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Minimal quasi-separable realizations for the inverse of a quasi-separable operator

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Abstract

We derive minimal quasi-separable (i.e. state space) representations for the upper and lower parts of the inverse of an invertible but otherwise general operator T which itself is given by its upper and lower minimal quasi-separable representations. We show that if the original representation is given in an adequate normal form, then the computation of the representation of the inverse can be done in a single downward or upward pass, involving only small, local computations. So called 'intrinsic factors' play an essential role in the derivation. We define them and show how they can be extracted. The results are given in closed form, provided one accepts the computation of a basis for a space and its orthogonal complement as numerically closed (QR type factorizations, common in 'array computations'). The central workhorse is the classical square root algorithm utilized here in a generalized form.

Keywords: Inner–outer factorization; Lyapunov–Stein equation; Time-varying systems; QR factorization; ULV-decomposition; Quasi-separable operators; Semi-separable operators; Intrinsic factors

1. Preliminaries

The object of this study are linear operators, and in particular matrices, given by *quasi-separable* or state space representations and acting between Hilbert spaces of *non-uniform* sequences. These operators can be characterized by doubly indexed block-matrices, in which each block has an additional characterization in terms of a 'state space model'. In the recent literature, such matrices have also been called 'semi-separable' or 'matrices of low Hankel rank'. Originally, a subclass

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was identified in the study of structural properties of the inverses of banded matrices [20]. In the eighties, low rank approximations of submatrices of semi-separable type appeared in the study of integral equations [16], where it was noted that an LU factorization can be obtained efficiently when semi-separability can be used systematically. In the early nineties, the connection with time-varying dynamical system theory was established and new ways of handling the numerics were proposed [6]. In particular, it was shown in [7] and [8] that backward stable numerical algorithms for the inversion of such structured systems can be obtained using orthogonal transformations, yielding transformation matrices or operators with the same efficient structural properties. These basic methods were later refined by a number of authors [21,9,19,22], yielding a variety of precise characterizations for efficient inversion of the system of equations.

In this paper we aim at completing these results by providing another way of characterizing the (Moore–Penrose or regular) inverse system of a system given by a quasi-separable (i.e. state space) representation, namely by a guaranteed minimal quasi-separable representation for the inverse system itself. The algorithmic procedure follows largely the method originally presented in [8] and later adopted by most authors, but it adds some essential components to it, namely the characterization of intrinsic inner factors and the state space realization of the actual inverse, instead of intermediaries. This approach provides closed formulas for the state space characterization of the inverse that are guaranteed to be minimal in state space complexity. The formulas so obtained have, to our knowledge, not yet been presented in the literature and hence complete the work of the authors stated above. Interestingly enough, the computations can be done entirely on the original representation using local computations of the same complexity as the matrix inversion presented elsewhere. In addition, the desirable property of providing for a 'one pass' algorithm is also maintained, provided the original representation satisfies some structural properties (it has to be in a specific normal form to start with), and the user can live with a final representation that has not been split in upper and lower parts (the splitting entails a recursion that necessarily runs in the opposite direction but is not necessary to achieve a minimal quasi-separable representation of the end result).

The operators we consider may be of infinite dimension without much added difficulty, we just assume that they possess finite dimensional local characterizations (so called 'locally finite systems'). Hence, their representations are given in terms of finite matrices (in most practical cases in terms of a finite set of matrices). Finite matrices themselves may also profitably be represented in this way, the theory is applicable to that important special case. The properties of quasi-separable systems and state space representations of time-varying systems has been extensively studied in the book [9], whose notation we adopt here throughout and refer reader to it for further information.

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ denote respectively the set of natural numbers, the set of integers and the set of complex numbers. A space of non-uniform sequences consists of row vectors of possibly infinite length $u = [\ldots, u_{-1}, \boxed{u_0}, u_1, \ldots]$, each component of which is a finite dimensional row vector (the box identifies the 0th element of the sequence). To each element u of a non-uniform space we adjoin an indexed collection of non-negative integers $\{N_i \in \mathbb{N}, i \in \mathbb{Z}\}$ such that N_i is the dimension of the vector u_i. The sequence N, $N = [\ldots, N_{-1}, \boxed{N_0}, N_1, \ldots]$ is called the *index sequence* of u. Define $\mathcal{N}_i = \mathbb{C}^{N_i}$, then u belongs to the space $\mathcal{N} = \cdots \times \mathcal{N}_{-1} \times \mathcal{N}_0 \times \mathcal{N}_1 \times \cdots$. Since it is possible that some of components of u have zero dimension (i.e. is empty), we put a marker or a placeholder $u_i = \cdot$ when this case occurs. Calculation rules with vectors or matrices that have blocks of dimension zero are defined as follows. The product of an "empty" matrix of dimension $m \times 0$ and an empty matrix of dimension $0 \times n$ is a matrix of dimension $m \times n$ with zero entries.

Other rules of block matrix multiplication, such as the rule of 'matching dimensions', remain valid. With $\ell_2^{\mathcal{N}}$, we denote the space of non-uniform sequences for which $\sum_{i=-\infty}^{\infty}\|u_i\|_2^2$ is finite. The inner product of two non-uniform sequence $f, g \in \mathcal{N}$ is defined as $(f, g) = \sum_i (f_i, g_i)$ where $(f_i, g_i) := f_i g_i^*$ and the '$*$' denotes the Hermitian conjugate. A linear operator $T : \mathcal{M} \to \mathcal{N} : y = uT$ is called bounded if and only if for each $u \in \ell_2^{\mathcal{M}}$ the result $y = uT$ is in $\ell_2^{\mathcal{N}}$ and the induced operator norm defined as $\|T\| = \sup\{\|uT\|,\ u \in \ell_2^{\mathcal{M}}, \|u\| = 1\}$ is bounded. The space of bounded linear operators will be denoted by $\mathcal{X}(\mathcal{M}, \mathcal{N})$ or shortly by $\mathcal{X}$ whenever it is clear what is meant. The operator can be represented by a doubly infinite matrix

$$T = \begin{bmatrix} \ddots & & \vdots & & \\ & T_{-1,-1} & T_{-1,0} & T_{-1,1} & \\ & T_{0,-1} & T_{0,0} & T_{0,1} & \\ & T_{1,-1} & T_{1,0} & T_{1,1} & \\ & & \vdots & & \ddots \end{bmatrix}$$

whose block entry on the i, j position is the matrix $T_{i,j} : \mathbb{C}^{M_i} \to \mathbb{C}^{N_j}$ of size $M_i \times N_j$. We define the following subspaces of bounded operators: $\mathcal{U} = \{T \in \mathcal{X} : T_{i,j} = 0 \text{ for } i > j\}$, $\mathcal{L} = \{T \in \mathcal{X} : T_{i,j} = 0 \text{ for } i < j\}$ and $\mathcal{D} = \mathcal{U} \cap \mathcal{L}$ which will be referred to as subspaces of *causal-*, *anti-causal* and operators reducing to a constant respectively. If T consists of both causal and anti-causal parts, it is called of a *mixed causality* operator.

To define the equivalent of reachability and observability spaces in the present setting, an extension of the input and output spaces is needed. On a block matrix $T \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ a Hilbert–Schmidt norm may be defined as

$$\|T\|_{\mathrm{HS}}^2 := \sum_{i,j} \|T_{ij}\|_{\mathrm{F}}^2,$$

where the term $\|T_{ij}\|_{\mathrm{F}}^2$ denotes the squared Frobenius norm of the matrix T_{ij} and is defined as the sum of the squared magnitudes of entries from $T_{i,j}$. We denote with $\mathcal{X}_2 = \{T \in \mathcal{X}(\mathcal{M}, \mathcal{N}) : \|T\|_{\mathrm{HS}}^2 < \infty\}$ the subspace of $\mathcal{X}$ consisting of operators bounded in the sense of the Hilbert–Schmidt norm. Subspaces $\mathcal{U}_2$, $\mathcal{L}_2$ and $\mathcal{D}_2$ of $\mathcal{X}_2$ are defined in a similar way.

On sequences of type $\ell_2(\mathcal{M})$ a 'causal shift operator' Z may be defined as $(uZ)_k = u_{k-1}$. Its (formal) adjoint $Z^* = Z^{-1}$ is the elementary anti-causal shift operator for which $(uZ^*)_k = u_{k+1}$. Because of the non-uniformity of the sequences, shifts are dependent on the definition of the space $\mathcal{M}$. This dependence is usually suppressed. We will also need diagonal shifts on an operator T, defined as $T^{(1)} = Z^*TZ$. It represents a one unit shift downwards in the South-East direction. Likewise, the upward shift in the North-West direction will be $T^{(-1)} = ZTZ^*$.

Related to the spaces $\mathcal{U}_2$, $\mathcal{L}_2$ and $\mathcal{D}_2$, we define orthogonal projections

$$\mathbf{P}(\cdot) : \mathcal{X}_2 \to \mathcal{U}_2, \quad \mathbf{P}'(\cdot) : \mathcal{X}_2 \to \mathcal{L}_2 Z^{-1}, \quad \mathbf{P}_0(\cdot) : \mathcal{X}_2 \to \mathcal{D}_2$$

on an operator from $\mathcal{X}_2$ to select its block upper, block lower and block diagonal parts respectively.

The notion of projection on upper or lower triangular spaces of operators does not generalize to the (Banach-) algebra of operators. The projection $\mathbf{P} : \mathcal{X} \to \mathcal{U}$ is only defined on a dense subset, it is not true that the upper part of a bounded operator is necessarily bounded itself. In our treatment, we shall hardly encounter this difficulty. However, the restriction to single diagonals is a bounded operator.

We say that $T \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ is *quasi-separable* if it is possible to find a set of finite dimensional matrices;

$$\{(A_c)_k, (B_c)_k, (C_c)_k, (A_a)_k, (B_a)_k, (C_a)_k, (D)_k\}, \quad k \in \mathbb{Z} \tag{1}$$

and a set of intermediate state subspaces $\mathcal{B}_{c,k}$ and $\mathcal{B}_{a,k}$, such that

$$\left\{\begin{array}{ll} (A_c)_k \in \mathcal{D}(\mathcal{B}_{c,k}, \mathcal{B}_{c,k+1}), & (A_a)_k \in \mathcal{D}(\mathcal{B}_{a,k+1}, \mathcal{B}_{a,k}), \\ (B_c)_k \in \mathcal{D}(\mathcal{M}_k, \mathcal{B}_{c,k+1}), & (B_a)_k \in \mathcal{D}(\mathcal{M}_k, \mathcal{B}_{a,k}), \\ (C_c)_k \in \mathcal{D}(\mathcal{B}_{c,k}, \mathcal{N}_k), & (C_a)_k \in \mathcal{D}(\mathcal{B}_{a,k+1}, \mathcal{N}_k), \\ (D)_k \in \mathcal{D}(\mathcal{M}_k, \mathcal{N}_k) & \forall k \in \mathbb{Z} \end{array}\right\} \tag{2}$$

and the entries of T have the following form:

$$\begin{cases} \text{if } i < j: & T_{i,j} = (B_c)_i (A_c)_{i+1} \cdots (A_c)_{j-1} (C_c)_j, \\ \text{if } i = j: & T_{i,i} = D_i, \\ \text{if } i > j: & T_{i,j} = (B_a)_j (A_a)_{j+1} \cdots (A_a)_{i-1} (C_a)_i. \end{cases} \tag{3}$$

The set of matrices defining the quasi-separable form are called a *realization*, following the tradition of dynamical system theory where these notions were first introduced. Also, the intermediate spaces $\mathcal{B}_{a,c}$ are called 'state spaces' as they contain intermediate data. These spaces can be chosen uniquely of minimal dimension, in which case one talks of 'minimal realizations'. These can be found by considering range and co-range of an associated operator called the Hankel operator. For further details we refer to [9].

Introducing the intermediate state space variables $x_k = \begin{bmatrix} x_{c,k} & x_{a,k} \end{bmatrix} \in \mathcal{B}_{c,k} \oplus \mathcal{B}_{a,k},\ k \in \mathbb{Z}$, at level k of the realization, then (3) is equivalent to a representation of the operator $T \in \mathcal{X}(\mathcal{M}, \mathcal{N})$ as an implicit sequence of equations

$$\langle T \rangle_k = \left[\begin{array}{cc|c} (A_c)_k & & (C_c)_k \\ & (A_a)_k & (C_a)_k \\ \hline (B_c)_k & (B_a)_k & (D)_k \end{array}\right], \tag{4}$$

$$\left[\begin{array}{cc|c} x_{c,k+1} & x_{a,k} & y_k \end{array}\right] = \left[\begin{array}{cc|c} x_{c,k} & x_{a,k+1} & u_k \end{array}\right] \langle T \rangle_k, \quad k \in \mathbb{Z}, \tag{5}$$

where $u_k \in \mathcal{M}_k$ and $y_k \in \mathcal{N}_k$ for $k \in \mathbb{Z}$. These equations are called 'state space equations'. The dimension of x_k in a minimal realization is called the state space dimension or *s-dim* at level k. Collecting them over k produces $\#T$, the sequence of minimal dimensions of the state spaces for T, also called the 'degree of T'. In the case of finite matrices, realizations and minimal realizations as defined will always exist, their use only becomes interesting when their dimensions are small. It turns out that in that case the computational complexity for most problems can be kept linear in the overall dimension of the matrix (and often quadratic or cubic in the state space dimension, depending on the problem).

The realization will be called uniformly exponentially stable (u.e.s.) if the non-negative quantities ℓ_{A_c} and ℓ_{A_a} defined by

$$\ell_{A_c} := \lim_{l \to \infty} \left(\sup_k \|(A_c)_{k+1} (A_c)_{k+2} \ldots (A_c)_{k+l}\| \right)^{1/l} \quad \text{and}$$

$$\ell_{A_a} := \lim_{l \to -\infty} \left(\sup_k \|(A_a)_{k-1} (A_a)_{k-2} \ldots (A_a)_{k-l}\| \right)^{1/l}$$

are smaller than one. We write $\ell_{A_c} < 1$ (resp. $\ell_{A_a} < 1$) to denote that the causal (resp. anti-causal) part of T, $\mathbf{P}(T)$ (resp. $\mathbf{P}'(T)$), has a u.e.s. minimal realization.

To keep notations as simple as possible, it is convenient to put the set of matrices of a realization into a diagonal form. For the set $\{(D)_k\}$, $k \in \mathbb{Z}$ we define $D = \mathrm{diag}[\ldots, D_{-1}, \boxed{D_0}, D_1, \ldots]$. Block diagonals A_c, B_c, C_c, A_a, B_a and C_a are defined similarly.

A (minimal) realization $\langle T\rangle$ for T is not unique. If $R_c, R_a \in \mathscr{D}$ are any two boundedly invertible block diagonals of appropriate s-dim sequence, then

$$\{R_c^{-1}A_cR_c^{(-1)},\ B_cR_c^{(-1)},\ R_c^{-1}C_c,\ R_a^{-(-1)}A_aR_a,\ B_aR_a,\ R_a^{-(-1)}C_a,\ D\}$$

is a minimal realization for T also. The opposite direction holds as well, given two minimal u.e.s. realizations for the operator under consideration, there exists a invertible state space transformation which relates them. In the case of uniform exponential stability $l_{A_c} < 1$ (resp. $l_{A_a} < 1$), it can be shown (see [9]) that the operator $(I - A_cZ)^{-1}$ (resp. $(I - A_aZ^*)^{-1}$) exists as a causal (resp. an anti-causal) operator (if $l_A < 1$ then the Neumann series $\sum_{i=0}^{\infty}(AZ)^i$ converges to $(I - AZ)^{-1}$). Then, the u.e.s. property for $\langle T\rangle$ implies the existence of a *transfer function representation* for T in the form

$$T = D + B_cZ(I - A_cZ)^{-1}C_c + B_aZ^*(I - A_aZ^*)^{-1}C_a.$$

Similarly as in the linear time-invariant system theory, a realization for a time-varying system is *minimal* iff it is both *reachable* and *observable*. Equivalently, a realization is minimal iff its reachability and observability *Gramians* being solutions of appropriate Lyapunov–Stein equations, are invertible. If (A, B) is only *partially reachable* then there is a properly partitioned unitary state space transform $Q \in \mathscr{D}$, $QQ^* = I$, $Q^*Q = I$ and $a_{11}, a_{21}, a_{22}, b_1 \in \mathscr{D}$ such that

$$a = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} = Q^*AQ^{(-1)}, \qquad b = \begin{bmatrix} b_1 & 0 \end{bmatrix} = BQ^{(-1)},$$

whereby (a_{11}, b_1) is a reachable pair and it holds that $l_{a_{ii}} < 1$, $i = 1, 2$. The pair (a, b) will be referred to as the *Kalman canonical form* for the pair (A, B). The Kalman canonical form for a *partially observable* (A, C) pair is defined dually.

We shall also make use of pseudo-inverses and summarize here the main properties we shall be using. Suppose that Y has linearly independent rows (i.e. YY^* is non-singular), then the Moore–Penrose pseudo-inverse of Y is given by

$$Y^{\mathrm{M}} := Y^*(YY^*)^{-1}. \tag{6}$$

The Moore–Penrose inverse solves the least squares problem in the Euclidean operator norm

$$\mathrm{argmin}\left(\|\mathrm{argmin}_x\|xY - b\|_2\|_2\right) = bY^{\mathrm{M}}$$

for given Y and b. A more general right pseudo-inverse for Y with linearly independent rows is any conformal matrix $Y^\dagger$ that satisfies $YY^\dagger = I$.

The following property holds:

Lemma 1. *Let the rows of $Y^\perp$ form an orthonormal basis for the orthogonal complement of the space generated by the rows of Y, and let Y^{M} be the Moore–Penrose inverse of Y, then any pseudo-inverse for Y is found as*

$$Y^\dagger = Y^{\mathrm{M}} + Y^{\perp *}X \tag{7}$$

in which X is a conformal but otherwise arbitrary matrix.

Next, we look at embeddings. We have the following lemma.

Lemma 2. *Suppose that* Y_1 *and* Y_2 *are matrices with rows of the same dimension and suppose that the row stack*

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

has independent rows. Then

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}^{\mathrm{M}} = \begin{bmatrix} Y_1^{\mathrm{M}} & Y_2^{\mathrm{M}} \end{bmatrix} \tag{8}$$

if and only if the row spans of Y_1 *and* Y_2 *are orthogonal.*

Proofs of the two lemmas are elementary.

2. System inversion and the inner–outer square-root algorithm

Our first main result concerns the minimal realization of the inverse of a bounded and block upper matrix (or operator) T:

$$T = D + BZ(I - AZ)^{-1}C,$$

given by the minimal u.e.s. realization

$$\langle T \rangle = \{A, B, C, D\}.$$

In this and the next section we assume that the inverse T^{-1} of T exists as a bounded, but not necessarily block upper operator. Our goal will be to derive efficient and backward stable quasi-separable representations for this inverse. Our main workhorse will be the square-root algorithm for inner–outer factorization, which we now introduce. We follow the notation of [9] throughout and refer to it for further information on the relation between semi-separable matrices and time-variant systems.

It is known from the work cited that the operator T can be factored as $T = UT_o$ in which U is a block upper unitary operator and T_o is a block upper operator with block upper inverse, a so-called 'outer operator' or 'minimal-phase'. Both U and T_o possess state space realizations that are at most of the same local degree (the local dimension of the state) as the degree of T. The factorization is unique except for a trivial unitary block-diagonal operator to right of U with conjugate left of T_o. These state space representations are found by the famous square root algorithm for inner–outer factorization [10]. This square root algorithm involves an intermediary block-diagonal matrix $Y = \mathrm{diag}(Y_k)$ and produces the state space realization matrices

$$\langle U \rangle = \begin{bmatrix} A_U & C_U \\ B_U & D_U \end{bmatrix} \quad \text{and} \quad \langle T_o \rangle = \begin{bmatrix} A & C \\ B_o & D_o \end{bmatrix}$$

as the solution of the forward recursion of block QL factorizations:

$$\begin{bmatrix} Y_k A_k & Y_k C_k \\ B_k & D_k \end{bmatrix} = \begin{bmatrix} A_{Uk} & C_{Uk} \\ B_{Uk} & D_{Uk} \end{bmatrix} \begin{bmatrix} Y_{k+1} & 0 \\ B_{ok} & D_{ok} \end{bmatrix} \quad k = \cdots - 1, 0, 1, \ldots. \tag{9}$$

In these equations the initial Y_{k_o} at some index k_o is assumed known throughout. Its determination depends on the type of the time-varying system involved. In case the system starts at a given point k_0, the initial T_{k_0} can be taken empty. This will be the case when a singly infinite system is considered. Another interesting case is the IV case, or invariant–variant case, where the initial value is found as a fixed point solution for the square root equation [10]. In [9] it is shown that even the more general case will always yield a bounded fixed point solution provided the original realization is u.e.s. It is not hard to see that there is a related Riccati equation for which Y^*Y is the positive definite solution, but in view of the fact that the solution can be found directly from the square root equations, it is not advisable to attempt a solution of the more complex Riccati equation, which moreover is often ill conditioned (while the square root system is well conditioned [10]).

Lemma 3. *If the original realization* $\{A, B, C, D\}$ *is in input normal form, then each* Y_k *in* (9) *is contractive.*

Proof. Premultiplying (9) with $\begin{bmatrix} A_U^* & B_U^* \\ C_U^* & D_U^* \end{bmatrix}$ and taking the (1, 1) entry produces

$$A_{Uk}^* Y_k A_k + B_{Uk}^* B_k = Y_{k+1}.$$

Solving for the fixed point diagonal solution $Y = \mathrm{diag}[Y_k]$ of this set of equations (see [9]) gives $Y = \mathbf{P}_0(\mathscr{R}_U \mathscr{R}^*)$, in which $\mathscr{R}_U$ is an orthonormal sliced basis for the reachability space of U, and $\mathscr{R}$ likewise for T, since the realization for T was assumed in input normal form. Hence Y is obtained as the projection of one orthonormal basis on another and has to be contractive. □

The system will contain an intrinsic inner factor, i.e. we will be able to write $T = T'V$ for some inner V such that $\#T = \#T' + \#V$ iff some Y_k contain an isometric part. We shall assume that if this is the case, then the intrinsic part V has been extracted, leaving a reduced T' for further processing. The inverse will then also contain an intrinsic anti-causal part V^* which can be handled separately. Hence we will be allowed to assume that the local Y_k is strictly contractive and no intrinsic part is present after extraction, which is handled by the following theorem.

Theorem 1. *Let T be a causal locally finite operator with u.e.s. realization* $\{A, B, C, D\}$ *in input normal form, and assume that Y is the solution of the square root algorithm given by* (9). *Then*: (1) *each Y_k can be written as*

$$Y_k = \begin{bmatrix} Y_{1,k} \\ Y_{2,k} \end{bmatrix}$$

in which $Y_{1,k}$ is isometric and $Y_{2,k}$ is strictly contractive; (2) *the co-isometric factor U can be factored as $U = U_1 U_2$ with U_1 inner and such that* (2a) $T = U_1 \widehat{T}$ *with* $\#T = \#U_1 + \#\widehat{T}$ *and* (2b) *the square root equation*

$$\begin{bmatrix} Y_{1,k} A_k & Y_{1,k} C_k \\ B_k & D_k \end{bmatrix} = \begin{bmatrix} A_{U_1,k} & C_{U_1,k} \\ B_{U_1,k} & D_{U_1,k} \end{bmatrix} \begin{bmatrix} Y_{1,k+1} & 0 \\ \widehat{B}_k & \widehat{D}_k \end{bmatrix} \tag{10}$$

for some $\widehat{B}_k$ and $\widehat{D}_k$ belonging to a minimal realization for $\widehat{T}$, and a unitary realization for U_1 holds.

Comments

A few comments are in order here:

- Y_k and any of its subdivisions $Y_{1,k}$ and $Y_{2,k}$ can be empty.

- The original square root algorithm given in (9) produces a Y_k that has maximal dimension under the condition that $\ker(\cdot Y_k) = 0$. In that case the companion factor will define an outer operator. Non-maximal square root equations are possible, but the companion factor will then correspond to an operator which is not outer.

Proof. Let

$$Y_k = P_k \left[\begin{array}{cc|c} I_{Y,k} & 0 & 0 \\ \hline 0 & \sigma_k & 0 \end{array}\right] \begin{bmatrix} Q_{k,1} \\ Q_{k,2} \\ Q_{k,3} \end{bmatrix}$$

be an SVD of Y_k. Since Y_k is contractive, it may have unit singular values (represented by the matrix $I_{Y,k}$), and the singular values in the remainder σ_k will be strictly less than one. There will be no singular values zero because of the kernel condition on Y_k. Hence also

$$P_k^* Y_k = \begin{bmatrix} Q_{k,1} \\ \sigma_k Q_{k,2} \end{bmatrix}$$

and hence $Y_{1,k} =: Q_{k,1}$ is isometric and $Y_{2,k} =: \sigma_k Q_{k,2}$ strictly contractive. Next, we perform state transformations on the data used in the square root algorithm, as follows:

$$\begin{bmatrix} A'_k & C'_k \\ B'_k & D'_k \end{bmatrix} = \begin{bmatrix} Q_k A_k Q^*_{k+1} & Q_k C_k \\ B_k Q^*_{k+1} & D_k \end{bmatrix}$$

and

$$\begin{bmatrix} A'_{U,k} & C'_{U,k} \\ B'_{U,k} & D_{U,k} \end{bmatrix} = \begin{bmatrix} P_k^* A_{U,k} P_{k+1} & P_k^* C_k \\ B_{U,k} P_{k+1} & D_{U,k} \end{bmatrix}.$$

These are orthonormal state transformations that will preserve normal forms. In the new representation, the square root algorithm transforms to (using conformal decompositions of the matrices)

$$\left[\begin{array}{ccc|c} I_{Y,k} & 0 & 0 & \\ 0 & \sigma_k & 0 & \\ \hline & & & I \end{array}\right] \left[\begin{array}{ccc|c} A'_{k,11} & A'_{k,12} & A'_{k,13} & C'_{k,1} \\ A'_{k,21} & A'_{k,22} & A'_{k,23} & C'_{k,2} \\ A'_{k,31} & A'_{k,32} & A'_{k,33} & C'_{k,3} \\ \hline B'_{k,1} & B'_{k,2} & B'_{k,3} & D'_k \end{array}\right]$$
$$= \langle U_k \rangle' \left[\begin{array}{ccc|c} I_{Y,k+1} & 0 & 0 & \\ 0 & \sigma_{k+1} & 0 & \\ \hline B_{o,k,1} & B_{o,k,2} & B_{o,k,3} & D'_{o,k} \end{array}\right]$$

in which

$$\langle U_k \rangle' = \left[\begin{array}{cc|c} A'_{U,k,11} & A'_{U,k,12} & C'_{U,k,1} \\ A'_{U,k,21} & A'_{U,k,22} & C'_{U,k,2} \\ \hline B'_{U,k,1} & B'_{U,k,2} & D'_{U,k} \end{array}\right].$$

Working the product out, and taking the first columns from both sides, one obtains

$$\left[\begin{array}{c} A'_{k,11} \\ \sigma_k A'_{k,21} \\ 0 \cdot A'_{k,31} \\ \hline B'_{k,1} \end{array}\right] = \langle U_k \rangle' \left[\begin{array}{c} I_{Y,k+1} \\ 0 \\ \hline B'_{o,k,1} \end{array}\right].$$

Premultiplying both sides with their adjoints and using the fact that $\begin{bmatrix} A'_{k,\cdot 1} \\ \hline B'_{k,1} \end{bmatrix}$ is co-isometric, because the prime realization for T is in input normal form, we obtain

$$A'^{*}_{k,21}[I - \sigma_k^* \sigma_k] A'_{k,21} + A'_{k,31} A_{k,31} = -B'^{*}_{o,k,1} B_{o,k,1}.$$

It follows immediately that $A'_{k,21} = A'_{U,k,21} = 0$, $A_{k,31} = 0$ and $B_{o,k,1} = 0$, because σ_k is strictly contractive, the first member of the equation must hence be positive definite and the second negative definite, and hence both have to be zero. It now remains to put the pieces back together to obtain the claims of the theorem. The realization for $\langle U_k \rangle'$ has been brought to the form

$$\langle U_k \rangle' = \left[\begin{array}{cc|c} A'_{U,k,11} & A'_{U,k,12} & C'_{U,k,1} \\ 0 & A'_{U,k,22} & C'_{U,k,2} \\ \hline B'_{U,k,1} & B'_{U,k,2} & D'_{U,k} \end{array}\right]$$

in which the first block column is isometric. This has as a direct consequence that U factors accordingly, to be seen as follows. Redefining $A'_{U,k,11} = A'_{U_1,k}$, $B'_{U,k,1} = B'_{U_1,k}$ and completing the first block column to unitary we obtain a unitary realization

$$\langle U_{1,k} \rangle' = \left[\begin{array}{c|c} A'_{U_1,k} & C'_{U_1,k} \\ \hline B'_{U_1,k} & D'_{U_1,k} \end{array}\right]$$

for a first factor U_1 of U. In this form the columns of $\begin{bmatrix} C'_{U_1,k} \\ D'_{U_1,k} \end{bmatrix}$ form and orthonormal basis for the orthogonal complement of the basis generated by the columns of $\begin{bmatrix} A'_{U_1,k} \\ \hline B'_{U_1,k} \end{bmatrix}$. It follows next that the realization for U factors as

$$\left[\begin{array}{cc|c} A'_{U_1,k} & & C'_{U_1,k} \\ & I & \\ \hline B'_{U_1,k} & & D'_{U_1,k} \end{array}\right] \left[\begin{array}{cc|c} I & & \\ & A'_{U_2,k} & C'_{U_2,k} \\ \hline & B'_{U_2,k} & D'_{U_2,k} \end{array}\right]$$

with $A'_{U_2,k} = A'_{U,k,22}$, $C'_{U_2,k} = C'_{U,k,2}$, and the other quantities following directly from the procedure. This is a consequence of the fact that the product

$$\left[\begin{array}{cc|c} A'_{U_1,k} & & C'_{U_1,k} \\ & I & \\ \hline B'_{U_1,k} & & D'_{U_1,k} \end{array}\right]^{-1} \langle U_k \rangle' = \left[\begin{array}{cc|c} I & * & * \\ 0 & A'_{U_2,k} & C'_{U_2,k} \\ \hline 0 & ** & ** \end{array}\right]$$

is unitary and has the form given. Hence the '$*$' quantities have to be zero, while the '$**$'s are determined by the procedure (the procedure given is classical, for more details see [9, p. 403]). Hence U factors as $U = U_1 U_2$ with realizations derived from the product, and the primed square root equation becomes, after dropping the second row:

$$\begin{aligned} &\left[\begin{array}{c|c} I_{Y,k} & \\ \hline & 0 \end{array}\right] \left[\begin{array}{c|c} A'_k & C'_k \\ \hline B'_k & D'_k \end{array}\right] \\ &\quad = \left[\begin{array}{c|c} A'_{U_1,k} & C'_{U_1,k} \\ \hline B'_{U_1,k} & D'_{U_1,k} \end{array}\right] \left[\begin{array}{ccc|c} I_{Y,k+1} & 0 & 0 & 0 \\ \hline 0 & B'_{U_2,k}\sigma_{k+1} + D'_{U_2,k} B'_{o,2,k} & D'_{U_2,k} B'_{o,3,k} & D'_{U_2,k} D'_{o,k} \end{array}\right]. \end{aligned}$$

Converting back to the original non-primed realization and using the definition for $Y_{1,k}$, the partial square root expression (10) is obtained. The fact that U_1 is an intrinsic factor follows from the realization for $\widehat{T}$, which in the primed realization used earlier is given by (dropping the dependence on k)

$$\left[\begin{array}{ccc|c} A'_{11} & A'_{12} & A'_{13} & C'_1 \\ 0 & A'_{22} & A'_{23} & C'_2 \\ 0 & A'_{32} & A'_{33} & C'_3 \\ \hline 0 & B'_{o,2} & B'_{o,3} & D'_o \end{array}\right].$$

The realization is obviously non-minimal. A minimal realization is obtained by dropping the first block row and the first block column, corresponding to states that are unreachable. The resulting realization has degree $\#T - \#U_1$ (the degree corresponds to the number of rows), and minimality follows directly from the fact that the degree of a product cannot be larger than the sum of the degrees of the factors. Explicit expressions for the reduced system are obtained by forcing the uncontrollable state to be zero, i.e. by putting $A'_{1,i} = 0$ for $i = 1, 2, 3$, and transforming back:

$$\begin{aligned} \widehat{A}_k &= \begin{bmatrix} Q^*_{k,2} & Q^*_{k,3} \end{bmatrix} \begin{bmatrix} A'_{k,22} & A'_{k,23} \\ A'_{k,32} & A'_{k,33} \end{bmatrix} \begin{bmatrix} Q_{k+1,2} \\ Q_{k+1,3} \end{bmatrix}, \\ \widehat{B}_k &= \begin{bmatrix} B'_{k,2} & B'_{k,3} \end{bmatrix} \begin{bmatrix} Q_{k+1,2} \\ Q_{k+1,3} \end{bmatrix}, \\ \widehat{C}_k &= \begin{bmatrix} Q^*_{k,2} & Q^*_{k,3} \end{bmatrix} \begin{bmatrix} C'_{k,2} \\ C'_{k,3} \end{bmatrix}, \\ \widehat{D}_k &= D_k, \end{aligned}$$

producing a reduced system after extraction of the intrinsic factor (in practice, however, subsequent computations as detailed next would be done on the primed system). □

The converse of the theorem is true also:

Theorem 2. *Let T be a locally finite causal operator with u.e.s. realization $\{A, B, C, D\}$ in input normal form and such that $T = U\widehat{T}$ with U isometric and $\#T = \#U + \#\widehat{T}$, and let Y be the bounded solution of the square root equation* (9), *then $Y_k = \begin{bmatrix} Y_{k,1} \\ Y_{k,2} \end{bmatrix}$ with $Y_{k,1}$ isometric of dimension $\#U$ and $Y_{k,2}$ contractive.*

The proof is straightforward. It retraces some of the steps of the previous theorem in a backward direction.

Assuming that intrinsic factors have been dealt with and that each Y_k resulting from the square root algorithm is contractive, our goal is now to find a closed and minimal mixed representation for T^{-1}. To that aim, we explore the properties of $\Delta_o = A - CD_o^{-1}B_o$ further. We refer to the square root algorithm given and the diagonal sequence of matrices Y_k defined therein (represented in the block diagonal operator diag(Y_k)). Let $Y_k^{\perp}$ form a (row) basis for the orthogonal complement of the space spanned by the rows of Y_k at each point k, and let $\Delta_{ok} = A_k - C_k D_{ok}^{-1} B_{ok}$ as before.

We have the following property related to the square root algorithm.

Lemma 4. *Let $Y_k^{\perp}$ be a matrix whose rows form an orthonormal basis for the orthogonal complement of the space generated by the rows of Y_k, then for all k*

$$Y_k \Delta_{ok} Y_{k+1}^{\perp *} = 0. \tag{11}$$

Proof. The square root algorithm in the present case gives

$$\begin{bmatrix} Y_k A_k & Y_k C_k \\ B_k & D_k \end{bmatrix} = \begin{bmatrix} A_{U,k} & C_{U,k} \\ B_{U,k} & D_{U,k} \end{bmatrix} \begin{bmatrix} Y_{k+1} & \\ B_{o,k} & D_{o,k} \end{bmatrix}.$$

By construction, Y_{k+1} has a right inverse, while $D_{o,k}$ is invertible (it is by construction right invertible, but since T is assumed invertible, the corresponding T_o will have an invertible main diagonal D_o). Taking a right inverse of the rightmost factor and introducing an arbitrary right inverse $Y_{k+1}^{\dagger}$ for Y_{k+1} we find

$$\begin{bmatrix} Y_k \Delta_{o,k} Y_{k+1}^{\dagger} & Y_k C_k D_{o,k}^{-1} \\ (B_k - D_{o,k}^{-1} B_{o,k}) Y_{k+1}^{\dagger} & D_k D_{o,k}^{-1} \end{bmatrix} = \begin{bmatrix} A_{U,k} & C_{U,k} \\ B_{U,k} & D_{U,k} \end{bmatrix}.$$

Since $Y_{k+1}^{\dagger}$ is an arbitrary pseudo-inverse, and because of Lemma (1) we can write $Y_{k+1}^{\dagger} = Y_{k+1}^{\mathrm{M}} + Y_{k+1}^{\perp *} X$ for a fitting but otherwise arbitrary matrix X and with Y_{k+1}^{M} the Moore–Penrose right inverse of Y_{k+1}

$$Y_k \Delta_{o,k} (Y_{k+1}^{\mathrm{M}} + Y_{k+1}^{\perp *} X) = Y_k \Delta_{o,k} Y_{k+1}^{\mathrm{M}}$$

for any conformal matrix X, hence $Y_k \Delta_{o,k} Y_{k+1}^{\perp *} X = 0$ for all fitting X and hence the statement of the lemma, since X is arbitrary. □

Lemma (4) allows for block triangularization of Δ_o. Let

$$R_k = \begin{bmatrix} Y_k^{\perp} \\ Y_k \end{bmatrix}$$

be a non-singular state transformation matrix of size $(m_k^{\perp} + m_k) \times \delta_k$ and applicable on the original state representation for T. Consider the transformed equivalent representation

$$\begin{bmatrix} A_k' & C_k' \\ B_k' & D_k' \end{bmatrix} = \begin{bmatrix} R_k A_k R_{k+1}^{-1} & R_k C_k \\ B_k R_{k+1}^{-1} & D_k \end{bmatrix}, \tag{12}$$

then the new representation will yield a realization of the inverse for which the state transition matrix is in block triangular form. To see this, it is instructive to detail the square root algorithm for the prime representation. We state it in the following lemma.

Lemma 5. *The square root algorithm corresponding to the prime representation given in Eq.* (12) *is given by*

$$\begin{bmatrix} \begin{bmatrix} 0 & I_{m_k} \end{bmatrix} A_k' & \begin{bmatrix} 0 & I_{m_k} \end{bmatrix} C_k' \\ B_k' & D_k' \end{bmatrix} = \begin{bmatrix} A_{U_k} & C_{U_k} \\ B_{U_k} & D_{U_k} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & I_{m_{k+1}} \end{bmatrix} & 0 \\ B_{ok} R_{k+1}^{-1} & D_{ok} \end{bmatrix}. \tag{13}$$

This leads to our first result.

Proposition 1. *The state transition matrix Δ_o for the inverse of T_o satisfies*

$$\Delta_{ok} = R_k^{-1} \begin{bmatrix} \Delta_{o,11,k}' & \Delta_{o,12,k}' \\ 0 & A_{U_{2,k}} \end{bmatrix} R_{k+1}, \tag{14}$$

where $\Delta_{o,11,k}' = Y_k^{\perp} \Delta_{ok} Y_{k+1}^{\perp *}$ and $\Delta_{o,12,k}' = Y_k^{\perp} \Delta_{ok} Y_{k+1}^{\mathrm{M}}$.

Hence we find a conformal primed representation

$$\left[\Delta'_{ok} \;\middle|\; C'_{ok}D_{ok}^{-1}\right] = \left[\begin{array}{cc|c} \Delta'_{o,11,k} & \Delta'_{o,12,k} & C'_{o,1,k}D_{ok}^{-1} \\ 0 & A_{U_k} & C_{U_k} \end{array}\right]$$

in which the second block row is isometric. The first block row however is not necessarily isometric.

Our next goal is to find a closed form, minimal representation for the mixed operator T^{-1}. This can be obtained in a straightforward way from the primed representations, which we relabel for simplicity of notation as the ongoing representation for T_o:

$$\mathbf{T}_o = \left[\begin{array}{cc|c} \Delta_{11} & \Delta_{12} & C_1D_o^{-1} \\ & A_U & C_U \\ \hline -D_o^{-1}B'_{o1} & -D_o^{-1}B'_{o2} & D_o^{-1} \end{array}\right], \quad \mathbf{U} = \begin{bmatrix} A_U & C_U \\ B_U & D_U \end{bmatrix}$$

(notice that the representation for U has not changed in the primed version due to the last lemma, but the other quantities have been transformed).

Theorem 3. *Let m be the diagonal operator solution of the Lyapunov–Stein equation*

$$m_{k-1} = \Delta_{11,k}m_kA^*_{U,k} + C_{1,k}D^{-1}_{o,k}D^*_{U,k} + \Delta_{12,k}A^*_{U,k}$$

then a minimal realization for the inverse of T is given by

$$\begin{aligned} T^{-1} = D_o^{-1}\big\{&(C_U^* - B_{o1}mA_U^* - B_{o2}A_U^*)(I - Z^*A_U^*)^{-1}Z^*B_U^* \\ &+ D_U^* - B_{o1}mB_U^* - B_{o2}B_U^* \\ &- B_{o1}Z(I - \Delta_{11}Z)^{-1}(C_1D_o^{-1}D_U^* + \Delta_{11}mB_U^* + \Delta_{12})\big\}. \end{aligned} \tag{15}$$

The proof follows by direct computation. The fact that the representation is minimal follows directly from the fact that the overall degree of T^{-1} equals the original degree of T.

3. Minimal inverse of a mixed invertible operator

In this section we derive a minimal quasi-separable (or time-varying) representation for the inverse of a mixed operator, assuming its existence (the formulas given will also be directly valid for the Moore–Penrose inverse of a left invertible operator, for a more general Moore–Penrose inverse they become slightly more complicated). We are given

$$T = B_aZ^*(I - A_aZ^*)^{-1}C_a + D + B_cZ(I - A_cZ)^{-1}C_c$$

as a minimal realization for the operator in additive form. Just as for the unilateral case we will write realizations for mixed forms as

$$\langle T\rangle = \left[\begin{array}{cc|c} A_a & & C_a \\ & A_c & C_c \\ \hline B_a & B_c & D \end{array}\right]$$

corresponding to the mixed state equations

$$\left[x_a \quad x_c^{(-1)} \middle| y\right] = \left[x_a^{(-1)} \quad x_c \middle| u\right] \left[\begin{array}{cc|c} A_a & & C_a \\ & A_c & C_c \\ \hline B_a & B_c & D \end{array}\right],$$

with x_a the upward propagating state and x_c the downward. For details see [9]. We are asked to produce a compact or even closed form for the realization of the inverse operator T^{-1}. In this section we shall provide for such a form, by exploitation of the properties of minimal external and inner–outer factorizations. The strategy will be to follow the general inversion plan of [9] in the form proposed in [17], but we shall focus, in contrast on realization with the aim of reducing their complexity by making them. It consists of the following steps:

(1) produce the minimal external factorization

$$UT = T' \in \mathcal{U};$$

(2) then produce the (automatically minimal) outer–inner factorization

$$T' = T_o'V.$$

T_o' is known to be at least locally invertible (it may not be u.e.s. in which case it would only have dense range). At this point realizations have been obtained for the factors in

$$T = U^*T_o'V.$$

A similar factorization for the inverse is found by direct inversion

$$T^{-1} = V^*T_o'^{-1}U.$$

and the final (new) step will consist in

(3) reduce the factorization of the inverse to a closed minimal form.

The factorization of the inverse has exactly the same form as the factorization of the original, with the roles of U and V reversed. The final reduction will parallel the initial construction steps.

Important in the subsequent derivation is the observation that $T_o'^{-1}U$ is a minimal outer–inner factorization in $\mathcal{U}$ and $V^*[T_o'^{-1}U]$ a minimal external factorization. We derive minimal realizations for the subsequent factors, beginning with T'. For ease of notation, we indicate realizations with '$\langle\,\rangle$' brackets. Assuming the anti-causal part in the original realization in 'Input Normal Form' (INF)—i.e. $A_a^*A_a + B_a^*B_a = I$, which we can always assume, we find:

$$\langle T'\rangle = \left[\begin{array}{cc|c} A_a^* & B_a^*B_c & A_a^*C_a + B_a^*D \\ & A_c & C_c \\ \hline B_U & D_UB_c & C_UC_a + D_UD \end{array}\right]. \tag{16}$$

Proof. Since the anti-causal part is given in INF form, a realization for U is obtained as [9]

$$U = D_U + C_UZ(I - A_a^*Z)^{-1}B_a^*$$

in which

$$\langle U\rangle = \begin{bmatrix} A_a^* & B_a^* \\ B_U & D_U \end{bmatrix}$$

is a unitary matrix. The realization follows then from working out the product $T' = UT$ (the computation is standard in time-varying system theory). □

Next, the outer–inner factorization $T' = T_o'V$ is performed in the same way as in the previous section (we skip details for the time being). As stated before, T_o' inherits the reachability data from T' and we obtain

$$\langle T_o'\rangle = \left[\begin{array}{cc|c} A_a^* & B_a^*B_c & C_{o1} \\ & A_c & C_{o2} \\ \hline B_U & D_UB_c & D_o \end{array}\right]$$

in which the data marked with the subscript 'o' is new data obtained by the outer–inner realization. We assume that intrinsic factors have been removed as in the previous section, and hence that the realization is minimal.

In the next step, we endeavor to find a minimal realization for the product U^*T_o'. It turns out that this product is easy to evaluate, many terms cancel, and collecting the others produces

$$\langle U^*T_o' \rangle = \left[\begin{array}{cc|c} A_a & & B_U^* D_o + A_a C_{o1} \\ & A_c & C_{o2} \\ \hline B_a & B_c & D_U^* D_o + B_a C_{o1} \end{array}\right]. \tag{17}$$

Now we know that U^*T_o' has the causal inverse $T_o'^{-1}U$, this fact allows us to determine a realization for it. The technique to find this realization is straightforward but a little involved, we document the important steps (notice the redefinition of terms for simplicity).

Lemma 6. *Suppose*

$$\langle T \rangle = \left[\begin{array}{cc|c} A_a & & C_a \\ & A_c & C_c \\ \hline B_a & B_c & D \end{array}\right]$$

is a minimal mixed realization of a mixed-causality invertible system, the inverse of which is causal and contains no intrinsic inner factors. Let the anti-causal part be in INF, and let B_U and D_U form the unitary completion

$$\begin{bmatrix} A_a^* & B_a^* \\ B_U & D_U \end{bmatrix}.$$

Define $\delta = (B_U C_a + D_U D)$.

Then δ is invertible and

$$\langle T^{-1} \rangle = \left[\begin{array}{cc|c} \Delta_{11} & \Delta_{12} & B_a^* - (A_a^* C_a + B_a^* D)\delta^{-1} D_U \\ \Delta_{21} & \Delta_{22} & -C_c \delta^{-1} D_U \\ \hline \delta^{-1} B_U & \delta^{-1} D_U B_c & \delta^{-1} D_U \end{array}\right],$$

where

$$\Delta = \begin{bmatrix} A_a^* & B_a^* B_c \\ & A_c \end{bmatrix} - \begin{bmatrix} A_a^* C_a + B_a^* D \\ C_c \end{bmatrix} \delta^{-1} \begin{bmatrix} B_U & D_U B_c \end{bmatrix}$$

is a minimal, u.e.s. realization for T^{-1}.

Proof. Let $U_a = D_U + B_U Z (I - A_a^* Z)^{-1} B_a^*$. As before, the external factorization produces

$$\langle U_a T \rangle = \left[\begin{array}{cc|c} A_a^* & B_a^* B_c & A_a^* C_a + B_a^* D \\ & A_c & C_c \\ \hline B_U & D_U B_c & B_U C_a + D_U D \end{array}\right].$$

The causality assumption for T^{-1} has as a consequence that $U_a T$ is actually outer. Hence $(\delta =)$ $(B_U C_a + D_U D)$ is invertible. Formally, $T^{-1} U_a^*$ then has the realization

$$\langle T^{-1} U_a^* \rangle = \left[\begin{array}{cc|c} \Delta_{11} & \Delta_{12} & -(A_a^* C_a + B_a^* D)\delta^{-1} \\ \Delta_{21} & \Delta_{22} & -C_c \delta^{-1} \\ \hline \delta^{-1} B_U & \delta^{-1} D_U B_c & \delta^{-1} \end{array}\right]$$

with Δ and δ as defined in the statement of the theorem. By the assumption that the causal T^{-1} contains no inner intrinsic factors, the realization $\langle T^{-1} U_a^* \rangle$ will be minimal. The transition operator Δ will actually be u.e.s. This fact follows from [9, Proposition 13.2]. The realization for T^{-1} is now easily deduced by reducing the direct realization for the product $[T^{-1} U_a^*] U_a$

$$\left[\begin{array}{ccc|c} \Delta_{11} & \Delta_{12} & -(A_a^*C_a + B_a^*D)\delta^{-1}B_U & -(A_a^*C_a + B_a^*D)\delta^{-1}D_U \\ \Delta_{21} & \Delta_{22} & -C_c\delta^{-1}B_U & -C_c\delta^{-1}D_U \\ 0 & 0 & A_a^* & B_a^* \\ \hline \delta^{-1}B_U & \delta^{-1}D_U B_c & \delta^{-1}B_U & \delta^{-1}D_U \end{array}\right].$$

Applying the state transformation $R := \begin{bmatrix} I & & -I \\ & I & \\ & & I \end{bmatrix}$ on the 'A, B and C terms' from the previous realization, respect. $R^{-1}(\cdot)R^{(-1)}$, $(\cdot)R^{(-1)}$, $R^{-1}(\cdot)$ produces the alternative (non-minimal) realization

$$\left[\begin{array}{ccc|c} \Delta_{11} & \Delta_{12} & 0 & B_a^* - (A_a^*C_a + B_a^*D)\delta^{-1}D_U \\ \Delta_{21} & \Delta_{22} & 0 & -C_c\delta^{-1}D_U \\ 0 & 0 & A_a^* & B_a^* \\ \hline \delta^{-1}B_U & \delta^{-1}D_U B_c & 0 & \delta^{-1}D_U \end{array}\right].$$

Leaving out the third column- and the third row block yields a minimal realization $\langle T^{-1}\rangle$ and completes the proof. □

The remainder of the determination hinges on the following lemma.

Lemma 7. *Suppose $T = V^*T''$ is a minimal external factorization of a mixed T and $\begin{bmatrix} A_V & C_V \\ B_V & D_V \end{bmatrix}$ is a unitary realization for V while $\begin{bmatrix} A'' & C'' \\ B'' & D'' \end{bmatrix}$ is one for T'', then there exists a state transformation $R \cdots R^{-(-1)}$ on $\langle T''\rangle$ such that*

$$RA''R^{-(-1)} = \begin{bmatrix} A_V & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B''R^{-(-1)} = \begin{bmatrix} B_V & B_2 \end{bmatrix}$$

for adequate minimal A_{12}, A_{22}, B_2.

Proof. The proof makes use of the property of reachability spaces as detailed in [9], whose notation we use here without further explanations. The assumptions immediately produce the containment

$$\mathscr{D}_2[B_V Z(I - A_V Z)^{-1}]^* \subset \mathscr{D}_2[B'' Z(I - A'' Z)^{-1}]^*.$$

Hence we may choose a sliced reachability basis for T'' which first consists of a reachability basis for V and then complete it with an orthogonal complement to produce a sliced basis for the reachability space of T''. Let $\mathbf{F}$ be the basis so obtained and $\mathbf{F}''$ the original reachability basis of T'', then, because of minimality, there will exist a state transformation R such that

$$\mathbf{F}'' = R^{(-1)*}\mathbf{F}.$$

The containment subdivides $\mathbf{F}$ as

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_V \\ \mathbf{F}_2 \end{bmatrix}.$$

In this basis the 'A' and 'B' matrices are defined by

$$\mathbf{P}'Z\mathbf{F} = \begin{bmatrix} A_V^* & 0 \\ A_{12}^* & A_{22}^* \end{bmatrix} \begin{bmatrix} \mathbf{F}_V \\ \mathbf{F}_2 \end{bmatrix}$$

and

$$\mathbf{P}_0 Z\mathbf{F} = \begin{bmatrix} B_V^* \\ B_2^* \end{bmatrix}.$$

It follows that there exists a state transformation as in the statement of the lemma. $\square$

The lemma admits a converse as well, in case the containment is satisfied a partial factorization will follow. The remainder of the determination consists in computing R and making the final result explicit.

Going back to our original case and utilizing the inversion result of lemma (6) on the data of Eq. (17) (the subsequent lemma's were put in a more generic notation), we obtain

$$\langle T_o'^{-1} U\rangle = \left[\begin{array}{cc|c} \Delta_{11} & \Delta_{12} & B_a^* - C_{o1} D_o^{-1} D_U \\ \Delta_{21} & \Delta_{22} & -C_{o2} D_o^{-1} D_U \\ \hline D_o^{-1} B_U & D_o^{-1} D_U B_c & D_o^{-1} D_U \end{array}\right] \tag{18}$$

in which Δ is now given by

$$\Delta = \begin{bmatrix} A_a^* & B_a^* B_c \\ & A_c \end{bmatrix} - \begin{bmatrix} C_{o1} \\ C_{o2} \end{bmatrix} D_o^{-1} \begin{bmatrix} B_U & D_U B_c \end{bmatrix}$$

(the term that was previously denoted by δ now reduces to D_o). Returning to the original outer–inner factorization of T' (we had postponed the discussion), it defines the inner operator V, the C_{o1}, C_{o2}, D_o matrices and a connecting recursive diagonal Y through the square root recursion, now filled with the quantities that are relevant here. Let

$$\langle T'\rangle = \begin{bmatrix} A' & C' \\ B' & D' \end{bmatrix}$$

with

$$A' = \begin{bmatrix} A_a^* & B_a^* B_c \\ & A_c \end{bmatrix}, \quad B' = \begin{bmatrix} B_U & D_U B_c \end{bmatrix}, \quad C' = \begin{bmatrix} A_a^* C_a + B_a^* D \\ C_c \end{bmatrix},$$

$D' = (C_U C_a + D_U D)$ and $C_o = \begin{bmatrix} C_{o1} \\ C_{o2} \end{bmatrix}$, then the square root algorithm for outer inner factorization produces a diagonal Y_1 and the realization of V by the recursion

$$\begin{bmatrix} A'Y_1 & C' \\ B'Y_1 & D' \end{bmatrix} = \begin{bmatrix} Y_1^{(1)} & C_o \\ & D_o \end{bmatrix} \begin{bmatrix} A_V & C_V \\ B_V & D_V \end{bmatrix},$$

with Y_1 left invertible and D_o invertible. Since $\Delta = A' - C_o D_o^{-1} B'$ and taking $B'' = \begin{bmatrix} B_a^* - C_{o1} D_o^{-1} D_U \\ -C_{o2} D_o^{-1} D_U \end{bmatrix}$ we also have

$$\langle T_o'^{-1} U\rangle = \begin{bmatrix} \Delta & B'' \\ D_o^{-1} B' & D_o^{-1} D_U \end{bmatrix}.$$

The transformation needed to put this realization in block triangular form so as to satisfy Lemma 7 now follows from the theory in Section 2, Lemma 4. Let

$$Y = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}$$

be an embedding of Y_1 into a fully invertible operator with $Y_2 = Y_1^{\perp}$, and choose for the R in Lemma 7

$$R = (Y^{(1)})^{-1}.$$

Because of the orthogonality property of Y_1 we can write

$$Y^{-1} = \begin{bmatrix} Y_1^{\mathrm{M}} \\ Y_2^{\mathrm{M}} \end{bmatrix},$$

where the superscript 'M' indicates the Moore–Penrose inverse. Applying the state transformation $Y^{-(1)} \cdots Y$ produces the realization

$$\langle T_o'^{-1} U \rangle_Y = \left[\begin{array}{cc|c} A_V & A_{12} & C_1 \\ 0 & A_{22} & C_2 \\ \hline B_V & B_2 & D_o^{-1} D_U \end{array}\right]$$

in which

$$\begin{aligned} B_2 &= \begin{bmatrix} D_o^{-1} B_U & D_o^{-1} B_U B_c \end{bmatrix} Y_2, \\ A_{12} &= Y_1^{(1)M} \varDelta Y_2, \\ A_{22} &= Y_2^{(1)M} \varDelta Y_2, \\ \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} &= Y^{(1)M} \begin{bmatrix} B_a^* - C_{o1} D_o^{-1} D_U \\ -C_{o2} D_o^{-1} D_U \end{bmatrix}, \end{aligned}$$

all known quantities. The promised minimal realization for T^{-1} is now found by simply working out the product $T^{-1} = V^*[T_o'^{-1} U]$. Since minimal realizations are known for the factors and since they are in a proper form to assure the necessary cancellations, we obtain by straight computation

$$\begin{aligned} T^{-1} = {} & C_V^* Z^* (I - A_V^* Z^*)^{-1} (A_V^* C_1 + B_V^* D_o^{-1} D_U) \\ & + C_V^* Z^* (I - A_V^* Z^*)^{-1} (A_V^* A_{12} + B_V^* B_2) Z (I - A_{22} Z)^{-1} C_2 \\ & + C_V^* C_1 + D_V^* D_o^{-1} D_U + (C_V^* A_{12} + D_V^* B_2) Z (I - A_{22} Z)^{-1} C_2. \end{aligned}$$

This form already corresponds to a minimal, mixed realization given by

$$\langle T^{-1} \rangle = \left[\begin{array}{cc|c} A_V^* & A_V^* A_{12} + B_V^* D_o^{-1} D_U & A_V^* C_1 + B_V^* D_o^{-1} D_U \\ & A_{22} & C_2 \\ \hline C_V^* & C_V^* A_{12} + D_V^* B_2 & C_V^* C_1 + D_V^* D_o^{-1} D_U \end{array}\right]. \tag{19}$$

A realization without mixed term is obtained by splitting it. This can only be done by solving an extra Lyapunov–Stein equation, which unfortunately runs in the opposite direction of the recursion for Y:

$$m = (A_V^* A_{12} + B_V^* B_2) + [A_V^* m A_{22}]^{(1)}.$$

Our final expression for the split version of $T^{\dagger}$ becomes

$$\langle T^{-1} \rangle_{\mathrm{split}} = \left[\begin{array}{cc|c} A_V^* & & \begin{array}{c} A_V^* (C_1 + m C_2) \\ + B_V^* D_o^{-1} D_U \end{array} \\ & A_{22} & C_2 \\ \hline C_V^* & \begin{array}{c} C_V^* (A_{12} + m A_{22}) \\ + D_V^* B_2 \end{array} & \begin{array}{c} C_V^* (C_1 + m C_2) \\ + D_V^* D_o^{-1} D_U \end{array} \end{array}\right]. \tag{20}$$

4. Discussion

Equation (19) shows that a minimal realization for the inverse of a mixed operator can be obtained simply by computing a left external factor and a right inner factor. These operations can be done in a one-pass backward algorithm. Given the result of these two recursions, a minimal realization is hence obtained in a one pass backward recursion combining the two. If a split representation is desired, then an extra Lyapunv–Stein recursion has to be executed that runs in the opposite direction. It can be shown by sensitivity arguments that this further step is unavoidable, because a complete decomposition necessarily involves the computation of the square root of a Gramian for which the Lyapunov–Stein equation runs in the opposite direction—but this step is unnecessary if only a minimal, but not split realization is desired. Of course, forward recursions are just as well possible, but then the order of factors has to be reversed, external to right and inner to the left. Care has to be excercized to handle intrinsic factors correctly, if only for numerical stability, but this step in the inner–outer factorization does not destroy the one-pass character nor does it increase the complexity otherwise than that it makes a local SVD unavoidable. The results given extend easily to Moore–Penrose inverses of general, non-invertible, mixed locally finite operators. In fact, if the operator is known to be left invertible, the formulas as given apply, because in that case the intermediate T_o is locally invertible. If not, then the inversion of T_o requires an extra step that we have left out for simplicity.

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