Static Field Estimation Using a Wireless Sensor Network Based on the Finite Element Method

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***Abstract*—In this paper, we propose a novel framework for field estimation in a wireless sensor network (WSN). The fundamental problem of estimating field values at locations where no WSN measurements are available is tackled by including a physical field model in the form of a partial differential equation (PDE). If the PDE is discretized in the spatiotemporal domain by use of the finite element method (FEM), then the physical field model reduces to a set of linear equations that can be elegantly combined with the WSN field measurements in a constrained optimization problem. In contrast to existing approaches, we do not require the driving source function or the locations of point sources to be known. Instead, we assume limited prior knowledge on the nature of the field and/or source functions, such as a sparsity or nonnegativity prior, for obtaining a unique solution of the otherwise underdetermined problem of joint field and source estimation. Within the proposed framework, we derive a cooperative estimation algorithm for static 2-D fields governed by a Poisson PDE. Simulation results illustrate that a significant improvement in field estimation accuracy can be obtained, compared to the cases when only WSN measurements (without a physical model) or only the FEM (without WSN measurements) are used.**

I. Introduction

Many physical phenomena are understood to be governed by a partial differential equation (PDE) that relates the spatiotemporal variation of a field to the underlying driving source function. The problem of field estimation is hence equivalent to the problem of solving a PDE subject to certain initial and/or boundary conditions. A particularly popular numerical method for solving such initial/boundary value problems is the finite element method (FEM), which has been extensively covered in literature, see, e.g., [1], [2]. In a nutshell, the FEM approximates the infinite-dimensional field in a finite-dimensional subspace, obtained by a discretization of the spatiotemporal domain. By enforcing the approximation error to be orthogonal to this subspace, the FEM reduces the boundary value problem to a square system of linear equations. Moreover, by choosing a subspace basis with small spatiotemporal support, the system of equations exhibits a high degree of sparsity and hence can be efficiently solved. There are, however, a number of drawbacks when considering the FEM for field estimation. First of all, the right-hand side of the system of equations consists of a subspace approximation of the source function, which makes the FEM unsuitable for problems with limited or no knowledge about the driving source. Second, the FEM approximation accuracy is linearly related to the resolution of the mesh (that is, the domain discretization) [1, Th. 1.10], which implies that a high accuracy can only be attained at the cost of a high dimensionality.

The recent advent of wireless sensor networks (WSNs) offers a significantly different yet attractive approach to field estimation. Indeed, the dense deployment of sensor nodes inside a spatially distributed field makes it possible to collect a large number of local field estimates which can then be gathered in a fusion center for global field reconstruction. However, a fundamental issue with this approach is how to estimate field values at locations different from the WSN sensor node locations. A naive approach would be to interpolate the field values estimated at the sensor node locations to obtain field values at arbitrary locations, yet the choice of the interpolant will then carry an implicit assumption on the spatiotemporal variation of the field. A more rigorous approach is to combine the WSN field measurements with a PDE-based field model. In [3] and [4], the field estimation problem is recast in a dynamic state estimation problem, in which the state equation is derived from a discretization of the PDE (using the FEM [3] or the finite difference method (FDM) [4]) while only a subset of the states (i.e., the field values at the sensor node locations) is propagated to the measurement equation. However, these methods still require significant knowledge about the driving source: in [3] it is assumed that a noisy observation of the (continuous) source function is available, while in [4] the source function is assumed to be composed solely of point source contributions at locations where sensor nodes have been deployed. A different yet related problem that has recently been considered concerns the estimation of the (initial) driving source function from WSN field measurements. This inverse problem has been tackled in [5]–[7] for the case of a source function composed of one or more point sources, by fitting the field measurements to a spatiotemporal discretization of the analytical PDE solution.

In this paper, we propose a novel framework for field estimation based on the combination of WSN field measurements with a physical model in the form of a PDE. We formulate the field estimation problem as a constrained optimization problem, in which the constraints originate from a finite element model of the PDE and its initial and/or boundary conditions, and the objective minimizes the misfit between the estimated and measured field values at the sensor node locations. In contrast to the approaches in [3], [4], we do not assume the driving source function or the locations of point sources to be known, however, the price we pay is that the optimization problem is highly underdetermined. To alleviate this deficiency, we propose to include additional constraints and/or regularization terms in the optimization problem, which express any prior knowledge that may be available on the nature of the field and/or source functions, such as sparsity or nonnegativity.

The proposed framework is derived here for one particular type of PDE, namely the two-dimensional (2-D) Poisson equation, which has applications in gravitation, electrostatics, fluid mechanics, and thermostatics, to name just a few. This restriction implies that we discard the time variable and only consider the estimation of static 2-D fields. We should stress, however, that the framework is believed to be suitable also for the estimation of dynamic and 3-D fields, yet this extension is postponed to future work. In addition, the field estimation algorithms presented in this paper are cooperative algorithms, in the sense that all processing is performed in a fusion center (FC) which

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gathers the available WSN field measurements. It will become clear, however, that the structure of the constrained optimization problem under consideration allows for an efficient distributed implementation due to the sparsity of the finite element model.

II. Problem statement

Consider the 2-D Poisson PDE (with $\nabla = [\partial/\partial x, \partial/\partial y]$)

$$-\nabla^2 u(x,y) = s(x,y) \tag{1}$$

where the source function $s(x,y)$ and the field function $u(x,y)$ are infinite-dimensional functions of the spatial variables (x,y) defined on a 2-D domain $\Omega \subset \mathbb{R}^2$. The field $u(x,y)$ is measured using a WSN with sensor nodes at J discrete locations $(x_j, y_j),\ j = 1,\ldots,J$. Each of the J sensor nodes provides N field measurements (with $\mathbf{1}_{N\times 1} = [1\ldots 1]^T$),

$$\mathbf{v}_j = \begin{bmatrix} v_j^{(1)} \\ \vdots \\ v_j^{(N)} \end{bmatrix} = u(x_j, y_j)\mathbf{1}_{N\times 1} + \begin{bmatrix} w_j^{(1)} \\ \vdots \\ w_j^{(N)} \end{bmatrix},\ j = 1,\ldots,J \tag{2}$$

which are obtained by sensing the field at successive time instants and/or equipping the WSN nodes with multiple sensors. The measurement noise $w_j^{(n)}$ at the jth sensor node is assumed to be i.i.d. with variance σ_j^2, and independent of the measurement noise at other sensor nodes. Additionally, the field may be subject to boundary conditions of the Dirichlet or Neumann type. Since the boundary conditions appear as additional (and known) terms on the right-hand side of the FEM system of equations [1, Ch. 1], we can assume zero boundary conditions without loss of generality to simplify notation.

Our aim is to estimate the field $u(x,y)$ at $J+P$ distinct locations, without assuming knowledge of the source function $s(x,y)$. These locations include the J sensor node locations $(x_j, y_j),\ j = 1,\ldots,J$ as well as the locations $(x_{J+p}, y_{J+p}),\ p = 1,\ldots,P$ of P points of interest (POIs) at which no sensor nodes have been deployed.

III. FEM for Poisson-type boundary value problems

A. Derivation of the Galerkin equations

The FEM involves the approximation of the infinite-dimensional field and source functions $u(x,y)$ and $s(x,y)$ in a finite-dimensional subspace, i.e.,

$$\tilde{u}(x,y) = \sum_{k=1}^{K_\Omega} u_k \phi_k(x,y), \quad \tilde{s}(x,y) = \sum_{k=1}^{K_\Omega} s_k \phi_k(x,y). \tag{3}$$

Here, K_Ω denotes the subspace order, $\phi_k(x,y), k = 1,\ldots,K_\Omega$ is a basis for the subspace, and $\{u_k, s_k\}, k = 1,\ldots,K_\Omega$ represent the basis expansion coefficients. Instead of directly substituting the above subspace approximations in the PDE in (1), the boundary value problem related to (1) is first transformed into its weak formulation. A weak solution of the above boundary value problem is a solution that obeys the boundary conditions and moreover satisfies

$$\int_\Omega \left[\nabla^2 u(x,y) + s(x,y)\right] g(x,y)dxdy = 0 \tag{4}$$

for an appropriate set of so-called test functions $g(x,y)$. By applying integration by parts and making use of the assumption of zero boundary conditions, (4) can be rewritten as follows,

$$\int_\Omega \nabla u(x,y) \cdot \nabla g(x,y)dxdy = \int_\Omega s(x,y)g(x,y)dxdy \tag{5}$$

where $\cdot$ denotes the dot product. The main motivation for considering the weak formulation (5) instead of the original boundary value problem is that the differentiability requirements on the subspace basis functions can be relaxed from second-order to first-order differentiability, which in particular allows the use of piecewise linear basis functions (see Section III-B).

The so-called Galerkin equations are obtained by enforcing the field approximation error to be orthogonal to the chosen subspace, which is equivalent to evaluating the weak formulation with the test function equal to each of the subspace basis functions. This results in a square system of linear equations,

$$\mathbf{Au} = \mathbf{Bs} \tag{6}$$

where the so-called stiffness and mass matrices are defined by

$$[\mathbf{A}]_{ij} = \int_\Omega \nabla\phi_j(x,y) \cdot \nabla\phi_i(x,y)dxdy \tag{7}$$

$$[\mathbf{B}]_{ij} = \int_\Omega \phi_j(x,y)\phi_i(x,y)dxdy \tag{8}$$

and the field and source vectors contain the corresponding basis expansion coefficients, i.e.,

$$\mathbf{u} = \left[u_1 \ldots u_{K_\Omega}\right], \quad \mathbf{s} = \left[s_1 \ldots s_{K_\Omega}\right]. \tag{9}$$

For a more profound treatment of the FEM, we refer to [1],[2].

B. Implementation issues

The practical implementation of the FEM requires the determination of three ingredients that are intimately related: nodes, elements, and basis functions. FEM nodes (not to be confused with sensor nodes) are points inside the domain Ω and on its boundary $\partial\Omega$ that are used in the definition of the basis functions. The elements are relatively small subdivisions of the domain Ω whose size and geometry is determined by the node locations (e.g., such that nodes coincide with element vertices or lie in the center of gravity of an element). The FEM basis functions are usually chosen to be piecewise polynomial functions possessing two particular properties. First of all, by ensuring that $\phi_i(x_k, y_k) = \delta(i-k),\ i = 1,\ldots,K_\Omega$ at all FEM node locations $(x_k, y_k),\ k = 1,\ldots,K_\Omega$, the basis expansion coefficients in the FEM subspace approximation are equal to the field/source values at these locations, i.e., $u_k = u(x_k, y_k),\ s_k = s(x_k, y_k)$. The FEM thus provides a spatial sampling of the field and source functions. Second, the basis functions are typically chosen to have small spatial support, in the sense that the kth basis function is non-zero only in the area covered by the elements containing the kth node. Consequently, the stiffness and mass matrices defined in (7) and (8) have a highly sparse structure, which is desirable in terms of computational efficiency and moreover facilitates a distributed implementation for solving the Galerkin equations (6). Different choices for these three ingredients are extensively discussed in [2, Ch. 3]. We will resort to the simplest combination, often referred to as the linear Lagrange element [2, Ch. 3] or the P1 element [1, Ch. 1], which consists of triangular elements with vertices at the node positions and piecewise linear "tent-shaped" basis functions.

It is important to point out that the FEM node locations cannot be chosen arbitrarily. In order to obtain a well-conditioned system of equations in (6), we need to define a high-quality "mesh" (which refers to the joint collection of nodes and triangles). FEM mesh generation software will typically attempt to produce uniformly distributed nodes with a density that varies according to the source function, and elements that maximally resemble equilateral triangles. However, for the field estimation problem considered in this paper, slightly different mesh properties are required. First of all, we need a mesh generator that does not require the specification of a source function, which is indeed considered unknown in our problem statement. Second, we must enforce the FEM node locations to include the sensor node and POI locations, which is crucial for allowing field estimation at these particular locations. The mesh generation algorithm developed in [8] possesses both these properties and will be used in the sequel.

Once the FEM mesh has been generated, the stiffness and mass matrices can be efficiently calculated as outlined in [1, Ch. 1]. A final step in the FEM implementation consists in reducing the dimension of the Galerkin system of equations from K_Ω to K, which corresponds to the number of interior nodes in the mesh. Since the coefficients in $\mathbf{u}$ and $\mathbf{s}$ corresponding to the FEM nodes on the domain boundary are known to be zero, the corresponding columns in $\mathbf{A}$ and $\mathbf{B}$ can be deleted. The squareness of the Galerkin system can then be restored by also removing the corresponding basis functions from the set of test functions, which comes down to deleting the appropriate rows in $\mathbf{A}$ and $\mathbf{B}$ [1, Ch. 1].

IV. FEM-CONSTRAINED COOPERATIVE FIELD ESTIMATION

We can now combine the WSN measurement model in (2) and the FEM system of equations in (6) into a single estimation problem. A rather straightforward approach is to minimize the sum of squared WSN measurement errors subject to the Galerkin equations,

$$\min_{\mathbf{u},\mathbf{s}} \quad \sum_{j=1}^{J} \|\mathbf{v}_j - u_j \mathbf{1}_{N\times 1}\|_2^2 \quad \text{s. t.} \quad \mathbf{Au} = \mathbf{Bs} \tag{10}$$

However, this approach is not capable of producing an accurate field estimate at the POIs. This can be understood by examining the Karush-Kuhn-Tucker (KKT) optimality conditions for the problem in (10), which can be reduced to

$$\mathbf{u}_s = \frac{1}{N}\begin{bmatrix}\mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_J^T\end{bmatrix} \mathbf{1}_{N\times 1} \text{ and } \begin{bmatrix}-\mathbf{A}_n \; \mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{u}_n \\ \mathbf{s}\end{bmatrix} = \mathbf{A}_s \mathbf{u}_s \tag{11}$$

where $\mathbf{u}^T = [\mathbf{u}_s^T \; \mathbf{u}_n^T]$ and $\mathbf{A} = [\mathbf{A}_s \; \mathbf{A}_n]$ have been partitioned such as to separate columns related to sensor nodes and non-sensor nodes. The field estimation at the sensor node locations consists of a simple measurement averaging, while the field estimation at the non-sensor nodes (including the POIs) requires the solution of an underdetermined system of equations.

The underdetermined nature requires the inclusion of additional objective functions or constraints in the optimization problem (10). In many signal processing applications, it makes sense to assume that the source function $s(x, y)$ is composed solely of point source contributions (which is indeed an assumption that is also exploited in [4]-[7]). This assumption naturally leads to the inclusion of a sparsity-inducing regularization term on the source vector $\mathbf{s}$, since the dimension K of the Galerkin system of equations will typically be much larger than the number of point sources. Moreover, a static point source is inherently positive-valued (otherwise it would be a sink), and consequently the Poisson PDE generates a nonnegative field if the boundary conditions are also nonnegative. Appending a sparsity-inducing regularization term and appropriate nonnegativity constraints to the optimization problem (10), results in the following constrained optimization problem,

$$\min_{\mathbf{u},\mathbf{s}} \quad \sum_{j=1}^{J} \|\mathbf{v}_j - u_j \mathbf{1}_{N\times 1}\|_2^2 + \lambda \|\mathbf{s}\|_1 \tag{12}$$

$$\text{s. t.} \quad \mathbf{Au} = \mathbf{Bs}, \;\; \mathbf{u} \geq \mathbf{0}_{K\times 1}, \;\; \mathbf{s} \geq \mathbf{0}_{K\times 1} \tag{13}$$

This is a convex problem, which can be readily solved using convex optimization software such as CVX [9].

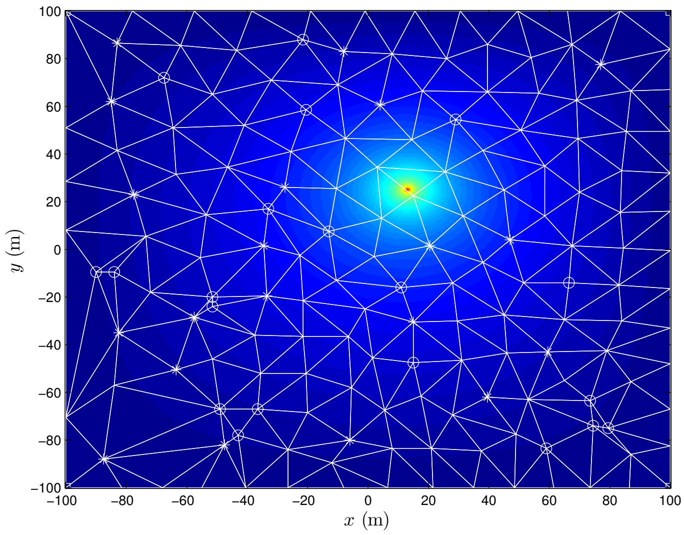

Fig. 1. Contour plot of the simulated field showing a FEM mesh generated for one realization of the random sensor node and POI deployment. A fixed subset of the FEM nodes consists of WSN sensor nodes (o), POIs (∗), and domain corners (□).

V. SIMULATION RESULTS

We simulate a static 2-D field governed by the Poisson PDE on a square domain of 200 x 200 m, with zero boundary conditions. The field is driven by a single point source with coordinates $(13, 25)$ m and unit amplitude. A WSN with $J = 20$ sensor nodes is randomly deployed in a square area of 180 x 180 m, maintaining a margin of 20 m to the domain boundary with the aim of avoiding ill-shaped boundary elements. Similarly, $P = 20$ POIs are randomly chosen in the area where the WSN sensor nodes are located. Each WSN sensor node provides $N = 10$ field measurements, corrupted by i.i.d. measurement noise with a variance that yields a local 0 dB signal-to-noise ratio (SNR). The mesh generation algorithm [8] is initialized by appending a set of equally spaced nodes at a mutual distance of $h_0 = 20$ m to the fixed subset of FEM nodes consisting of the sensor nodes, POIs, and domain corners. The resulting mesh is shown on a contour plot of the simulated field in Fig. 1.

The FEM-constrained cooperative field estimation (FCE) algorithm proposed in Section IV is evaluated with only a nonnegativity constraint (FCE-NN, $\lambda = 0$) and with both a nonnegativity constraint and a sparsity-inducing regularization term (FCE-ℓ1-NN, $\lambda = 1$). Two benchmark algorithms are also evaluated for comparison with the proposed algorithm: a FEM with *known* source vector that does not employ WSN measurements, and a measurement averaging and interpolation (MAI) method that produces local field estimates by measurement averaging at the WSN sensor nodes and linear interpolation at the POIs. The algorithms are compared in terms of the mean squared relative field estimation error (MSE) at the sensor nodes and at the POIs, which is calculated by averaging the squared relative error over $N_{\text{MC}} = 100$ Monte Carlo trials,

$$\text{MSE (sensors)} = \sum_{i=1}^{N_{\text{MC}}} \sum_{j=1}^{J} \left(\frac{u(x_j^{(i)}, y_j^{(i)}) - u_j^{(i)}}{u(x_j^{(i)}, y_j^{(i)})} \right)^2 \tag{14}$$

$$\text{MSE (POIs)} = \sum_{i=1}^{N_{\text{MC}}} \sum_{p=1}^{P} \left(\frac{u(x_{J+p}^{(i)}, y_{J+p}^{(i)}) - u_{J+p}^{(i)}}{u(x_{J+p}^{(i)}, y_{J+p}^{(i)})} \right)^2 . \tag{15}$$

Fig. 2 diplays the MSE behavior when one of the simulation parameters (SNR, initial FEM node distance h_0, or number of

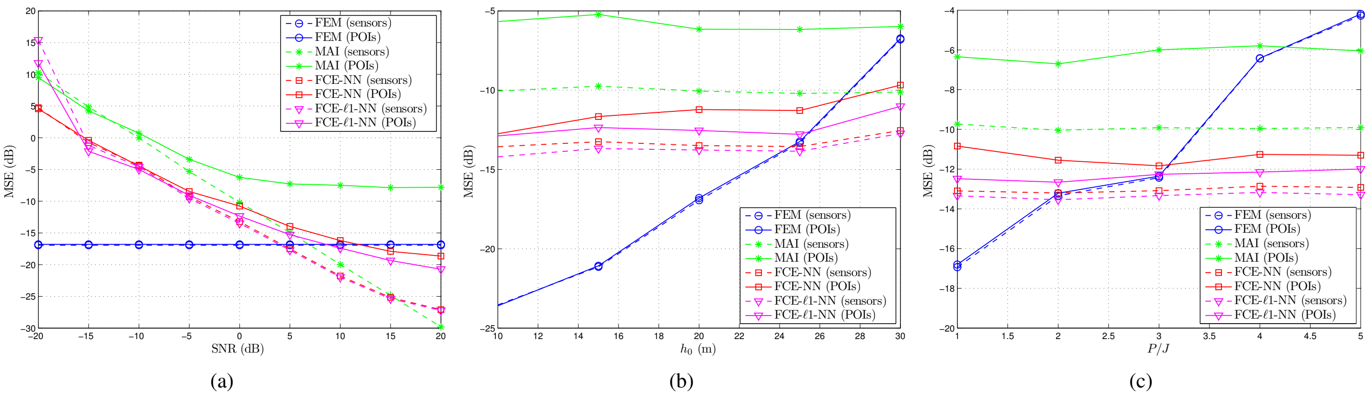

Fig. 2. Comparison of field estimation MSE at sensor node locations (- -) and POI locations (–) for different estimation algorithms, plotted vs. (a) SNR of WSN measurements, (b) initial node distance $h0$ in mesh generation, (c) ratio P/J of number of POIs and sensor nodes.

POIs P) is varied while the other parameters are kept fixed at the values given earlier. A first observation is that the proposed FCE algorithm consistently performs better than the MAI algorithm, both at sensor node and POI locations. This confirms our intuition that the use of a physical model is indeed beneficial for field estimation at sensor nodes providing noisy field measurements, and moreover offers a decent alternative to "naive" field interpolation at POIs. A second important observation is that the FCE algorithm behaves very differently from the FEM for varying simulation parameters. While the FEM performance is seen to deteriorate for a decreasing mesh quality (i.e., both for an increasing initial node distance h_0 and for an increasing number of fixed mesh points $P + J$), the FCE performance appears to be independent of the mesh properties. On the other hand, the WSN measurement SNR is the only parameter that has a fundamental influence on the FCE performance (note that a behavior similar to Fig. 2(a) is obtained when fixing the SNR and varying the number of measurements N per sensor node). Remarkably, under certain conditions the FCE algorithm outperforms the FEM, although the latter requires full knowledge of the source vector while the former does not. This observation holds either when high SNR measurements are available, when using a coarse mesh, or for a large number of POIs. A final overall observation is that the inclusion of a sparsity-inducing regularization term slightly improves the performance of the FCE algorithm with nonnegativity constraints.

VI. Conclusion and Future Work

In this paper, we have proposed a new framework for field estimation in which WSN measurements are combined with a FEM-based physical field model. This framework provides an appealing solution to the fundamental problem of estimating field values at locations where no WSN measurements are available. By formulating the discretized field estimation problem on a mesh that includes these sensorless locations, and by appending an appropriate sparsity-inducing regularization term and nonnegativity constraints, we end up with a well-determined and convex optimization problem that can be readily solved in a cooperative fashion. Simulations for the case of a static 2-D field governed by a Poisson PDE illustrate that the proposed FEM-constrained estimation algorithm consistently outperforms an estimation method based on WSN measurements only, and under certain conditions even performs better than a FEM that assumes full knowledge of the driving source function.

In our future work we will focus on two particular research challenges that have not been dealt with in this paper. First of all, the proposed framework needs to be generalized to the case of dynamic fields governed by PDEs that also include time derivatives. The FEM-constrained optimization problem will then feature evolutionary constraints, which naturally leads to the use of a Kalman filter for dynamic field estimation. A second challenge is to convert the cooperative FEM-constrained field estimation algorithm into a distributed estimation algorithm. As pointed out earlier, the sparse structure of the Galerkin system of equations is expected to facilitate an efficient distributed implementation.

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