The Effect of Local Scattering on the Gain and Beamwidth of a Collaborative Beampattern for Wireless Sensor Networks

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***Abstract*—Collaborative beamforming is an approach where sensor nodes in a wireless sensor network, deployed randomly in an area of interest, transmit a common message by forming a beampattern towards a destination. Previous statistical analysis of the averaged power beampattern considered multipath-free conditions. Herein, we express the averaged power beampattern when the signal is observed at the destination in the presence of local scattering. Assuming the spreading angles are uniformly distributed around the destination direction, we derive closed-form expressions for the maximum gain and numerically examine the beamwidth as a function of the number of nodes, the cluster size, and the scattering parameters, for node positions with a uniform distribution or a Gaussian distribution.**

***Index Terms*—Collaborative beamforming, array beampattern, wireless sensor network, local scattering.**

1. Introduction

In a wireless sensor network (WSN), a large number of small-sized sensor nodes are deployed within an area of interest, organized into clusters, and monitor environmental or physical activities (e.g., temperature, pressure or motion) according to the sensing task of the WSN [1], [2]. The collected data has to be transmitted to a remote destination (e.g., a ground station, an unmanned aerial vehicle or a nearby cluster). To overcome the difficulty of an individual transmission by each node, due to its limited communication range and battery lifetime, a collaborative beamforming technique was recently proposed by Ochiai et al. [3], Tummala et al. [4] and Mudumbai et al. [5], where randomly deployed sensors in a cluster cooperate as a random array, and transmit a common message by forming a beam towards the destination. This joint transmission requires the nodes to be time and frequency synchronized [3], [6]. A similar idea was discussed in the past by Vespoli et al. [7] for military applications where randomly deployed elements act as a relay transmitter. However, the focus in [7] was on system parameters (e.g., signal to noise ratio) and practical designs, and not on the statistical behavior of the random array. As is mentioned in [3], random arrays were also researched in the past in the field of array processing [8]–[10].

The statistical properties of the averaged power beampattern for a collaborative beamforming are analyzed by Ochiai et al. [3] where the node positions are uniformly distributed over a disk cluster, and by Ahmed and Vorobyov [11] where a Gaussian distribution is used instead[1]. It was shown that the power received by the destination is proportional to the number of nodes. Hence, not only that the transmission range and communication rate are increased as more nodes participate in the beamforming, but also the transmission power of each sensor node is reduced.

Whereas the analysis in [3], [11] assumes no reflection or scattering of the transmitted signal, we consider multipath propagation. Such propagation may cause angular spreading of the transmitted signal due to local scattering in the vicinity of the destination [12], [13]. The received signal is modeled as a superposition of a large number of independent, and identically distributed (i.i.d.) rays. Depending on the distribution of the locations of the scatters, the spatial distribution of the angles of the incident scattered rays can be modeled, for example, as a Gaussian distribution, a Laplacian distribution, or a uniform distribution [12], [13] around the nominal angle of arrival. It is noteworthy to mention that the problem of receiving a signal by a passive random array in a multipath environment was previously discussed by Haber in his unpublished report [14], where the application was the deployment of freely drifting sonobuoys. The position of each array element was modeled as a two-dimensional random walk, which after a time, approaches a Gaussian distribution. The mean and variance of the averaged power beampattern were then derived, and the effect of the drift variance was numerically examined.

Herein, the goal is to analyze the effect of local scattering on the maximum gain (i.e., normalized power) and the 3dB beamwidth of the averaged beampattern achieved by a collaborative transmit beamforming. The main contributions of the work are: The averaged maximum gain is expressed as a function of: the number of nodes, the averaged pattern of each sensor node, and the spatial distribution of the spreading angles of the rays; Assuming the spreading angles are uniformly distributed, and for node positions with a uniform distribution or a Gaussian distribution: i) We derive closed-form polynomial expressions for the maximum gain; ii) We show that the maximum gain and the 3dB beamwidth obtained for node positions with a Gaussian distribution are larger than those obtained with a uniform distribution; iii) We provide a simple formula for the value of the scattering radius, given a cluster size, for which the maximum gain equals half the maximum gain achieved with scattering-free conditions.

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[1]The practical case of a truncated Gaussian distribution is also discussed in [11]. However, the averaged power beampattern is analyzed only for the standard Gaussian distribution with infinite support.

2. Problem Formulation

Consider N sensor nodes randomly deployed in the $x-y$ plane. The coordinate vector in polar space of the nth node is (r_n, ψ_n), $n = 1, \ldots, N$. Each coordinate vector has an identical distribution (e.g., uniform [3] or Gaussian [11]) and is independent with all the other coordinate vectors. The coordinate vector in spherical space of the destination is (A_s, ϕ_s, θ_s) where $\phi_s \in [-\pi, \pi]$ and $\theta_s \in [0, \pi]$. We assume the following hold [3]: i) each node is equipped with a single isotropic antenna; ii) all sensors transmit identical energies; iii) the nodes are sufficiently separated such that mutual coupling effects are negligible; iv) the nodes are perfectly time and frequency synchronized. Define by $\mathbf{a}(A, \phi, \theta) \triangleq [e^{j\frac{2\pi}{\lambda}d_1(A,\phi,\theta)}, \ldots, e^{j\frac{2\pi}{\lambda}d_N(A,\phi,\theta)}]^T$ the steering vector of the random array, where λ is the wavelength of the radio frequency carrier, and the distance between the nth node and an arbitrary position (A, ϕ, θ) in spherical space is $d_n(A, \phi, \theta) \triangleq (A^2 + r_n^2 - 2Ar_n \sin(\theta)\cos(\phi - \psi_n))^{1/2}$. Assuming the arbitrary position is located at the far-field region of the array (i.e., $A \gg r_n$) the distance is approximated as $d_n(A, \phi, \theta) \cong A - r_n \sin(\theta)\cos(\phi - \psi_n)$, and thus $\mathbf{a}(A, \phi, \theta) \cong e^{j\frac{2\pi}{\lambda}A}\mathbf{a}(\phi, \theta)$ where $\mathbf{a}(\phi, \theta) \triangleq [a_1(\phi, \theta), \ldots, a_N(\phi, \theta)]^T$ and $a_n(\phi, \theta) \triangleq e^{-j\frac{2\pi}{\lambda}r_n \sin(\theta)\cos(\phi - \psi_n)}$, $n = 1, \ldots, N$. We assume that the array is focused on the desired destination direction (ϕ_s, θ_s) by synchronizing the phases of the nodes.

The collaborative transmission is detailed in [16]. For simplicity assume that at time k one of the nodes (e.g., node 1) wishes to communicate with a destination located at the far-field region at a direction (ϕ_s, θ_s). All the nodes in the cluster collaborate to transmit to the destination the packet of node 1 consisting of Q data symbols $\mathbf{s}(k) = [s(k; 1), \ldots, s(k; Q)]^T$. At first, node 1 broadcasts the packet in the cluster. As a result, node n hears the (noiseless) signal $\mathbf{x}_n(k) = c_n(k)\mathbf{s}(k)$, $n = 1, \ldots, N$, where $c_n(k) = \xi(k)e^{j\chi_n(k)}$ is the complex channel gain at time k between node 1 and node n [16, Eq. (7)], assumed to be modeled as a circularly symmetric complex Gaussian variable with zero mean and variance σ_ξ^2. In the next time slot, node n transmits the signal $\tilde{\mathbf{x}}_n(k+1) = c_n^*(k)\mathbf{x}_n(k)e^{-j\frac{2\pi}{\lambda}r_n \sin(\theta_s)\cos(\phi_s - \psi_n)}$ [16, Eq. (8)][2]. Assuming a flat fading channel between the cluster and the destination with a complex gain denoted by b, and a line of sight and no scattering, we get that the (noiseless) received signal at time k at an arbitrary direction (ϕ, θ) is [16, Eq. (9)]

$$\begin{aligned} \mathbf{y}(\phi, \theta; k) &= b\sum_{i=n}^{N} \tilde{\mathbf{x}}_n(k+1)e^{j\frac{2\pi}{\lambda}r_n \sin(\theta)\cos(\phi - \psi_n)} \\ &= b\xi^2(k)\mathbf{s}(k)\mathbf{a}^H(\phi, \theta)\mathbf{a}(\phi_s, \theta_s) \end{aligned} \quad (1)$$

For simplicity, we denote by $\mathbf{h}_{\text{no,sc}}(\phi, \theta) \triangleq \mathbf{a}(\phi, \theta)$ the line-of-sight (scattering-free) channel between the cluster and an arbitrary destination located at direction (ϕ, θ). We see that in case of a line of sight and no reflections or scattering, the received power at an arbitrary direction (ϕ, θ) is proportional to

[2]It is assumed that each node uses cross-correlation techniques to determine $c_n(k)$ [17, comment before Eq. (8)].

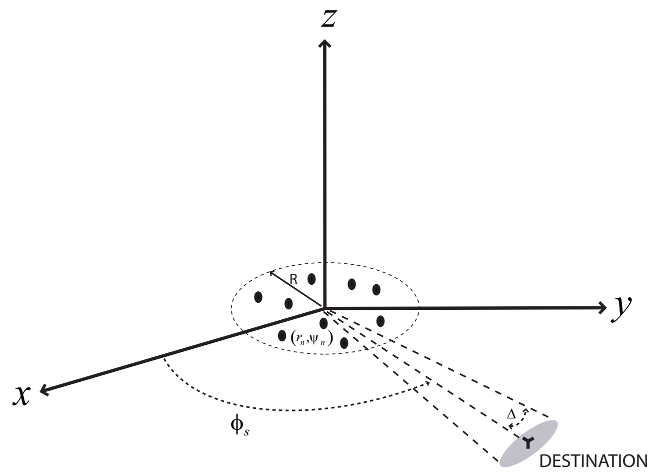

Fig. 1. The geometry of the model.

the power beampattern $P(\phi, \theta) = \left|\mathbf{a}^H(\phi_s, \theta_s)\mathbf{h}_{\text{no,sc}}(\phi, \theta)\right|^2 = \left|\mathbf{a}^H(\phi_s, \theta_s)\mathbf{a}(\phi, \theta)\right|^2$. The statistical analysis of this power beampattern is presented in [3], [11].

Herein, we assume the destination is subjected to local scattering. The goal is to determine the maximum gain and 3dB beamwidth of the power beampattern under such scattering conditions.

3. The Averaged Beampattern In The Presence Of Local Scattering

We assume that the transmitted signal is observed at an arbitrary far-field destination (ϕ, θ) in the presence of local scattering (see Figure 1). The received signal is then modeled as a superposition of L i.i.d. rays. The azimuth and elevation angles of each ray at an arbitrary direction (ϕ, θ) are denoted by $\phi + \tilde{\phi}_\ell$ and $\theta + \tilde{\theta}_\ell$, $\ell = 1, \ldots, L$, respectively, where $\tilde{\phi}_\ell$ and $\tilde{\theta}_\ell$ are i.i.d. random variables distributed with probability density functions (pdf's) $p(\tilde{\phi})$ and $p(\tilde{\theta})$, respectively. Each ray is characterized by a complex amplitude $\alpha_\ell \triangleq g_\ell e^{j\rho_\ell}$ where the phases $\{\rho_\ell\}_{\ell=1}^L$ are i.i.d. random variables, and each phase is uniformly distributed over $[-\pi, \pi]$. The amplitudes $\{g_\ell\}_{\ell=1}^L$ are i.i.d. random variables, independent of the phases and angles. We also assume that all rays have equal power, i.e., $E[|\alpha_\ell|^2] = \frac{1}{L}$ [12]. There are several spatial probability distributions to describe the locations of the local scatters. A simple model assumes that the locations of the scatters are uniformly deployed in a disc around the destination. Another model assumes their locations are evenly spaced on a circular ring (known as Lee's model) or that they have a Gaussian distribution [13, Eq. (8)-(9)]. Depending on the spatial distribution of the scatters, different angle of arrival distributions are considered in the literature including: Gaussian, and Laplacian [13, Eq. (10), (12)].

The channel between the cluster and an arbitrary destination (ϕ, θ) is then modeled as

$$\mathbf{h}_{\text{sc}}(\phi, \theta) = \sum_{\ell=1}^{L} \alpha_\ell \mathbf{a}(\phi + \tilde{\phi}_\ell, \theta + \tilde{\theta}_\ell) \quad (2)$$

Following the model given in (1), we can now express the received signal as,

$$\mathbf{y}(\phi,\theta;k) = b\xi^2(k)\mathbf{s}(k)\mathbf{h}_{\mathrm{sc}}^H(\phi,\theta)\mathbf{a}(\phi_s,\theta_s) \tag{3}$$

We see that the received power at an arbitrary direction (ϕ,θ) for a certain realization of $\{\tilde{\phi}_\ell,\tilde{\theta}_\ell,\rho_\ell,g_\ell\}$ and $\{r_n,\psi_n\}$ is proportional to,

$$\begin{aligned} P_{sc}(\phi,\theta) &= \left|\mathbf{a}^H(\phi_s,\theta_s)\mathbf{h}_{\mathrm{sc}}(\phi,\theta)\right|^2 \\ &= \left|\sum_{\ell=1}^{L}\alpha_\ell A(\phi+\tilde{\phi}_\ell,\theta+\tilde{\theta}_\ell)\right|^2 \end{aligned} \tag{4}$$

where $A(\phi',\theta') \triangleq \mathbf{a}^H(\phi_s,\theta_s)\mathbf{a}(\phi',\theta')$ represents the single-path array beampattern. We are interested in expressing the averaged power beampattern. We start by taking the expectation with respect to (w.r.t.) the distributions of $\{\rho_\ell,g_\ell\}$ and $\{r_n,\psi_n\}$. Since $\{\rho_\ell\}$ are i.i.d. random variables, and thus $E[e^{j(\rho_\ell-\rho_{\ell'})}]=1$ only if $\ell=\ell'$, we get that the averaged power beampattern given $\tilde{\phi}_\ell$ and $\tilde{\theta}_\ell$ is

$$\begin{aligned} P_{sc,av}(\phi,\theta) &= \sum_{\ell=1}^{L}E\left[|\alpha_\ell|^2\right]E\left[\left|A(\phi+\tilde{\phi}_\ell,\theta+\tilde{\theta}_\ell)\right|^2\right] \\ &= \frac{1}{L}\sum_{\ell=1}^{L}P(\phi+\tilde{\phi}_\ell,\theta+\tilde{\theta}_\ell) \end{aligned} \tag{5}$$

where $P(\phi',\theta')$ is the single-path power beampattern averaged over the node coordinates,

$$\begin{aligned} P(\phi',\theta') &\triangleq E\left[\left|A(\phi',\theta')\right|^2\right] \\ &= E\left[\left|\sum_{n=1}^{N}a_n^*(\phi_s,\theta_s)a_n(\phi',\theta')\right|^2\right] \\ &= N + N(N-1)\left|\beta(\phi',\theta')\right|^2 \end{aligned} \tag{6}$$

where $\beta(\phi',\theta') \triangleq E\left[a_1^*(\phi_s,\theta_s)a_1(\phi',\theta')\right]$ is the averaged pattern of each sensor node (observe that in the last passing we used the knowledge that the coordinate vectors are i.i.d. random vectors). Note that $|\beta(\phi',\theta')|<1$ and therefore the maximum value of $P(\phi',\theta')$ is N^2. Since $\tilde{\phi}_\ell$ and $\tilde{\theta}_\ell$ are i.i.d. random variables, the averaged gain beampattern, obtained by taking the expectation of (5) over the distributions of $\tilde{\phi}_\ell$ and $\tilde{\theta}_\ell$ and normalizing by N^2, is

$$\begin{aligned} G_{sc,av}(\phi,\theta) &= \frac{1}{N}+\left(1-\frac{1}{N}\right) \\ &\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p(\tilde{\phi})p(\tilde{\theta})\left|\beta(\phi+\tilde{\phi},\theta+\tilde{\theta})\right|^2 d\tilde{\phi}d\tilde{\theta} \end{aligned} \tag{7}$$

The received power at a direction (ϕ,θ) is therefore the result of averaging the averaged pattern of a node w.r.t. the distribution of the scattering angles around that direction.

4. Scattering Rays with A Uniform Spreading Distribution

We analyze the maximum gain and beamwidth of the averaged beampattern assuming that the pdf of the scattering angles is uniform. It should be emphasized that, as mentioned in [13, comment following Eq. (9)] it is difficult to physically justify spatial distributions of scatters that result in a uniform angle of arrival distribution. However, this type of a uniform distribution is widely used in the literature mainly due to mathematical considerations (i.e., obtaining closed-form expressions). For simplicity we assume that the rays are only scattered in the azimuth axis, i.e., $p(\tilde{\phi})=\frac{1}{2\Delta}, -\Delta\leq\tilde{\phi}\leq\Delta$, where $\Delta\leq\pi$ (we refer to Δ as the scattering radius) [18, Section IV.B]. We consider the case where the destination direction is $\theta_s=\frac{\pi}{2}$ and $\phi_s=0$ [3] and set $\theta=\theta_s$ to examine the received power in the azimuth axis only. We consider two distributions of node deployments, i.e., uniform [3] and Gaussian [11]. Using the result in (7) we get that for a uniform deployment [3, Eq. (15)],

$$\begin{aligned} G_{sc,av}^{(Unif)}\left(\phi,\frac{\pi}{2}\right) &= \frac{1}{N}+\left(1-\frac{1}{N}\right)\frac{1}{2\Delta} \\ &\times\int_{-\Delta}^{\Delta}\left(2\gamma^{-1}(\phi+\tilde{\phi})J_1(\gamma(\phi+\tilde{\phi}))\right)^2 d\tilde{\phi} \end{aligned} \tag{8}$$

where $\gamma(\phi)\triangleq 4\pi\tilde{R}\sin\left(\frac{\phi}{2}\right)$, $\tilde{R}\triangleq\frac{R}{\lambda}$, and R is the radius of the disk cluster. Also, $J_1(x)\triangleq\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{2k+1}}{k!\Gamma(k+2)}$ is the first order Bessel function of the first kind [15, p. 22] where $\Gamma(n)$ is the Gamma function. For a Gaussian deployment [11, Eq. (6)] we get that[3],

$$G_{sc,av}^{(Gauss)}\left(\phi,\frac{\pi}{2}\right) = \frac{1}{N}+\left(1-\frac{1}{N}\right)\frac{1}{2\Delta}\int_{-\Delta}^{\Delta}e^{-\gamma^2(\phi+\tilde{\phi})\sigma^2/\tilde{R}^2}d\tilde{\phi} \tag{9}$$

where $\sigma^2\triangleq\frac{\sigma_0^2}{\lambda^2}$ is the normalized variance, and σ_0^2 is the variance of the Gaussian distribution. The integrand in both cases is a symmetric function around zero only if $\phi=0$. Therefore, the maximum gain of each averaged beampattern occurs for $\phi=0$.

To illustrate this behavior we plot in Figure 2 the averaged beampattern with and without local scattering for uniform and Gaussian deployments. The number of nodes is $N=128$. In order to have a fair comparison between the two node deployments, we need to assume that σ_0 equals the standard deviation of the node positions under a uniform deployment, that is, $\sigma_0=\frac{R}{3}$, and thus $\sigma=\frac{\tilde{R}}{3}$. Then, as indicated in [11, p. 640] we obtain that 99.73% of all nodes are deployed in a disk of radius R. We assume that $\Delta=10$ [degrees] or $\Delta=20$ [degrees], and $\tilde{R}=1$ or $\tilde{R}=4$. The destination azimuth is $\phi_s=0$ [degrees]. To clarify the exhibition, we show the range of $\phi\in[-90,90]$ [degrees] only. As can be

[3] Notice that the difference between the parameter $\gamma(\phi)$ used here and the one that is defined in [11, text following Eq. (2)] is the multiplication by the parameter $\tilde{R}$. This is the reason for the normalization by $\tilde{R}$ in the integrand of the integral in (9).

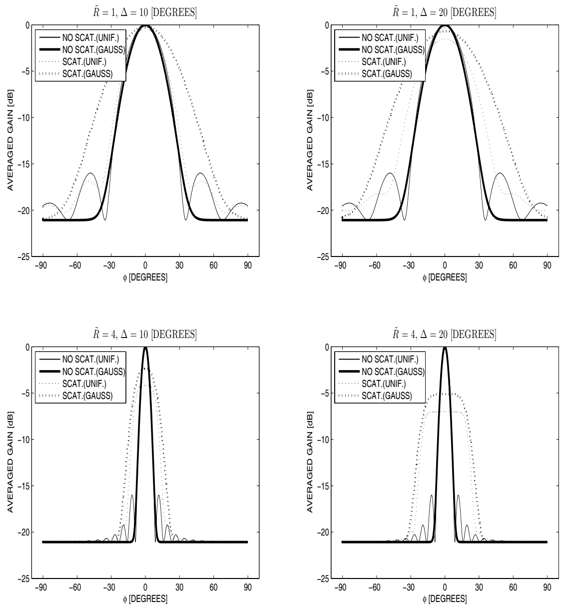

Fig. 2. The averaged gain beampattern for $N = 128$, $\tilde{R} = \{1, 4\}$, and $\Delta = \{10, 20\}$ [degrees] with and without local scattering, and for uniform and Gaussian deployments of nodes.

seen, in both cases, as $\tilde{R}$ and Δ are increased, the mainlobe of the averaged beampattern becomes more flat around the destination direction, and the maximum gain of the averaged beampattern is reduced.

A. The Maximum Gain

We analyze the maximum gains of (8) and (9), denoted by $G_{max}^{(Unif)} \triangleq G_{sc,av}^{(Unif)}\left(0, \frac{\pi}{2}\right)$ and $G_{max}^{(Gauss)} \triangleq G_{sc,av}^{(Gauss)}\left(0, \frac{\pi}{2}\right)$, respectively.

1) Uniform deployment: The maximum of the averaged beampattern in (8) is,

$$\begin{aligned} G_{max}^{(Unif)} &= \frac{1}{N} + \left(1 - \frac{1}{N}\right) \frac{4}{\Delta} \int_0^{\Delta} \left(\gamma^{-1}(\tilde{\phi}) J_1(\gamma(\tilde{\phi}))\right)^2 d\tilde{\phi} \\ &= \frac{1}{N} + \left(1 - \frac{1}{N}\right) \frac{1}{2\pi^2 \tilde{R}^2 \Delta} \\ &\quad \times \int_0^{\sin\left(\frac{\Delta}{2}\right)} \frac{1}{t^2\sqrt{1-t^2}} J_1^2(4\pi \tilde{R} t) dt \end{aligned} \tag{10}$$

where in the second passing we substitute $\gamma(\cdot)$ and defined $t \triangleq \sin(\tilde{\phi}/2)$. Using Taylor series expansion we obtain that $\left(1 - t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} (-1)^n \frac{\sqrt{\pi}}{\Gamma(n+1)\Gamma(\frac{1}{2}-n)} t^{2n}$. We then express (10) as,

$$\begin{aligned} G_{max}^{(Unif)} &= \frac{1}{N} + \left(1 - \frac{1}{N}\right) \frac{1}{2\pi^2 \tilde{R}^2 \Delta} \\ &\quad \times \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{\pi}}{n!\Gamma(\frac{1}{2}-n)} \int_0^{\sin\left(\frac{\Delta}{2}\right)} t^{2(n-1)} J_1^2(4\pi \tilde{R} t) dt \end{aligned} \tag{11}$$

Applying the results in [15, p. 259, Eq. (27)] and [15, p. 18, Eq. (1)] leads to

$$\begin{aligned} &\int_0^{\sin\left(\frac{\Delta}{2}\right)} t^{2(n-1)} J_1^2(4\pi \tilde{R} t) dt = \\ &\sum_{k=0}^{\infty} \frac{(-1)^k (2\pi)^{2(k+1)} (2k+2)!}{k!\,(3+2(n-1)+2k)\,((k+1)!)^2 (k+2)!} \\ &\qquad \times \tilde{R}^{2(k+1)} [\sin(\Delta/2)]^{2(n+k)+1} \end{aligned} \tag{12}$$

Substituting (12) into (11) yields that the maximum gain can be expressed as a series of powers of $\sin\left(\frac{\Delta}{2}\right)$, that is,

$$G_{max}^{(Unif)} = \frac{1}{N} + \left(1 - \frac{1}{N}\right) \frac{1}{\Delta} \sum_{n,k=0}^{\infty} b_{n,k} \tilde{R}^{2k} \left[\sin(\Delta/2)\right]^{2(n+k)+1} \tag{13}$$

where the coefficients of the series are,

$$b_{n,k} \triangleq \frac{(-1)^{n+k}\, 2^{2k+1} \pi^{2k+\frac{1}{2}} (2k+2)!}{\Gamma(\frac{1}{2}-n) n! k! ((k+1)!)^2 (k+2)!\, (3+2(n-1)+2k)} \tag{14}$$

We further assume that N is large, and that Δ is small enough such that $\sin\left(\frac{\Delta}{2}\right) \cong \frac{\Delta}{2}$. Define by $\tilde{b}_{n,k} \triangleq \left(\frac{1}{2}\right)^{2(n+k)+1} b_{n,k}$. The maximum gain in (13) is then approximately given by,

$$\begin{aligned} G_{max}^{(Unif)} &\cong \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \tilde{b}_{n,k} \tilde{R}^{2k} \Delta^{2(n+k)} \\ &= 1 + \sum_{n=1}^{\infty} \left[\sum_{k=0}^{n} \tilde{b}_{n-k,k} \tilde{R}^{2k}\right] \Delta^{2n} \end{aligned} \tag{15}$$

By numerically verifying the values of the coefficients $\{\tilde{b}_{n,k}\}$, it can be shown that $|\tilde{b}_{0,n}| \gg \{|\tilde{b}_{n-k,k}|\}_{k=0}^{n-1}$. Thus, we can further approximate (15) as,

$$\begin{aligned} G_{max}^{(Unif)} &\cong 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n} (2n+2)!}{n!((n+1)!)^2 (n+2)!\,(1+2n)} (\tilde{R}\Delta)^{2n} \\ &= 1 - \frac{\pi^2}{3} (\tilde{R}\Delta)^2 + \frac{\pi^4}{12} (\tilde{R}\Delta)^4 - \cdots \end{aligned} \tag{16}$$

As can be seen, the maximum gain asymptotically reduces w.r.t. even powers of the product of the normalized cluster radius and the scattering radius.

2) Gaussian deployment: The maximum of the averaged beampattern in (9) is,

$$\begin{aligned} G_{max}^{(Gauss)} &= \frac{1}{N} + \left(1 - \frac{1}{N}\right) \frac{1}{\Delta} \int_0^{\Delta} e^{-\gamma^2(\tilde{\phi})\sigma^2/\tilde{R}^2} d\tilde{\phi} \\ &= \frac{1}{N} + \left(1 - \frac{1}{N}\right) \frac{1}{\Delta} \\ &\quad \times \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n 2^{4n} \sigma^{2n} \pi^{2n} \int_0^{\Delta} \sin^{2n}\left(\tilde{\phi}/2\right) d\tilde{\phi} \end{aligned} \tag{17}$$

where we substitute $\gamma(\cdot)$ and express $e^{-\gamma^2(\tilde{\phi})\sigma^2/\tilde{R}^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n (4\sigma\pi \sin(\tilde{\phi}/2))^{2n}$ using the Taylor series expansion. Let $t \triangleq \frac{\tilde{\phi}}{2}$. Applying the integration by parts technique yields that

$$\int_0^{\Delta/2} \sin^{2n}(t)\,dt = -\frac{\Gamma(n+1/2)}{\sqrt{\pi}n!}$$
$$\left[\cos(\Delta/2)\sum_{k=0}^{n-1}\frac{\Gamma(k+1)\Gamma(3/2)}{\Gamma(k+3/2)}\sin^{2k+1}(\Delta/2) - \Delta/2\right],$$

and thus,

$$G_{max}^{(Gauss)} = \frac{1}{N} + \left(1 - \frac{1}{N}\right)$$
$$\times[1 + \sum_{n=1}^{\infty} q_n\sigma^{2n}(1 - \frac{2}{\Delta}\cos(\Delta/2)$$
$$\times\sum_{k=0}^{n-1}\frac{\Gamma(k+1)\Gamma(3/2)}{\Gamma(k+3/2)}\sin^{2k+1}(\Delta/2))] \quad (18)$$

where the coefficients of the series are defined as,

$$q_n \triangleq \frac{\Gamma(n+1/2)}{(n!)^2}(-1)^n 2^{4n}\pi^{2n-\frac{1}{2}} \quad (19)$$

Assume that Δ is small. By applying the Leibniz's rule for differentiation of an integral we get that $\lim_{\Delta/2\to 0}\frac{2}{\Delta}\int_0^{\Delta/2}\sin^{2n}(t)\,dt = \sin^{2n}(\Delta/2) \cong \Delta^{2n}/2^{2n}$. Therefore, we can approximate $\int_0^{\Delta/2}\sin^{2n}(t)\,dt \cong \frac{\Delta^{2n+1}}{(2n+1)2^{2n+1}}$. Substituting this approximation into (17) and assuming that N is large, results in,

$$G_{max}^{(Gauss)} \cong 1 + \sum_{n=1}^{\infty}\frac{(-1)^n(2\pi)^{2n}}{n!(2n+1)}(\sigma\Delta)^{2n} \quad (20)$$

For $\sigma = \frac{\tilde{R}}{3}$ we get that (20) is

$$P_{max}^{(Gauss)} \cong 1 + \sum_{n=1}^{\infty}\frac{(-1)^n\pi^{2n}2^{2n}}{n!(2n+1)3^{2n}}(\tilde{R}\Delta)^{2n}$$
$$= 1 - \frac{4\pi^2}{27}(\tilde{R}\Delta)^2 + \frac{8\pi^4}{405}(\tilde{R}\Delta)^4 + \cdots \quad (21)$$

Similarly to the result in (16), we see that the maximum gain in this case also reduces w.r.t. even powers of $\tilde{R}\Delta$.

In Figure 3 we demonstrate the dependency of the maximum gain versus the scattering radius Δ [degrees] for a uniform deployment and a Gaussian deployment (with $\sigma = \frac{\tilde{R}}{3}$). We consider three values of the number of sensors in the array, i.e. $N = 16, 128$ and 256, and three values of the normalized cluster radius $\tilde{R}$, i.e., $\tilde{R} = 1, 4$ and 16. The scattering radius Δ varied from 0.01 [degrees] (almost scattering-free conditions) to 10 [degrees]. For each pair $(N, \tilde{R})$ we show the maximum gain obtained by numerically calculating (10) and (17), using the closed-form analytical expressions in (13) and (18), and their approximated expressions in (16) and (21). As can be observed, as $\tilde{R}$ increases the maximum gain reduces. The maximum gain of a Gaussian deployment is larger than that of a uniform deployment. Also, for large N the dependency of the maximum gain is weak. The lines associated with $\tilde{R} = 16$ end at $\Delta \cong 3$ [degrees] since these lines diverge above this value due to the use of a finite number of coefficients (we used only 10) in calculating (13), (16), (18), and (21). We note that numerical simulations show that by using more than 10 terms in these series, the discrepancy between the analytical line and the approximated line, and the numerical line (which occurs for large values of $\tilde{R}$ and Δ) appears at higher values of Δ than the current value of 3 [degrees] (depending on the number of terms used).

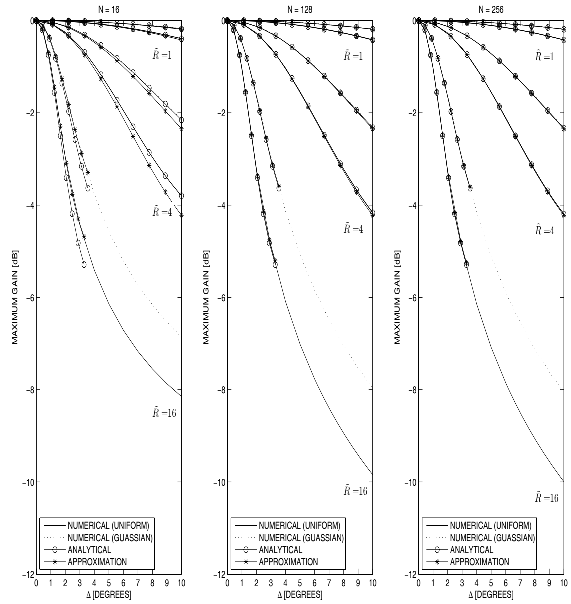

Fig. 3. The maximum gain versus Δ for $N = \{16, 128, 256\}$ and $\tilde{R} = \{1, 4, 16\}$ using numerical calculations, closed-form analytical expressions, and their approximations, for uniform and Gaussian deployments of nodes.

It is of interest to obtain the value of Δ, given $\tilde{R}$, for which the maximum gain equals half the maximum gain assuming scattering-free conditions. This value can be termed as the 3dB maximum gain point, and is obtained by solving the equations $P_{max}^{(Unif)} = 0.5$ or $P_{max}^{(Gauss)} = 0.5$. Using mathematical software such as MATLAB we obtain that the solution of (16) is $(\tilde{R}\Delta)_{Unif} \cong 0.54$ while the solution of (21) is $(\tilde{R}\Delta)_{Gauss} \cong 0.83$ (assuming large but finite number of coefficients). In other words, for a given normalized cluster size, $\tilde{R}$, the value of the scattering radius for which the maximum gain is reduced to half is $\Delta_{3dB}^{(Unif)} \cong \frac{0.54}{\tilde{R}}$ [radians] for a uniform deployment, and $\Delta_{3dB}^{(Gauss)} \cong \frac{0.83}{\tilde{R}}$ [radians] for a Gaussian deployment. In Figure 4 we plot $\Delta_{3dB}^{(Unif)}$ and $\Delta_{3dB}^{(Gauss)}$ versus $\tilde{R}$ where we varied $\tilde{R}$ from 1 to 4 with a step of 0.25. We also plot the values of these 3dB points determined numerically using the expressions in (10) and (17). As can be seen the theoretical results agree with the numerical results for any value of $\tilde{R}$.

B. The 3dB Mainlobe Beamwidth

We now discuss another important characteristic of the averaged beampattern which is the 3dB beamwidth. This beamwidth is defined as the azimuth angle at which the gain is reduced by 3dB below the gain in the direction of the desti-

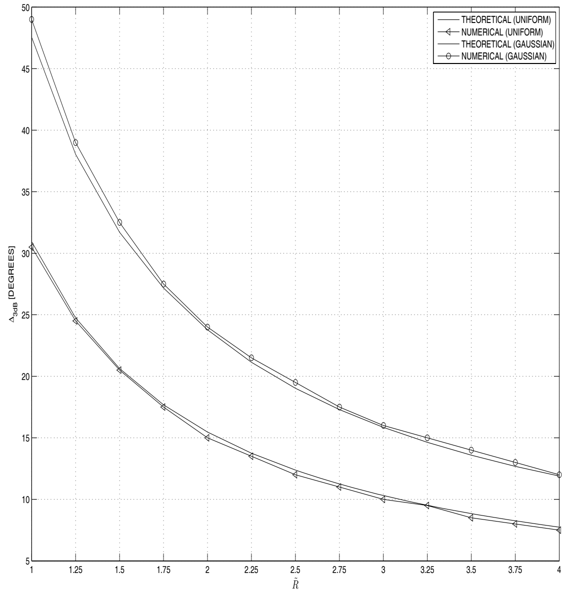

Fig. 4. The value of the scattering radius Δ required to obtain a maximum gain equals to half, versus $\tilde{R}$ (the normalized cluster radius), obtained by numerical calculations and closed-form expressions, for uniform and Gaussian deployments of nodes.

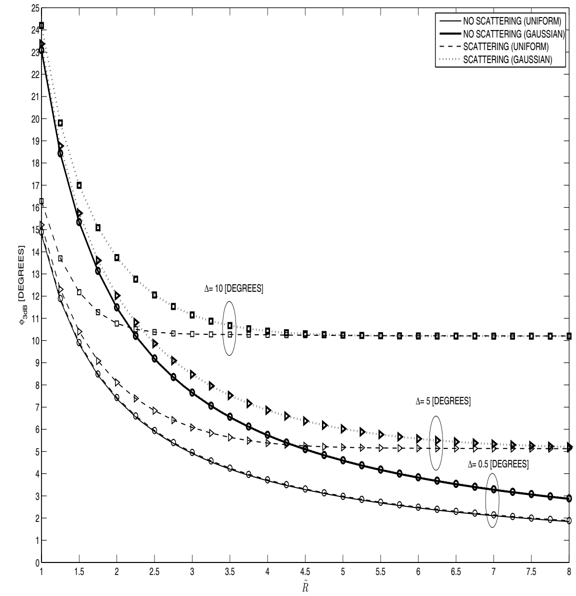

Fig. 5. The 3dB beamwidth versus $\tilde{R}$ (the normalized cluster radius) for $\Delta = 0.5, 5, 10$ [degrees], with and without local scattering, and for uniform and Gaussian deployments of nodes.

nation, and is mathematically expressed as $G_{sc,av}(\phi_{3dB}, \frac{\pi}{2}) = \frac{1}{2}\max\left\{G_{sc,av}(0, \frac{\pi}{2})\right\}$. We numerically determine the 3dB beamwidth as a function of the scattering radius Δ. In Figure 5 we show the dependency of the 3dB beamwidth as a function of $\tilde{R}$ for $\Delta = 0.5, 5$ and 10 [degrees], and for both uniform and Gaussian deployments. We varied $\tilde{R}$ from 1 to 8 with a step of 0.25. We also show the 3dB beamwidth in case of no scattering which is $\phi_{3dB} = 2\arcsin(0.1286/\tilde{R})$ [radians] for a uniform deployment [3, Eq. (23)], and is $\phi_{3dB} = 2\arcsin(0.2/\tilde{R})$ [radians] for a Gaussian deployment [11, Eq. (11)]. As can be observed: i) For a small scattering radius ($\Delta = 0.5$) the 3dB beamwidth is almost the same as the 3dB beamwidth when no scattering occurs; ii) For large scattering radius ($\Delta = 10$), as $\tilde{R}$ decreases, the 3dB beamwidth increases, however as $\tilde{R}$ increases the 3dB beamwidth is almost constant; iii) The 3dB beamwidth of a Gaussian deployment approaches that of a uniform deployment as $\tilde{R}$ increases.

5. Conclusions

We have examined the maximum gain and beamwidth of the averaged beampattern of collaborative beamforming in a wireless sensor network when the signal is received at the destination in the presence of local scattering. Assuming a uniform distribution of the spreading incident angles, we derive closed-form polynomial expressions for the maximum gain of the beampattern, and numerically demonstrate the dependency of the 3dB beamwidth on the number of nodes, the cluster size and the scattering radius for the cases where the node positions are with a uniform distribution or a Gaussian distribution.

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