we obtain that the first order error term $\tilde{\mathbf{H}}$ is given by

$$\tilde{\mathbf{H}} \triangleq \begin{bmatrix} \boldsymbol{\epsilon} & \mathbf{0}_{M-1} \\ \boldsymbol{\xi} & \boldsymbol{\epsilon} \end{bmatrix} \tag{41}$$

Note that by using (41) we get that the product $\tilde{\mathbf{H}}\mathbf{d}_{1,s}$ is given by,

$$\tilde{\mathbf{H}}\mathbf{d}_{1,s} = \begin{bmatrix} d_{1,s}\mathbf{I} & \mathbf{0}_{M-1}\mathbf{0}_{M-1}^T \\ \dot{d}_{1,s}\mathbf{I} & d_{1,s}\mathbf{I} \end{bmatrix} \boldsymbol{\delta} \tag{42}$$

By substituting (42) into (40) we get the expression of $\tilde{\bar{\mathbf{p}}}_s$ given in (23). This concludes the appendix.

APPENDIX B

COMPLEXITIES OF THE PROPOSED ESTIMATOR AND THE TWO-STEP ESTIMATOR

We derive the computational complexities of the proposed method and the two-step method.

A. Proposed estimator

Note that according to (19) we need to compute: $\hat{\tilde{\mathbf{u}}}$, $\hat{\mathbf{Q}}$ and their product, in order to estimate the vector $\bar{\mathbf{p}}_s$. We now discuss each component separately.

1) Complexity of computing $\hat{\tilde{\mathbf{u}}}$: According to (7) and (9) we see that we need $M-1$ RMs to calculate $\hat{\mathbf{u}}$ and the same amount of RMs to calculate $\hat{\dot{\mathbf{u}}}$ (note that the norm of the sensor position is assumed to be known). Therefore, $\text{RM}(\hat{\tilde{\mathbf{u}}}) = 2(M-1)$.

2) Complexity of computing $\hat{\mathbf{Q}}$: The calculation of $\hat{\mathbf{Q}}$ involves several steps. We first need to calculate $\hat{\mathbf{P}}^\perp$ in (16). To compute $\hat{\mathbf{H}}^T\hat{\mathbf{H}}$ we need $4q(M-1)^2$ RMs, and to further compute its inverse we need 8 RMs. The product $\hat{\mathbf{H}}\left(\hat{\mathbf{H}}^T\hat{\mathbf{H}}\right)^{-1}$ involves $2q^2(M-1)$ RMs, and finally to multiply $\hat{\mathbf{H}}\left(\hat{\mathbf{H}}^T\hat{\mathbf{H}}\right)^{-1}$ by $\hat{\mathbf{H}}^T$ we need $4q(M-1)^2$ RMs. Therefore, to summarize, $\text{RM}(\hat{\mathbf{P}}^\perp) = 8 + 2q^2(M-1) + 8q(M-1)^2$. Given $\hat{\mathbf{P}}^\perp$ we calculate $\hat{\mathbf{Q}}$ according to (20). The product $\hat{\mathbf{P}}^\perp\mathbf{F}$ involves $8q(M-1)^2$ RMs. The product

	space dimensionality		
Step	$q = 2$	$q = 3$	Subsection
1. Compute $\hat{\mathbf{u}}$	$2(M - 1)$	$2(M - 1)$	B-A1
2. Compute $\hat{\mathbf{Q}}$	$32M^2 - 8M + 48$	$48M^2 - 4M + 180$	B-A2
3. Compute $\hat{\mathbf{Q}}\hat{\mathbf{u}}$	$8(M - 1)$	$12(M - 1)$	B-A3

TABLE I

COMPLEXITY OF THE PROPOSED ALGORITHM.

of $\mathbf{F}^T\hat{\mathbf{P}}^\perp$ by $\mathbf{F}$ requires $8q^2(M - 1)$ RMs. Performing the inverse $(\mathbf{F}^T\hat{\mathbf{P}}^\perp\mathbf{F})^{-1}$ involves $8q^3$ RMs. Multiplying this inverse with $\mathbf{F}^T\hat{\mathbf{P}}^\perp$ involves $8q(M - 1)$ RMs. To summarize, $\text{RM}(\hat{\mathbf{Q}}) = 8(1 + q^3) + (10q^2 + 8q)(M - 1) + 16q(M - 1)^2$.

3) Complexity of estimating $\bar{\mathbf{p}}_s$: Given $\hat{\mathbf{u}}$ and $\hat{\mathbf{Q}}$, the computation of $\hat{\bar{\mathbf{p}}}_s$ involves multiplying $\hat{\mathbf{u}}$ and $\hat{\mathbf{Q}}$. The complexity of this step is $\text{RM}(\hat{\mathbf{Q}}\hat{\mathbf{u}}) = 4q(M - 1)$.

The complexity of each component is summarized in Table I.

B. Two-step estimator

In Table II we detail the complexity of each step for a two-dimensional geometry ($q = 2$) and a three-dimensional geometry ($q = 3$). For notation simplicity we define: $C_{\theta_1}^{(2)} = 24M^2 + 108M + 84$, $C_{\theta_1}^{(3)} = 32M^2 + 208M + 272$, and $C_{w_1} = 24(M - 1)^3$. The complexity of each term in this table is detailed in the following subsections using the same vector and matrix notation used in [9].

1) Complexity of computing $\mathbf{W}_1$ [9, Eq. (11)] : Calculating $\mathbf{W}_1$ involves computing: i) $\mathbf{B}_1^{-1}$, ii) $\mathbf{B}_1^{-1}\mathbf{Q}^{-1}$; iii) $\mathbf{B}_1^{-1}\mathbf{Q}^{-1}\mathbf{B}_1^{-1}$, where each requires $8(M - 1)^3$ RMs. Summing i)-iii) involves $24(M - 1)^3$ RMs.

2) Complexity of computing $\boldsymbol{\theta}_1$ [9, Eq. (10)]: Calculating $\boldsymbol{\theta}_1$ involves computing: i) $\mathbf{G}_1^T\mathbf{W}_1$ ($8(M - 1)^2(q + 1)$ RMs) , ii) $\mathbf{G}_1^T\mathbf{W}_1\mathbf{G}_1$ ($8(M - 1)(q + 1)^2$ RMs); iii) $\left(\mathbf{G}_1^T\mathbf{W}_1\mathbf{G}_1\right)^{-1}$ ($8(q + 1)^3$ RMs); iv) $\left(\mathbf{G}_1^T\mathbf{W}_1\mathbf{G}_1\right)^{-1}\mathbf{G}_1^T\mathbf{W}_1$ ($8(M - 1)(q + 1)^2$ RMs); v) $\boldsymbol{\theta}_1 = \left(\mathbf{G}_1^T\mathbf{W}_1\mathbf{G}_1\right)^{-1}\mathbf{G}_1^T\mathbf{W}_1\mathbf{h}_1$ ($4(M - 1)(q +$

Step	Eq. in [9]	space dimensionality		Subsection
		$q = 2$	$q = 3$	
1. First step				
1.1 initialize $\mathbf{W}_1 = \mathbf{Q}^{-1}$	(32)	0	0	
1.1 calculate $\boldsymbol{\theta}_1$	(10)	$C_{\theta_1}^{(2)}$	$C_{\theta_1}^{(3)}$	B-B2
1.2 Near field (repeat twice)				
1.2.1 calculate $\mathbf{W}_1$	(11)	$C_{w_1} \times 2$	$C_{w_1} \times 2$	B-B1
1.2.2 calculate $\boldsymbol{\theta}_1$	(10)	$C_{\theta_1}^{(2)} \times 2$	$C_{\theta_1}^{(3)} \times 2$	B-B2
2. Second step				
2.1 compute $\text{cov}(\theta_1)$	(13)	0	0	
2.2 form $\mathbf{W}_2$	(19)	648	1536	B-B3
2.3 calculate $\boldsymbol{\theta}_2$	(18)	280	840	B-B4
2.4 calculate $\boldsymbol{\theta}$	(21)-(22)	4	6	
2.5 Near field (repeat twice)				
2.5.1 calculate $\mathbf{B}_2$	(37)	0	0	
2.5.2 calculate $\mathbf{W}_2$	(19)	648×2	1536×2	B-B3
2.5.3 calculate $\boldsymbol{\theta}_2$	(18)	280×2	840×2	B-B4
2.5.4 calculate $\mathbf{p}_s$ and $\dot{\mathbf{p}}_s$	(21)-(22)	4×2	6×2	

TABLE II

COMPLEXITY OF THE TWO-STEP METHOD [9, SECTION IV, P. 2458].

1)RMs). Summing i)-v) involves $24M^2 + 108M + 84$ RMs ($q = 2$), and $32M^2 + 208M + 272$ RMs ($q = 3$).

3) Complexity of computing $\mathbf{W}_2$ [9, Eq. (19)]: Calculating $\mathbf{W}_2$ involves computing: i) $\mathbf{B}_2^{-1}$, ii) $\mathbf{B}_2^{-1}\text{cov}(\theta_1)^{-1}$; iii) $\mathbf{B}_2^{-1}\text{cov}(\theta_1)^{-1}\mathbf{B}_2^{-1}$, where each requires $8(q+1)^3$ RMs. Summing i)-iii) involves 648 RMs ($q = 2$), and 1536 RMs ($q = 3$).

4) Complexity of computing $\boldsymbol{\theta}_2$ [9, Eq. (18)]: Calculating $\boldsymbol{\theta}_2$ involves computing: i) $\mathbf{G}_2^T\mathbf{W}_2$ (which does not require calculations and therefore this operation is represented by 0 RMs); ii) $\mathbf{G}_2^T\mathbf{W}_2\mathbf{G}$ ($8q^2(q+1)$ RMs); iii) $\left(\mathbf{G}_2^T\mathbf{W}_2\mathbf{G}_2\right)^{-1}$ ($8q^3$ RMs); iv) $\left(\mathbf{G}_2^T\mathbf{W}_2\mathbf{G}_2\right)^{-1}\mathbf{G}_2^T\mathbf{W}_2$ ($8q^2(q+1)$ RMs); v) $\boldsymbol{\theta}_2 = \left(\mathbf{G}_2^T\mathbf{W}_2\mathbf{G}_2\right)^{-1}\mathbf{G}_2^T\mathbf{W}_2\mathbf{h}_2$ ($4q(q+1)$ RMs). Summing i)-v) involves 280 RMs ($q = 2$), and 840

RMs ($q = 3$).

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