Convergence Analysis of Downstream VDSL Adaptive Multichannel Partial FEXT Cancellation

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***Abstract*—In this paper we analyze an adaptive downstream multichannel VDSL precoder that is based on error signal feedback. The analysis presents sufficient conditions for precoder convergence and an upper bound on the precoder steady state error. The paper also considers and analyzes the case of partial FEXT cancellation. The analysis shows that in some scenarios (and in particular in mixed-length binders) the use of partial FEXT cancellation is crucial to achieve precoder convergence in a reasonable time. Based on this analysis we determine that convergence is achievable in most practical channels. These bounds also allow for the proper setting of the convergence parameters. The paper presents several simulations to demonstrate the theoretical results. In these simulations, setting the precoder parameters according to the analysis leads to convergence in less than 400 OFDM symbols.**

***Index Terms*—VDSL, VDSL2, digital subscriber line, vectored transmission, crosstalk mitigation, multichannel transmission, linear precoding.**

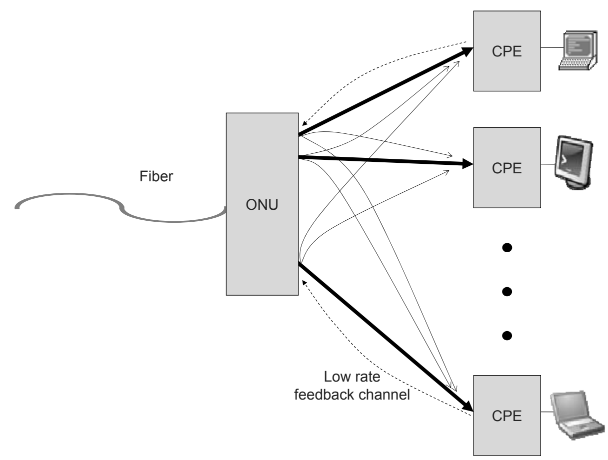

Fig. 1. Network topology with fiber to the point (FTTB/FTTC/FTTN) topology. Bold lines from ONU to CPE are the channels, thin lines are the far end crosstalk (FEXT) coupling channels.

I. Introduction

THE second generation of Very High bit-rate DSL (VDSL2) is the most advanced DSL technology. However, it is still limited by the crosstalk between copper pairs in the same binder. In a typical deployment, the fiber optic network terminates at an optical network unit (ONU). Data are further distributed over the existing copper infrastructure to the various users. Due to the distributed nature of customer premises equipment (CPE) any joint processing must take place at the optical network termination (ONU). The architecture is depicted in Figure 1. In [1], it is shown that there is a need to process at least half of the operating pairs to obtain a substantial gain in achievable rate. Recently several papers have dealt with full coordination of downstream VDSL transmission.

To cancel far end crosstalk (FEXT) in the downstream precancellation or precoding is required [2], [3]. Ginis and Cioffi [2] proposed a Tomlinson Harashima type non-linear FEXT precoding. Cendrillon et. al [4], [5], noted that it is sufficient to use linear precoding due to the diagonal dominance property of the FEXT coupling matrix, still their ZF solution requires matrix inversion at each tone of the multichannel DSL system. Leshem and Li [6], [7] proposed a simplified approximate precoder, based on a first or second order approximation of the inverse of the ZF precoding matrix, which significantly reduces the memory requirements. Cendrillon et al. [8] proposed another approach for complexity reduction through a partial FEXT cancellation scheme, which only compensates for the FEXT of dominant interferers. All the solutions above assume sufficiently good channel estimates. In [9] the effect of precoder quantization and channel estimation is discussed, assuming that crosstalk estimation takes place at the customer premises modem.

As new VDSL technologies are often deployed in a gradual manner, a practical precoder should be able to operate in the presence of legacy VDSL2 modems. In such scheme the precoder must rely only on error signals from upgraded modems, which are transmitted back to the ONU through the VDSL operation control channel (and on the precoder knowledge of all transmitted data to all modems)[1]. The ONU receives and processes the error signals from CPE and adapts the precoder until it converges to the desired precoder.

This approach was suggested in [10] where it was tested in simulations. However, no analysis of this technique has been given.

In this paper we present a theoretical performance analysis

[1]A current standardization effort aims to allow for full channel estimation using orthogonal pilot symbols. But, such a scheme would still require an upgrade to all CPE.

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both for precoder convergence and for the steady state error. The analysis provides sufficient conditions for the precoder convergence. These conditions show that in some cases (and in particular in mixed-length binders) the use of partial FEXT cancellation can significantly increase the convergence speed, with a negligible effect on steady state performance. The analysis also presents an expression for the precoder steady state error that can be used to further tune the precoder performance.

The structure of the paper is as follows: In section II we describe the mathematical model for multi-pair DSL systems and the analyzed precoder. Section III presents the performance analysis of the adaptive precoder. Simulations are reported in section IV, and our concluding remarks are in section V.

II. Signal Model

In this section we describe the signal model of a multi-channel precoded system. We concentrate on discrete multi-tone (DMT) systems where transmission takes place independently over many narrow sub-bands. Consider a system that coordinates the transmission of all p twisted pairs in a binder. Following the conventions of the VDSL2 standard, we assume that the system operates in a frequency division duplex mode (FDD), where upstream and downstream transmissions take place at separate frequency bands, and that all transmissions in the binder are synchronized (and therefore near end crosstalk (NEXT) is eliminated). The received signal at frequency bin f for all pairs can be written in vector form as

$$\mathbf{x}(f) = \mathbf{H}(f)\mathbf{F}(f)\mathbf{s}(f) + \boldsymbol{\nu}(f), \qquad f = 1, \ldots, M \tag{1}$$

where

$$\mathbf{H}(f) = \begin{bmatrix} h_{11}(f) & \cdots & h_{1p}(f) \\ \vdots & \ddots & \vdots \\ h_{p1}(f) & \cdots & h_{pp}(f) \end{bmatrix}$$

is the channel frequency response at frequency bin f, $\mathbf{F}(f)$ is the precoding matrix (for transmission with no precoding we use $\mathbf{F}(f) = \mathbf{I}$), $\mathbf{s}(f) = [s_1(f), \ldots, s_p(f)]^T$ are the symbols transmitted over all pairs and $\boldsymbol{\nu}(f)$ is a vector of independent identically distributed (iid) additive white Gaussian noise (AWGN) samples with power P_{N}. When the specific frequency is not relevant for the discussion we suppress the explicit dependence on f and use the following notation

$$\mathbf{x} = \mathbf{HFs} + \boldsymbol{\nu}. \tag{2}$$

The signals at each frequency bin are typically QAM modulated with modulation level determined by the signal to noise ratio (SNR) at the receiver at the given frequency. The modulation level varies from BPSK up to 2^{15} QAM when the signal to noise ratio is sufficiently good. We assume that all users have identical PSD and denote the transmitted power at the analyzed frequency bin by P_{S} (this assumption simplifies the notation, but the proposed technique can be implemented with any PSD constraint). We also assume that the signals transmitted to all users are statistically independent and have zero mean. Thus, the signal vector satisfies:

$$E\left[\mathbf{s}\mathbf{s}^{\mathrm{H}}\right] = P_{\mathrm{S}}\mathbf{I}, \qquad E\left[\mathbf{s}\mathbf{s}^{\mathrm{H}}\mathbf{s}\mathbf{s}^{\mathrm{H}}\right] = KP_{\mathrm{S}}^2\mathbf{I}. \tag{3}$$

Typical values for K range from p (QPSK modulation) to $p+1$ (Gaussian modulation).

The precoder aims to minimize the interference; i.e., for each user, minimize the power received from all other users. The popular ZF linear precoder [4] uses $\mathbf{F}_{\mathrm{ZF}} = \mathbf{H}^{-1}\mathbf{D}$ where $\mathbf{D}$ is the diagonal matrix

$$\mathbf{D} = \begin{bmatrix} h_{11} & & \\ & \ddots & \\ & & h_{pp} \end{bmatrix}. \tag{4}$$

The received signal now becomes

$$\mathbf{x} = \mathbf{Ds} + \boldsymbol{\nu} \tag{5}$$

resulting in a FEXT free channel. Furthermore since $\mathbf{H}$ is diagonally dominant no significant power is added since the precoding matrix is almost an identity matrix.

In this paper we study the general case in which only some of the users require FEXT cancellation. Partial FEXT cancellation[2] may be a necessity for practical reasons, but also for analytical concerns. As will be shown in the following section, partial FEXT cancellation can significantly shorten precoder convergence time. In the following we assume that $u \leq p$ users apply FEXT cancellation, and without loss of generality, assume that these users are labeled $1, \ldots, u$. (The case of complete FEXT cancellation, i.e., traditional ZF precoding, will be analyzed as the special case in which $u = p$).

Define by

$$\mathbf{H} = \left[\begin{array}{c|c} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \hline \mathbf{H}_{21} & \mathbf{H}_{22} \end{array}\right] \tag{6}$$

the division of the channel matrix into four sub-matrices in which $\mathbf{H}_{11}$ is of size $u \times u$. We will also use the corresponding division of the channel precoding matrix, $\mathbf{F}$ and the diagonal of the channel matrix $\mathbf{D}$. For partial FEXT cancellation we assume that $\mathbf{F}_{21} = \mathbf{0}$ and $\mathbf{F}_{22} = \mathbf{I}$. The resulting precoder is given by:

$$\mathbf{F}_{\mathrm{P}} = \left[\begin{array}{c|c} \mathbf{H}_{11}^{-1}\mathbf{D}_{11} & -\mathbf{H}_{11}^{-1}\mathbf{H}_{12} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right]. \tag{7}$$

Note that in the specific case of complete FEXT cancellation ($u = p$) this solution is identical to the ZF precoder ($\mathbf{F}_{\mathrm{P}} = \mathbf{F}_{\mathrm{ZF}}$).

However, the calculation of F_{ZF} or F_{P} requires good knowledge of the channel matrix $\mathbf{H}$ at the transmitter, and a matrix inversion operation each time that this matrix is updated. In [10] it was suggested to use an adaptive precoder, based on error signal feedback from the CPE. The error signal measured by the i-th user is given by:

$$\epsilon_i = x_i - h_{ii}s_i \tag{8}$$

where x_i and h_{ii} are the i-th user received signal and channel gain respectively. In this paper we assume that each user has exact knowledge of its direct channel coefficient. The study of

[2]Note that the partial FEXT cancellation scheme in this work is different from the one presented in [8]. In this work, only some of the users use FEXT cancellation, but, these users cancel the FEXT from all other users. On the other hand in [8], each user can only cancel the FEXT generated by some of the users.

the effect of channel estimation errors is left for future study. The user sends feedback consisting of the index of a quantized version of the error signal. This quantized error is given by:

$$\hat{\epsilon}_i = h_{ii}^{-1}\epsilon_i + w_i \tag{9}$$

and w_i is the quantization error resulting from the feedback quantization. In the following we will not assume a specific quantization scheme, but only that the quantization errors are statistically independent random variables with zero mean and variance of σ_w^2. Collecting all error signals into a vector we have:

$$\begin{aligned}\hat{\boldsymbol{\epsilon}} &= \mathbf{D}^{-1}(\mathbf{x} - \mathbf{D}\mathbf{s}) + \mathbf{w} \\ &= (\mathbf{D}^{-1}\mathbf{H}\mathbf{F} - \mathbf{I})\mathbf{s} + \mathbf{D}^{-1}\boldsymbol{\nu} + \mathbf{w}.\end{aligned} \tag{10}$$

In this paper we use a slightly simplified version of the precoder presented in [10]. The precoder update equation is given by:

$$\mathbf{F}_{k+1} = \mathbf{F}_k - \alpha\boldsymbol{\Gamma}\hat{\boldsymbol{\epsilon}}_k\mathbf{s}_k^{\mathrm{H}}, \tag{11}$$

where $\boldsymbol{\Gamma}$ is a selection matrix ($\Gamma_{ij} = 1$ if $i = j \le u$, and zero otherwise) that guarantees that only the upper part of the precoder is updated while the lower part keeps its original value. This can also be seen as an LMS estimation of the channel matrix followed by a first order approximation of the channel inversion [7]. However, it can be shown that the precoder suggested above can converge even for channel matrices in which the first order approximation is not accurate.

III. Performance Analysis

A. Convergence analysis

In this section we analyze the performance of the precoder presented in the previous section (11). The crucial part in the analysis of any adaptive technique is the derivation of conditions that guarantee system convergence. In this work, we say that the system has converged when the precoder is close enough to the ZF precoder as defined in the following lemma:

Lemma 3.1: A sufficient condition for the precoder to converge is that:

$$\beta_{\max} = \max\left(\max_{1\le i\le u}\sum_{\substack{j=1\\ j\ne i}}^{u}\frac{|h_{ij}|}{|h_{ii}|}, \max_{1\le j\le u}\sum_{\substack{i=1\\ i\ne j}}^{u}\frac{|h_{ij}|}{|h_{ii}|}\right) < 1 \tag{12}$$

and

$$\alpha < \frac{2}{KP_{\mathrm{S}}}\frac{1}{1+\beta_{\max}}. \tag{13}$$

If these conditions are met, after sufficiently long convergence time, the Frobenius norm of the difference between the desired precoder and the actual precoder is upper bounded by:

$$\begin{aligned}&\lim_{k\to\infty}\mathrm{tr}\left\{(\mathbf{F}_k - \mathbf{F}_{\mathrm{P}})(\mathbf{F}_k - \mathbf{F}_{\mathrm{P}})^H\right\} \\ &\quad\le \frac{\alpha p\sum_{i=1}^{u}\left(\sigma_w^2 + \frac{P_{\mathrm{N}}}{|h_{ii}|^2}\right)}{2 - \alpha KP_{\mathrm{S}}(1+\beta_{\max}^2) - 2|1-\alpha KP_{\mathrm{S}}|\beta_{\max}}.\end{aligned} \tag{14}$$

This lemma guarantees that if condition (12) is satisfied, we will be able to achieve any level of FEXT cancellation by choosing small enough α, at the price of slow convergence. Note that downstream VDSL systems are typically considered to be row-wise diagonal dominant (RWDD) [7]. A more accurate characterization of this property is that the matrix $\mathbf{D}^{-1}\mathbf{H}$ is diagonal dominant. Hence condition (12) is satisfied in almost all VDSL channels. If we are interested in fast convergence, we can set α according to condition (13), and achieve fast and yet guaranteed convergence.

Proof of Lemma 3.1: We define the precoder error as:

$$\boldsymbol{\Delta}_k = \mathbf{F}_k - \mathbf{F}_{\mathrm{P}}. \tag{15}$$

In the case of partial FEXT cancellation, note that the last $p-u$ rows of $\boldsymbol{\Delta}_k$ are always zero. Therefore, it is convenient to define $\tilde{\boldsymbol{\Delta}}_k$ as the matrix containing the top u rows of $\boldsymbol{\Delta}_k$, or mathematically $\tilde{\boldsymbol{\Delta}}_k = \tilde{\boldsymbol{\Gamma}}\boldsymbol{\Delta}_k$, where $\tilde{\boldsymbol{\Gamma}} = [\mathbf{I}, \mathbf{0}]$ is a $u \times p$ selection matrix (In the case of traditional ZF we have $\tilde{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma} = \mathbf{I}$). Now we can write the precoder error in the k-th iteration as:

$$\begin{aligned}\tilde{\boldsymbol{\Delta}}_{k+1} &= \tilde{\boldsymbol{\Delta}}_k - \alpha\tilde{\boldsymbol{\Gamma}}\hat{\boldsymbol{\epsilon}}_k\mathbf{s}_k^{\mathrm{H}}, \\ &= \tilde{\boldsymbol{\Delta}}_k - \alpha\tilde{\boldsymbol{\Gamma}}(\mathbf{D}^{-1}\mathbf{H}\mathbf{F}_k - \mathbf{I})\mathbf{s}_k\mathbf{s}_k^{\mathrm{H}} - \alpha\tilde{\boldsymbol{\Gamma}}(\mathbf{D}^{-1}\boldsymbol{\nu}_k + \mathbf{w}_k)\mathbf{s}_k^{\mathrm{H}} \\ &= \tilde{\boldsymbol{\Delta}}_k - \alpha\tilde{\boldsymbol{\Gamma}}\mathbf{D}^{-1}\mathbf{H}\tilde{\boldsymbol{\Gamma}}^T\tilde{\boldsymbol{\Delta}}_k\mathbf{s}_k\mathbf{s}_k^{\mathrm{H}} - \alpha\tilde{\boldsymbol{\Gamma}}(\mathbf{D}^{-1}\boldsymbol{\nu}_k + \mathbf{w}_k)\mathbf{s}_k^{\mathrm{H}} \\ &= \tilde{\boldsymbol{\Delta}}_k - \alpha\mathbf{D}_{11}^{-1}\mathbf{H}_{11}\tilde{\boldsymbol{\Delta}}_k\mathbf{s}_k\mathbf{s}_k^{\mathrm{H}} - \alpha\tilde{\boldsymbol{\Gamma}}(\mathbf{D}^{-1}\boldsymbol{\nu}_k + \mathbf{w}_k)\mathbf{s}_k^{\mathrm{H}}\end{aligned} \tag{16}$$

where in the second line we substituted the error signal (10); In the third line we used the fact that $\tilde{\boldsymbol{\Gamma}}\mathbf{D}^{-1}\mathbf{H}\mathbf{F}_{\mathrm{P}} = \tilde{\boldsymbol{\Gamma}}$ and the equality $\boldsymbol{\Delta}_k = \tilde{\boldsymbol{\Gamma}}^T\tilde{\boldsymbol{\Delta}}_k$; The last line directly follows from the fact that $\tilde{\boldsymbol{\Gamma}}\mathbf{A}\tilde{\boldsymbol{\Gamma}}^T = \mathbf{A}_{11}$ selects the upper left $u \times u$ sub-matrix of $\mathbf{A}$. For simplicity, we assume that the error vector, $\hat{\boldsymbol{\epsilon}}_k$, is always of length p. In the case of partial crosstalk cancellation, the matrix $\tilde{\boldsymbol{\Gamma}}$ ensures that the last $p-u$ elements of $\hat{\boldsymbol{\epsilon}}_k$ will have no effect (and hence the transmitter does not even need to know these elements).

Let:

$$\mathbf{W}_k = E\left[\tilde{\boldsymbol{\Delta}}_k\tilde{\boldsymbol{\Delta}}_k^{\mathrm{H}}\right], \quad \text{and} \quad \delta_k = \mathrm{tr}\left[\mathbf{W}_k\right]. \tag{17}$$

Note that δ_k is the Frobenius norm of $\tilde{\boldsymbol{\Delta}}_k$. Hence, it places an upper bound on the difference between each element in the precoder matrix and the ZF precoder matrix in mean square error (MSE). Substituting (16) into (17) we can write:

$$\begin{aligned}\mathbf{W}_{k+1} = &E\left[\tilde{\boldsymbol{\Delta}}_k\tilde{\boldsymbol{\Delta}}_k^{\mathrm{H}}\right] - \alpha\mathbf{D}_{11}^{-1}\mathbf{H}_{11}E\left[\tilde{\boldsymbol{\Delta}}_k\mathbf{s}_k\mathbf{s}_k^{\mathrm{H}}\tilde{\boldsymbol{\Delta}}_k^{\mathrm{H}}\right] \\ &- \alpha E\left[\tilde{\boldsymbol{\Delta}}_k\mathbf{s}_k\mathbf{s}_k^{\mathrm{H}}\tilde{\boldsymbol{\Delta}}_k^{\mathrm{H}}\right]\mathbf{H}_{11}^{\mathrm{H}}\mathbf{D}_{11}^{-\mathrm{H}} \\ &+ \alpha^2\mathbf{D}_{11}^{-1}\mathbf{H}_{11}E\left[\tilde{\boldsymbol{\Delta}}_k\mathbf{s}_k\mathbf{s}_k^{\mathrm{H}}\mathbf{s}_k\mathbf{s}_k^{\mathrm{H}}\tilde{\boldsymbol{\Delta}}_k^{\mathrm{H}}\right]\mathbf{H}_{11}^{\mathrm{H}}\mathbf{D}_{11}^{-\mathrm{H}} \\ &+ \alpha^2\tilde{\boldsymbol{\Gamma}}E\left[(\mathbf{D}^{-1}\boldsymbol{\nu}_k + \mathbf{w}_k)\mathbf{s}_k^{\mathrm{H}}\mathbf{s}_k(\mathbf{D}^{-1}\boldsymbol{\nu}_k + \mathbf{w}_k)^{\mathrm{H}}\right]\tilde{\boldsymbol{\Gamma}}^{\mathrm{H}}\end{aligned} \tag{18}$$

where we removed terms that involved the first moment of the noise or quantization error as their expectation equals 0. Noting that $\tilde{\boldsymbol{\Delta}}_k$ is statistically independent of $\mathbf{s}_k$ (since $\mathbf{F}_k$ is calculated based on $\mathbf{s}_{k-1}$), we use (3) to obtain:

$$\begin{aligned}\mathbf{W}_{k+1} = &\mathbf{W}_k - \alpha P_{\mathrm{S}}\mathbf{D}_{11}^{-1}\mathbf{H}_{11}\mathbf{W}_k - \alpha P_{\mathrm{S}}\mathbf{W}_k\mathbf{H}_{11}^{\mathrm{H}}\mathbf{D}_{11}^{-\mathrm{H}} \\ &+ \alpha^2KP_{\mathrm{S}}^2\mathbf{D}_{11}^{-1}\mathbf{H}_{11}\mathbf{W}_k\mathbf{H}_{11}^{\mathrm{H}}\mathbf{D}_{11}^{-\mathrm{H}} \\ &+ \alpha^2pP_{\mathrm{S}}\left(P_{\mathrm{N}}\mathbf{D}_{11}^{-1}\mathbf{D}_{11}^{-\mathrm{H}} + \sigma_w^2\mathbf{I}\right) \\ = &\frac{1}{K}\left(\mathbf{I} - \alpha KP_{\mathrm{S}}\mathbf{D}_{11}^{-1}\mathbf{H}_{11}\right)\mathbf{W}_k\left(\mathbf{I} - \alpha KP_{\mathrm{S}}\mathbf{D}_{11}^{-1}\mathbf{H}_{11}\right)^{\mathrm{H}} \\ &+ \frac{K-1}{K}\mathbf{W}_k + \alpha^2pP_{\mathrm{S}}\left(P_{\mathrm{N}}\mathbf{D}_{11}^{-1}\mathbf{D}_{11}^{-\mathrm{H}} + \sigma_w^2\mathbf{I}\right).\end{aligned} \tag{19}$$

Using $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and the fact that $\mathbf{W}_k$ is positive definite, and defining $\mathbf{Q} = \left(\mathbf{I} - \alpha K P_\mathrm{S} \mathbf{D}_{11}^{-1}\mathbf{H}_{11}\right)$ we can write the trace of $\mathbf{W}_{k+1}$ using (19) as:

$$\begin{aligned}\delta_{k+1} &= \mathrm{tr}\left[\mathbf{W}_k\left\{\frac{1}{K}\mathbf{Q}^\mathrm{H}\mathbf{Q} + \frac{K-1}{K}\mathbf{I}\right\}\right] \\ &\quad + \alpha^2 p P_\mathrm{S}\cdot \mathrm{tr}\left[P_\mathrm{N}\mathbf{D}_{11}^{-1}\mathbf{D}_{11}^{-\mathrm{H}} + \sigma_w^2\mathbf{I}\right] \\ &\le \delta_k\left[\frac{1}{K}\rho\left(\mathbf{Q}^\mathrm{H}\mathbf{Q}\right) + \frac{K-1}{K}\right] \\ &\quad + \alpha^2 p P_\mathrm{S}\left[u\sigma_w^2 + \sum_{i=1}^{u}\frac{P_\mathrm{N}}{|h_{ii}|^2}\right]\end{aligned} \tag{20}$$

where $\rho\left(\mathbf{A}\right)$ is the spectral radius (or maximal eigenvalue) of the matrix $\mathbf{A}$.

We first note that convergence is guaranteed if the spectral radius of $\mathbf{Q}^\mathrm{H}\mathbf{Q}$ is smaller than 1. The spectral radius of $\mathbf{Q}^\mathrm{H}\mathbf{Q}$ is upper bounded by ([11] page 223):

$$\begin{aligned}\rho\left(\mathbf{Q}^\mathrm{H}\mathbf{Q}\right) &\le |||\mathbf{Q}^\mathrm{H}|||_1|||\mathbf{Q}|||_1 \\ &= \left[\max_i\sum_{j=1}^{u}|q_{ij}|\right]\left[\max_j\sum_{i=1}^{u}|q_{ij}|\right].\end{aligned} \tag{21}$$

Testing the elements of the matrix $\mathbf{Q}$ we have:

$$|q_{ij}| = \begin{cases} |1 - \alpha K P_\mathrm{S}| & i = j \\ \alpha K P_\mathrm{S}\frac{|h_{ij}|}{|h_{ii}|} \le \alpha K P_\mathrm{S}\beta_{\max} & i \ne j. \end{cases} \tag{22}$$

Substituting in (21) we require

$$\begin{aligned}&\max_{1\le i\le u}\left(|1-\alpha K P_\mathrm{S}| + \alpha K P_\mathrm{S}\sum_{\substack{j=1\\ j\ne i}}^{u}\frac{|h_{ij}|}{|h_{ii}|}\right) \\ &\cdot\max_{1\le j\le u}\left(|1-\alpha K P_\mathrm{S}| + \alpha K P_\mathrm{S}\sum_{\substack{i=1\\ i\ne j}}^{u}\frac{|h_{ij}|}{|h_{ii}|}\right) < 1.\end{aligned} \tag{23}$$

The proof of convergence is completed by noticing that conditions (12) and (13) imply (23).

Assuming that the convergence conditions are satisfied, and looking at the bound (20) as time goes to infinity we can write the convergence bound as:

$$\delta_\infty \le \frac{K\alpha^2 p P_\mathrm{S}\sum_{i=1}^{u}\left(\sigma_w^2 + \frac{P_\mathrm{N}}{|h_{ii}|^2}\right)}{1 - \rho\left(\mathbf{Q}^\mathrm{H}\mathbf{Q}\right)}. \tag{24}$$

The proof is completed by substituting $\rho\left(\mathbf{Q}^\mathrm{H}\mathbf{Q}\right) \le \left(|1-\alpha K P_\mathrm{S}| + \alpha K P_\mathrm{S}\beta_{\max}\right)^2$ (which results from the substitution of (12) in (21)) in (24), resulting in (14).

Although the bounds derived in lemma 3.1 are not tight, they provide an important insight into the precoder convergence. Due to the diagonal dominance of the channel matrix, we expect the desired precoder to be quite close to the identity matrix, and hence, the Frobenius norm of the desired precoder should be around p $\left(\mathrm{tr}[\mathbf{F}_\mathrm{ZF}\mathbf{F}_\mathrm{ZF}^H] \simeq p\right)$. If the precoder actually converges we expect $\delta_\infty \ll p$. Inspecting (14) we see that convergence can be achieved whenever $\beta_{\max} < 1$, for sufficiently small α. Note that for $\alpha \le 1/(K P_\mathrm{S})$ (14) can be conveniently bounded by:

$$\delta_\infty \le \frac{\alpha p}{1-\beta_{\max}^2}\sum_{i=1}^{u}\left(\sigma_w^2 + \frac{P_\mathrm{N}}{|h_{ii}|^2}\right). \tag{25}$$

It is important to note that condition (14) depends on the users' signal to noise ratio (SNR), and is actually dominated by the user with the worst SNR, whereas condition (12) depends solely on the normalized channel response. Hence, partial FEXT cancellation only has a minor effect on the ability of the precoder to converge, but can significantly reduce the steady state error. If the system includes some users with poor SNR, although it may converge according to conditions (12) and (13), the value of α required to achieve a predetermined steady state error according to (14) can be quite small. Using partial FEXT cancellation of all other users will remove these poor users from the sum in (14) and make it possible to choose a much higher value for α, and hence, will significantly shorten the convergence time. On the other hand, the performance of users with poor SNR are limited by the received noise; thus, such users cannot benefit much from FEXT cancellation. This phenomenon is somewhat similar to the noise enhancement effect of ZF precoders in channels with high attenuation (e.g., bridged taps) or channels with strong received noise (e.g., external RFI from AM transmitters).

We therefore suggest defining a threshold SNR, ρ_L, so that the system will not execute FEXT cancellation for users that have lower SNR ($|h_{ii}|^2/\sigma_\mathrm{T}^2 < \rho_\mathrm{L}$). Users with low SNR are less sensitive to FEXT, and therefore the operation of FEXT cancellation for such users will waste system resources with almost no gain. Note that users' SNR changes with frequency, and can be above the threshold for some of the frequency bins and below the threshold for other frequency bins. Therefore, the proposed method can apply FEXT cancellation for such users only on some of the frequency bins.

B. Steady state analysis

Following the results of the previous subsections, and assuming that α and u were chosen so that convergence is achieved, we now turn to test the steady state error for each of the users. The covariance matrix of the first u elements of the error signal, (8), is given by:

$$\begin{aligned}\mathbf{R}_{\tilde{\boldsymbol{\epsilon}}_k} &= E\left[\tilde{\boldsymbol{\Gamma}}\boldsymbol{\epsilon}_k\boldsymbol{\epsilon}_k^\mathrm{H}\tilde{\boldsymbol{\Gamma}}^T\right] \\ &= E\left[\tilde{\boldsymbol{\Gamma}}(\mathbf{H}\mathbf{F}_k - \mathbf{D})\mathbf{s}_k\mathbf{s}_k^\mathrm{H}(\mathbf{H}\mathbf{F}_k - \mathbf{D})^\mathrm{H}\tilde{\boldsymbol{\Gamma}}^T\right] + P_\mathrm{N}\mathbf{I} \\ &= P_\mathrm{S}\tilde{\boldsymbol{\Gamma}}\mathbf{H}\tilde{\boldsymbol{\Gamma}}^T E\left[\tilde{\boldsymbol{\Delta}}\tilde{\boldsymbol{\Delta}}^\mathrm{H}\right]\tilde{\boldsymbol{\Gamma}}\mathbf{H}^\mathrm{H}\tilde{\boldsymbol{\Gamma}}^T + P_\mathrm{N}\mathbf{I} \\ &= P_\mathrm{S}\mathbf{H}_{11}\mathbf{W}_k\mathbf{H}_{11}^\mathrm{H} + P_\mathrm{N}\mathbf{I}.\end{aligned} \tag{26}$$

To evaluate (26), we first need to evaluate $\mathbf{W}$, the steady state value of $\mathbf{W}_k$. Inspecting equation (19) and setting $\mathbf{W}_{k+1} = \mathbf{W}_k = \mathbf{W}$, the steady state matrix must satisfy:

$$\begin{aligned}0 &= -\mathbf{D}_{11}^{-1}\mathbf{H}_{11}\mathbf{W} - \mathbf{W}\mathbf{H}_{11}^\mathrm{H}\mathbf{D}_{11}^{-\mathrm{H}} \\ &\quad + \alpha K P_\mathrm{S}\mathbf{D}_{11}^{-1}\mathbf{H}_{11}\mathbf{W}\mathbf{H}_{11}^\mathrm{H}\mathbf{D}_{11}^{-\mathrm{H}} + \alpha p\left(P_\mathrm{N}\mathbf{D}_{11}^{-1}\mathbf{D}_{11}^{-\mathrm{H}} + \sigma_w^2\mathbf{I}\right).\end{aligned} \tag{27}$$

Multiplying by $\mathbf{D}_{11}$ from the left and by $\mathbf{D}_{11}^\mathrm{H}$ from the right, and assuming that $\mathbf{H}_{11}$ is non singular, we rewrite (27) as:

$$\mathbf{V}\mathbf{A}^\mathrm{H} + \mathbf{A}\mathbf{V} = \alpha p\left(P_\mathrm{N}\mathbf{I} + \sigma_w^2\mathbf{D}_{11}\mathbf{D}_{11}^\mathrm{H}\right), \tag{28}$$

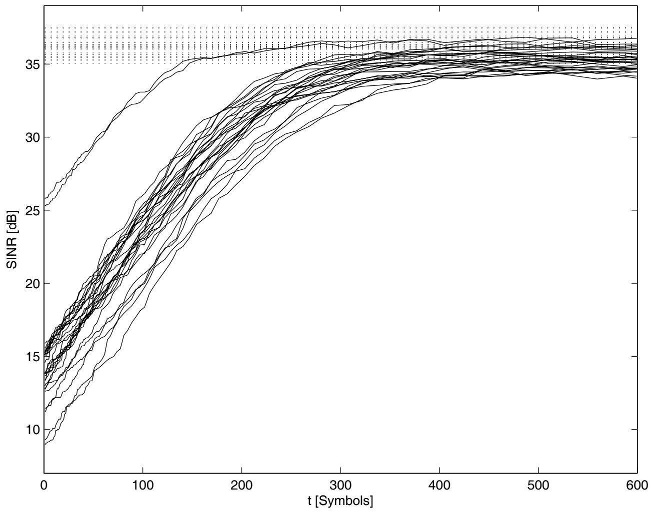

Fig. 2. SINR Convergence at a distance of 300m and a frequency of 27.4MHz (28 users).

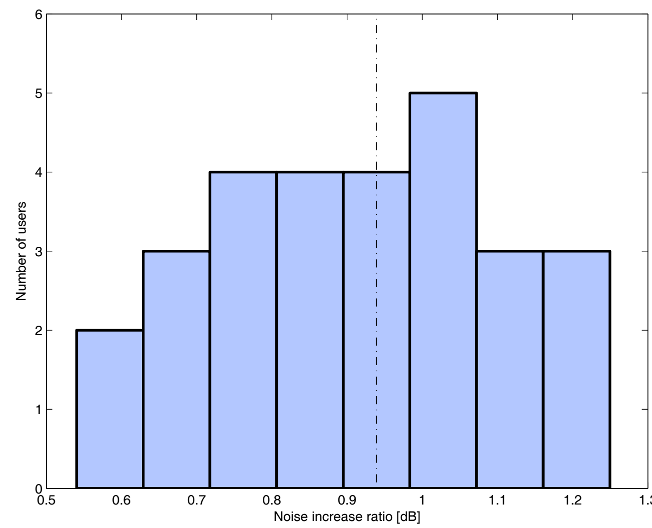

Fig. 3. Histogram of steady state SINR loss at a distance of 300m and a frequency of 27.4MHz.

where $\mathbf{V} = \mathbf{H}_{11}\mathbf{W}\mathbf{H}_{11}^{\mathrm{H}}$, and $\mathbf{A} = \mathbf{D}_{11}\mathbf{H}_{11}^{-1} - \alpha K P_{\mathrm{S}}\mathbf{I}/2$.

The existence of a steady state solution is guaranteed if the real part of all eigenvalues of the matrix $\mathbf{A}$ is positive ([11] Theorem 4.4.6). This will happen if and only if the real part of all eigenvalues of $\mathbf{B} = \mathbf{A}^{-1}$ is positive. Using the Gershgorin theorem to bound the real part of the eigenvalues of $\mathbf{B}$ we write:

$$\mathrm{Re}\{\lambda(\mathbf{B})\} \geq \max\left(\min_i\left[\mathrm{Re}\{b_{i,i}\} - \sum_{\substack{j=1\\ j\neq i}}^{p}|b_{ij}|\right], \min_j\left[\mathrm{Re}\{b_{j,j}\} - \sum_{\substack{i=1\\ i\neq j}}^{p}|b_{ij}|\right]\right) \quad (29)$$

where $\lambda(\mathbf{B})$ and $b_{i,j}$ are an eigenvalue and the i,j element of the matrix $\mathbf{B}$respectively. Writing the elements $b_{i,j}$ explicitly and defining:

$$\gamma_{\max} = \min\left(\max_{1\leq i\leq u}\left[\sum_{\substack{j=1\\ j\neq i}}^{u}\frac{|h_{ij}|}{|h_{ii}|}\right], \max_{1\leq j\leq u}\left[\sum_{\substack{i=1\\ i\neq j}}^{u}\frac{|h_{ij}|}{|h_{ii}|}\right]\right) \quad (30)$$

a steady state solution is guaranteed if:

$$1 - \frac{\alpha K P_{\mathrm{S}}}{2} - \gamma_{\max} \geq 0 \quad (31)$$

or equivalently:

$$\alpha < \frac{2}{K P_{\mathrm{S}}}(1 - \gamma_{\max}). \quad (32)$$

It is interesting to compare condition (32) with condition (13). On one hand, we have $\gamma_{\max} \leq \beta_{\max}$ and therefore for some channel matrices, condition (32) is more flexible (there are values of α that satisfy condition (32) but do not satisfy condition (13)). On the other hand, if $\gamma_{\max} = \beta_{\max}$ (for example if the channel matrix is symmetric) condition (13) is more flexible (using $1/(1+x) \geq 1-x$). But, it is more important to remember that both conditions are not tight, and that both are satisfied in almost all cases of practical interest.

The steady state error can be evaluated by finding the matrix $\mathbf{V}$ that solves (28), and substituting it in (26). The solution to (28) can be obtained by rewriting it as:

$$[(\mathbf{I}\otimes\mathbf{A}) + (\mathbf{A}^{*}\otimes\mathbf{I})]\,\mathrm{vec}\mathbf{V} = \alpha p\cdot\mathrm{vec}\left(P_{\mathrm{N}}\mathbf{I} + \sigma_w^2\mathbf{D}_{11}\mathbf{D}_{11}^{\mathrm{H}}\right), \quad (33)$$

where $\mathbf{A}^{*}$ is the complex conjugate of $\mathbf{A}$, $\otimes$ denotes the Kronecker product and $\mathrm{vec}\mathbf{V}$ denotes the reordering of the elements of the matrix $\mathbf{V}$ as a vector. Unfortunately, equation (33) has very high dimension, and its solution is not practical. We therefore produce an approximated solution of the error power, by substituting $\mathbf{I}$ instead of $\mathbf{D}_{11}\mathbf{H}_{11}^{-1}$ and solving (28):

$$\mathbf{V} \simeq \frac{\alpha p}{2 - \alpha K P_{\mathrm{S}}}\left(P_{\mathrm{N}}\mathbf{I} + \sigma_w^2\mathbf{D}_{11}\mathbf{D}_{11}^{\mathrm{H}}\right). \quad (34)$$

Substituting in (26) we get:

$$\mathbf{R}_{\boldsymbol{\epsilon}_k} \simeq \left(1 + \frac{\alpha p P_{\mathrm{S}}}{2 - \alpha K P_{\mathrm{S}}}\right)P_{\mathrm{N}}\mathbf{I} + \frac{\alpha p P_{\mathrm{S}}}{2 - \alpha K P_{\mathrm{S}}}\sigma_w^2\mathbf{D}_{11}\mathbf{D}_{11}^{\mathrm{H}}. \quad (35)$$

Note that the above covariance matrix is composed of two terms. The first is the error increase due to the use of the iterative precoder (recall that the error signal covariance matrix with the ideal FEXT cancellation is $P_{\mathrm{N}}\mathbf{I}$). We can see that if $\alpha p P_{\mathrm{S}} \ll 1$, then the error increase is negligible. The second term is due to the quantization error, and does not depend on the noise power.

IV. Simulations

To demonstrate the results derived in the previous section we conducted several simulations using channel measurements carried out by France Telecom[3].

We start by demonstrating the convergence of the suggested algorithm for FEXT cancellation of all 28 users in a binder at a distance of 300 meters. Figure 2 depicts the signal to noise

[3]The authors would like to thank M. Ouzzif, R. Tarafi, H. Marriott and F. Gauthier of France Telecom R&D, who conducted the VDSL channel measurements under the auspices of the U-BROAD project.

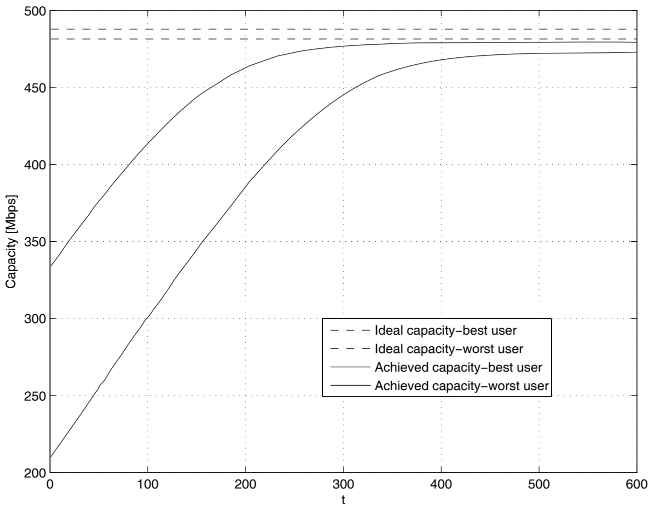

Fig. 4. Convergence of best and worst users' capacities at a distance of 300m.

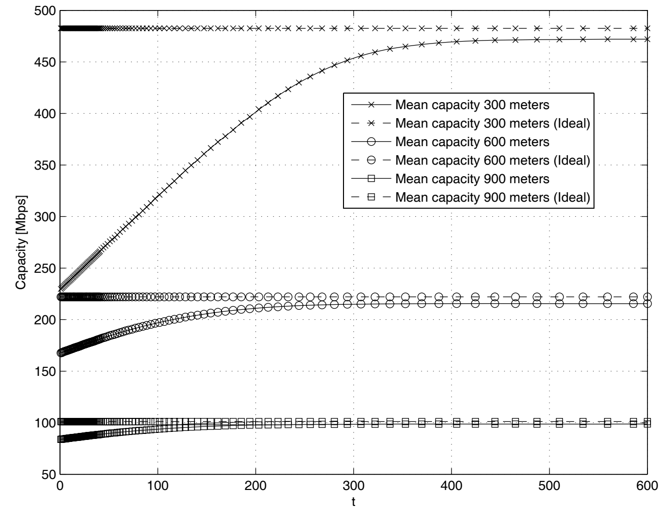

Fig. 5. Convergence of users' capacities at distances of 300, 600 and 900 meters in a mixed binder.

plus interference ratio (SINR) achieved by each of the users at a frequency of 27.4MHz over a time interval of 600 symbols. The transmitted PSD is set to -60dBm/Hz and the noise PSD is -140dBm/Hz. In this simulation we did not take into account the error signal quantization. Calculating the characteristics of the measured channel matrix we get: $\beta_{\max} = 0.91$ and $\gamma_{\max} = 0.86$, and we set $\alpha P_{\mathrm{S}} = 0.014$. The SINR at time 0 indicates the SINR achieved without FEXT cancellation, and the dashed lines show the SNRs achievable using ideal ZF precoding. It can be seen that after 400 symbols, all users' SINR converged to within 1.5dB from the ideal SNR.

To further study this scenario, Figure 3 depicts the histogram of all users' steady state SINR losses. The approximated steady state error increase calculated from (35) in this case was 0.94dB (shown by the dashed-dotted line in the figure). It can be seen that the approximation is very good, and provides a useful tool for system design.

Extending the simulation to the whole system bandwidth (30MHz), Figure 4 depicts the user capacity over the interval of the same 600 symbols. The figure shows only the capacity of the best and worst users (out of the 28 users simulated). The worst user starts with a capacity of only 210Mbps (without FEXT cancellation). Both users converge to a capacity that is less than 10Mbps of the ideal ZF capacity (480Mbps for the worst user).

Next, we studied the performance in a mixed length binder. We simulated the algorithm over a binder with 10 pairs of 300m, 9 pairs of 600m and 9 pairs of 900m. We set the threshold SNR to $\rho_{\mathrm{L}} = 15$dB (so that users that achieve an SNR lower than 15dB at a frequency bin will not use FEXT cancellation for that frequency bin). The mean user capacity for each group of users is depicted in Figure 5. As can be seen, the capacity for all distances converged to nearly its optimal value within less than 600 symbols. As expected the users located further away from the ONU benefited less from the FEXT cancellation, but converged faster.

Note that in this simulation the 900 meter users achieve an SINR which was almost equal to their ideal SNR. This is not trivial, since for many frequency bins these users did not use FEXT cancellation. In general the FEXT cancellation used by other users may even increase the FEXT for the non-canceled users. However, due to the diagonal dominance of VDSL channels, the precoder matrix is typically close to the unity matrix, and any FEXT increase is negligible (as can be verified in this simulation).

V. Conclusions

In this paper we presented an analysis of an adaptive multichannel downstream VDSL precoder. Analysis shows that with the correct choice of parameters, the precoder will converge in almost all practical channels. Furthermore, the analysis also presented bounds that allow for the proper choice of parameters, and predict the expected performance. The proposed precoder can be used for FEXT cancellation for all users or only some of them. Partial FEXT cancellation has important implication due to practical constraints (e.g., availability of appropriate end-user equipment or limited system complexity). We also showed that FEXT cancellation for users with low SNR is not advisable analytically.

These results are also supported by numerical simulations which showed precoder convergence within less than 400 OFDM symbols.

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