Power Spectrum Blind Sampling

Geert Leus and Dyonisius Dony Ariananda

***Abstract*—Power spectrum blind sampling (PSBS) consists of a sampling procedure and a reconstruction method that is capable of perfectly reconstructing the unknown power spectrum of a signal from the obtained samples. In this letter, we propose a solution to the PSBS problem based on a periodic sampling procedure and a simple least squares (LS) reconstruction method. For this PSBS technique, we derive the lowest possible average sampling rate, which is much lower than the Nyquist rate of the signal. Note the difference with spectrum blind sampling (SBS) where the goal is to perfectly reconstruct the spectrum and not the power spectrum of the signal, in which case sub-Nyquist rate sampling is only possible if the spectrum is sparse. In the current work, we can perform sub-Nyquist rate sampling without making any constraints on the power spectrum, because we try to reconstruct the power spectrum and not the spectrum. In many applications, such as spectrum sensing for cognitive radio, the power spectrum is of interest and estimating the spectrum is basically overkill.**

***Index Terms*—Cognitive radio, , compressive sampling, power spectrum estimation.**

I. Introduction

Spectrum estimation is a well-studied problem in the field of signal processing. Recently, it has gained renewed interest due to its importance in the field of cognitive radio networks. In such networks, radios opportunistically look for holes in the licensed spectrum, which can subsequently be exploited for setting up a communication link. In general, a wide spectral range has to be sensed, requiring power-hungry high-rate analog-to-digital converters (ADCs). That is why recent research has focused on reducing the requirements of the ADCs by exploiting specific properties of the licensed spectrum. Some features that are often considered are the sparsity of the spectrum, or of its derivative, the edge spectrum [1]–[4]. This allows one to reduce the sampling rate of the signal without sacrificing perfect reconstruction in the noiseless case. Popular ways to decrease the sampling rate are multi-coset sampling [5], [2], [3] or the modulated wideband converter [4], both of which are periodic sampling devices that can be casted into a compressive sampling framework. Reconstruction can be carried out using your favorite sparse reconstruction method, or even with more traditional methods, such as multiple signal classification (MUSIC), or the minimum variance distortionless response (MVDR) method [6]. The approaches developed in [2]–[4] are labeled spectrum blind sampling (SBS), where the goal is to enable minimum-rate sampling of the signal and reconstruction of the unknown spectrum from these samples, given the spectrum is sparse. From these works, it turns out that for most signals, the minimum average sampling rate is the same as if the sparse support of the spectrum would be known (as studied in [5]), and it is given by the Landau lower bound, which is equal to the Nyquist rate multiplied with the spectral occupancy. However, if all cases have to be covered, the minimum average sampling rate increases and it is given by the minimum of twice the Landau lower bound and the Nyquist rate [3].

All the above works are basically concerned with spectral estimation, but for the spectrum sensing application at hand, only the power spectral density, or power spectrum in short, is required. That is why we will focus on power spectrum blind sampling (PSBS) in this work, with as goal to enable minimum-rate sampling of the signal and reconstruction of the unknown power spectrum from these samples, and this without any sparsity constraints on the power spectrum. We again make use of a periodic sub-Nyquist rate sampler, like multi-coset sampling or the modulated wideband converter. Both of these samplers can be viewed as a bank of different branches, where each branch modulates the signal with a periodic waveform followed by a low-rate ADC. The key ingredient of this work is that we will make use of all the different cross spectral densities, or cross-spectra in short, between the outputs of the different branches in order to reconstruct the power spectrum of the original signal. We show that if we adopt a modulated wideband converter, then we can reduce the average sampling rate to a rate that is given by twice the Nyquist rate divided by the number of branches in the periodic sampler plus one. This can be a significant reduction even for a limited number of branches.

Note that some earlier attempts have been recorded to estimate the power spectrum of a signal with a reduced average sampling rate [7], [8], but those works do not fully exploit the complete knowledge of the different cross-spectra between the outputs of the branches in the periodic sampler. As a result, these methods again have to rely on the sparsity of the power spectrum in order to reduce the average sampling rate. Our method on the other hand can allow for a substantial reduction of the average sampling rate over the Nyquist rate, without putting any sparsity constraints on the power spectrum, which makes it very flexible and low cost.

II. Periodic Sampling

Consider a spectrum sensing application, where the task is to sense the power spectrum of a wide-sense stationary signal $x(t)$. We will assume here that $x(t)$ is real-valued, but it is easy to generalize this approach to complex-valued signals (e.g., the complex envelope of the observed real-valued signal). Further, assume $x(t)$ is bandlimited with double-sided bandwidth

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The authors are with the Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, 2628 CD Delft, The Netherlands (e-mail: g.j.t.leus@tudelft.nl; d.a.dyonisius@tudelft.nl).

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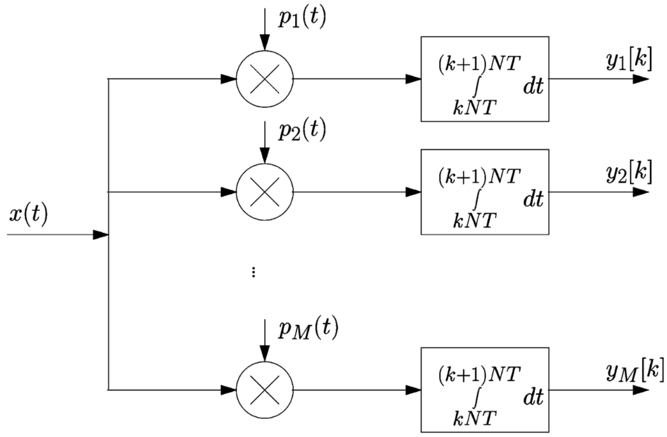

Fig. 1. Considered sampling device, consisting of M branches, where each branch modulates the signal with a real-valued periodic waveform followed by an integrate-and-dump process.

or Nyquist rate $1/T$. As illustrated in Fig. 1, we then apply a practical sampling device with M branches, where the ith branch modulates the signal $x(t)$ with a real-valued periodic waveform $p_i(t)$ of period NT followed by an integrate-and-dump device with period NT (thus with rate equal to $1/N$ times the Nyquist rate)[1]. As a result, the output of the ith branch at sampling index k can be written as

$$\begin{aligned} y_i[k] &= \int_{kNT}^{(k+1)NT} p_i(t)x(t)\,dt \\ &= \int_{kNT}^{(k+1)NT} c_i(t-kNT)x(t)\,dt \end{aligned} \tag{1}$$

where $c_i(t)$ yields one period of $p_i(t)$, i.e., $c_i(t) = p_i(t)$, for $0 \le t < NT$ and $c_i(t) = 0$, elsewhere. If we now assume that $c_i(t)$ is a piecewise constant function with constant values in every interval of length T, i.e., $c_i(t) = c_i[-n]$ for $nT \le t < (n+1)T$ with $n = 0, 1, \dots, N-1$, then we can rewrite (1) as

$$\begin{aligned} y_i[k] &= \sum_{n=0}^{N-1} c_i[-n] \int_{(kN+n)T}^{(kN+n+1)T} x(t)\,dt \\ &= \sum_{n=1-N}^{0} c_i[n]x[kN-n] \end{aligned} \tag{2}$$

where $x[n]$ can be interpreted as the output of an integrate-and-dump process with period T (thus with rate equal to the Nyquist rate) applied to $x(t)$, which is not explicitly carried out due to the high complexity. The average sampling rate of the proposed periodic sampler is equal to the Nyquist rate multiplied by M/N, and hence to save complexity we will assume here that $M \ll N$.

The considered sampler is similar to the modulated wideband converter of [4], where the $c_i[n]$ values are randomly selected, e.g., from a ± 1 distribution, although the latter sampler is a bit more general allowing for a difference in period between the real-valued periodic waveforms and the integrate-and-dump processes (taking them equal is often advantageous though [4]). Note that our sampling device can also be used to implement multi-coset sampling, simply by setting for every branch i, one different, yet random, entry of $c_i[n]$ to one and the others to zero, i.e., $c_i[n] = 1$ if $-n = n_i$ and $c_i[n] = 0$ if $-n \ne n_i$, where $n_i \ne n_j$ whenever $i \ne j$.

[1]Also other types of filters than the integrate-and-dump device are possible [4].

An important observation that will turn out to be useful later on, is that (2) can be viewed as a digital filtering operation of $x[n]$ by the filter $c_i[n]$ of length N followed by a downsampling operation with factor N, i.e., $y_i[k] = z_i[kN]$, where

$$z_i[n] = c_i[n] \star x[n] = \sum_{m=1-N}^{0} c_i[m]x[n-m]$$

with $\star$ denoting the convolution operator.

The goal of the considered PSBS problem now is to use the obtained samples to estimate the power spectrum of $x(t)$, which is equivalent to estimating the related power spectrum of $x[n]$. Note that the power spectrum or power spectral density of $x[n]$ is defined as

$$P_x(\omega) = \sum_{n=-\infty}^{\infty} r_x[n]e^{-jn\omega},\ 0 \le \omega < 2\pi$$

where $r_x[n]$ is the autocorrelation function of $x[n]$, given by $r_x[n] = E\{x[m]x[m-n]\}$. Hence, reconstructing the power spectrum $P_x(\omega)$ is equivalent to reconstructing the autocorrelation function $r_x[n]$. The original contribution of this work is that we will exploit all the $(M+1)M/2$ different cross-spectra of $y_i[k]$ with $y_j[k]$ for $i \le j$. Note that the cross-spectrum or cross spectral density of $y_i[k]$ with $y_j[k]$ is defined as

$$P_{y_i,y_j}(\omega) = \sum_{k=-\infty}^{\infty} r_{y_i,y_j}[k]e^{-jk\omega}, \quad 0 \le \omega < 2\pi$$

where $r_{y_i,y_j}[k]$ is the cross-correlation function of $y_i[k]$ with $y_j[k]$, given by $r_{y_i,y_j}[k] = E\{y_i[l]y_j[l-k]\}$. Hence, having prior knowledge about the $(M+1)M/2$ different cross-spectra $P_{y_i,y_j}(\omega)$ for $i \le j$, is the same as knowing the $(M+1)M/2$ different cross-correlation functions $r_{y_i,y_j}[k]$ for $i \le j$.

III. Reconstruction Approach

In this section, we will describe a method to reconstruct $r_x[n]$ given $r_{y_i,y_j}[k]$ for $i \le j$. In the next section, we will then discuss what this method can tell us about the minimum required average sampling rate for perfect reconstruction.

Since $y_i[k] = z_i[kN]$, we can write the cross-correlation function of $y_i[k]$ with $y_j[k]$ as the N-fold downsampled version of the cross-correlation function of $z_i[n]$ with $z_j[n]$:

$$\begin{aligned} r_{y_i,y_j}[k] &= E\{y_i[l]y_j[l-k]\} \\ &= E\{z_i[lN]z_j[(l-k)N]\} = r_{z_i,z_j}[kN]. \end{aligned} \tag{3}$$

It is well-known that $r_{z_i,z_j}[n]$ can be written as

$$\begin{aligned} r_{z_i,z_j}[n] &= r_{c_i,c_j}[n] \star r_x[n] \\ &= \sum_{m=-N+1}^{N-1} r_{c_i,c_j}[m]r_x[n-m] \end{aligned} \tag{4}$$

where $r_{c_i,c_j}[n]$ is the "deterministic" cross-correlation function between $c_i[n]$ and $c_j[n]$:

$$r_{c_i,c_j}[n] = c_i[n] \star c_j[-n] = \sum_{m=1-N}^{0} c_i[m]c_j[m-n].$$

From (3) and (4), we can thus write

$$r_{y_i,y_j}[k] = r_{z_i,z_j}[kN] = \sum_{m=-N+1}^{N-1} r_{c_i,c_j}[m] r_x[kN-m] = \sum_{l=0}^{1} \mathbf{r}_{c_i,c_j}^T[l]\mathbf{r}_x[k-l] \quad (5)$$

where we have

$$\mathbf{r}_{c_i,c_j}[k] = [r_{c_i,c_j}[kN], r_{c_i,c_j}[kN-1], \ldots, r_{c_i,c_j}[(k-1)N+1]]^T,$$
$$\mathbf{r}_x[k] = [r_x[kN], r_x[kN+1], \ldots, r_x[(k+1)N-1]]^T.$$

Stacking the $(M+1)M/2$ different cross-correlation functions $r_{y_i,y_j}[k]$, i.e., $\mathbf{r}_y[k] = [\ldots, r_{y_i,y_j}[k], \ldots]^T$, for $i,j = 1,\ldots,M$ with $i \le j$, we finally obtain

$$\mathbf{r}_y[k] = \sum_{l=0}^{1} \mathbf{R}_c[l]\mathbf{r}_x[k-l] \quad (6)$$

where $\mathbf{R}_c[k]$ is the $(M+1)M/2 \times N$ matrix given by $\mathbf{R}_c[k] = [\ldots, \mathbf{r}_{c_i,c_j}[k], \ldots]^T$, for $i,j = 1,\ldots,M$ with $i \le j$.

Assuming $\mathbf{r}_y[k]$ has a support limited to $-L \le k \le L$, the support of $r_x[n]$ should be limited to $-LN \le n \le LN$, which means that the support of $\mathbf{r}_x[k]$ should be limited to $-L \le k \le L$. As a result, all the information can be gathered into the vectors

$$\mathbf{r}_y = \left[\mathbf{r}_y^T[0], \mathbf{r}_y^T[1], \ldots, \mathbf{r}_y^T[L], \mathbf{r}_y^T[-L], \ldots, \mathbf{r}_y^T[-1]\right]^T, \quad (7)$$
$$\mathbf{r}_x = \left[\mathbf{r}_x^T[0], \mathbf{r}_x^T[1], \ldots, \mathbf{r}_x^T[L], \mathbf{r}_x^T[-L], \ldots, \mathbf{r}_x^T[-1]\right]^T. \quad (8)$$

From (6), and the fact that the first column of $\mathbf{R}_c[1]$ as well as the last $N-1$ entries of $\mathbf{r}_x[L]$ are zero, the relation between $\mathbf{r}_y$ and $\mathbf{r}_x$ can finally be expressed as

$$\mathbf{r}_y = \mathbf{R}_c\mathbf{r}_x \quad (9)$$

where $\mathbf{R}_c$ is the $(2L+1)(M+1)M/2 \times (2L+1)N$ matrix given by

$$\mathbf{R}_c = \begin{bmatrix} \mathbf{R}_c[0] & & & & \mathbf{R}_c[1] \\ \mathbf{R}_c[1] & \mathbf{R}_c[0] & & & \\ & \mathbf{R}_c[1] & \mathbf{R}_c[0] & & \\ & & \ddots & \ddots & \\ & & & \mathbf{R}_c[1] & \mathbf{R}_c[0] \end{bmatrix}.$$

Assuming that $\mathbf{R}_c$ has full column rank, we can solve (9) using LS.

However, the above computation can be simplified by exploiting the fact that $\mathbf{R}_c$ is a block circulant matrix with blocks of size $(M+1)M/2 \times N$, which can easily be turned into a block diagonal matrix with again blocks of size $(M+1)M/2 \times N$. More specifically, defining

$$\mathbf{Q}_c(\omega) = \sum_{k=0}^{1} \mathbf{R}_c[k]e^{-jk\omega}, \quad (10)$$
$$\mathbf{q}_y(\omega) = \sum_{k=-L}^{L} \mathbf{r}_y[k]e^{-jk\omega}, \quad (11)$$
$$\mathbf{q}_x(\omega) = \sum_{k=-L}^{L} \mathbf{r}_x[k]e^{-jk\omega} \quad (12)$$

we can rewrite (9) as a set of $2L+1$ matrix equations:

$$\mathbf{q}_y\left(2\pi\frac{l}{2L+1}\right) = \mathbf{Q}_c\left(2\pi\frac{l}{2L+1}\right)\mathbf{q}_x\left(2\pi\frac{l}{2L+1}\right), \quad l = 0,1,\ldots,2L. \quad (13)$$

Assuming $\mathbf{Q}_c(2\pi l/(2L+1))$ has full column rank for $l = 0,1,\ldots,2L$, we can solve (13) for $l = 0,1,\ldots,2L$ using LS.

IV. Discussion

We can solve (9) or (13) if the related system matrices have full column rank. For random modulating waveforms, as used in the modulated wideband converter, this occurs with high probability as soon as $(M+1)M/2 \ge N$, which can happen for M much smaller than N. For multi-coset sampling, on the other hand, the situation is a bit different. Assuming that $c_i[-n]$ has a one in the position n_i and zeros elsewhere, then $r_{c_i,c_j}[n]$ will have a one in position $n_{i,j} = n_i - n_j$ and zeros elsewhere. As a result, every row of $\mathbf{R}_c$ has a single one and thus $\mathbf{R}_c$ will have full column rank if it has a one in every column. This condition is actually related to the design condition of so-called minimal sparse rulers and the results of such a multi-coset sampler construction will be reported elsewhere due to space limitations. Suffice it to say that the condition for multi-coset sampling is not as beneficial as the one for the modulated wideband converter, and thus from that point of view, we would like to advocate the latter for PSBS.

From the condition for the modulated wideband converter, we see that the compression ratio M/N is lower bounded by $2/(M+1)$. So the minimal compression ratio is only determined by the number of branches that we can afford in our periodic sampler. The higher the number of branches, the more we can reduce the average sampling rate. Note though that at this optimal point where $M/N \approx 2/(M+1)$, there is a hardware tradeoff between the number of branches M, and the rate $1/(NT)$ of the ADCs. The higher the number of branches M, the lower the rate $1/(NT)$ of the ADCs, which are two effects that oppositely affect the hardware complexity. Hence, assuming $M/N \approx 2/(M+1)$, there will be some optimal compression ratio in terms of hardware complexity, satisfying $0 < M/N < \infty$. Finally, note that in practice, the cross-correlation functions have to be computed based on a finite-length sensing period, so reducing the rate $1/(NT)$ of the ADCs will introduce an additional estimation error. This is an issue that has

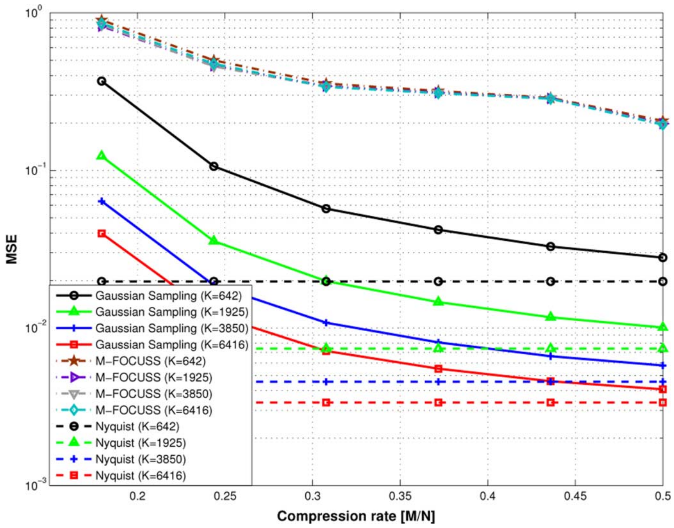

Fig. 2. MSE between the PSBS-based power spectrum and the theoretical one.

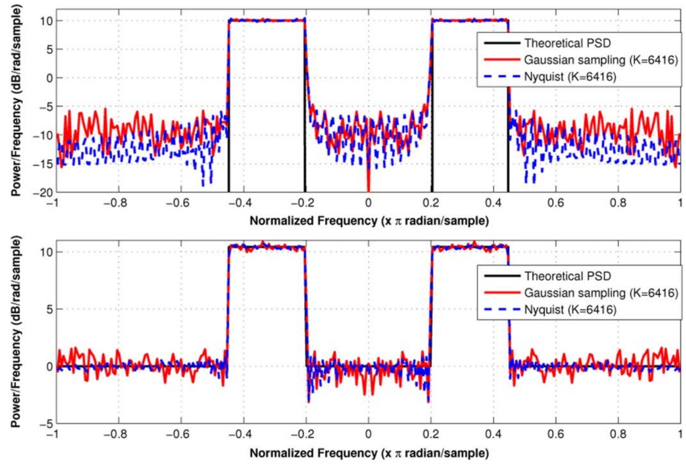

Fig. 3. Power spectrum estimated based on PSBS with $M/N = 0.5$ is plotted together with the theoretical and Nyquist-rate based power spectrum; top: noise-free; bottom: noisy.

not been touched upon in this work, and we will divert this to future research.

V. Numerical Example

In this section, we illustrate our approach with a numerical example. To simplify the presentation, we directly generate a discrete-time real-valued wide-sense stationary signal $x[n]$ and we assume that it is a bandpass signal with a frequency support between 0.2π and 0.45π. We set N to $N = 78$ and we assume that L is limited to $L = 2$ (above $L = 2$, the correlations are very small). In Fig. 2, we compute the normalized mean squared error (MSE) between the estimated power spectrum and the theoretical one having a perfectly rectangular shape. The signal power is set to 10 dB and no noise is considered. As a benchmark, we also present the normalized MSE between the estimated power spectrum produced by Nyquist-rate sampling and the theoretical one. This Nyquist-rate based power spectrum is obtained from our approach by setting $M = N$ and $c_i[-n] = \delta[n-i+1]$. It is clear from the figure that the quality of the estimation tends to increase with M/N and it slowly converges towards that of the Nyquist rate. We further notice that as the number of measurement vectors, denoted by K, increases, the MSE improves, and this is to be expected as our estimated cross-spectra get closer to the true ones. We also check what happens if we replace the true Nyquist-rate samples by the Nyquist-rate samples reconstructed from the measurements using M-FOCUSS [9] with $p = 0.3$, thereby exploiting the joint sparsity in the frequency domain of K snapshots of $N = 78$ Nyquist-rate samples. We observe that this approach does not work very well, even for a large K, which is most likely due to the fact that the snapshot length N is not very large.

In Fig. 3, the estimated power spectrum based on PSBS (with a compression ratio of $M/N = 0.5$) is depicted together with the theoretical and the Nyquist-rate based power spectrum for both the noiseless and noisy case (noise power is 0 dB). While there is an obvious degradation due to the reduction of the sampling rate, our PSBS method is clearly able to locate the presence of the occupied band making it a promising candidate for power spectrum sensing in a cognitive radio environment.

VI. Conclusion

In this letter, we have introduced the concept of power spectrum blind sampling (PSBS), a novel approach to estimate the power spectrum of a signal based on samples obtained from a periodic sampler that samples below the Nyquist rate. The presented solution consists of a set of simple LS problems that can easily be solved provided that some rank conditions are satisfied. If the considered periodic sampler is constructed using random periodic modulating waveforms, we have finally shown that the minimum average sampling rate for perfect reconstruction is given by twice the Nyquist rate divided by the number of branches in the periodic sampler plus one.

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