ON THE HANKEL-NORM APPROXIMATION OF UPPER-TRIANGULAR OPERATORS AND MATRICES

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A matrix $T = [T_{ij}]_{i,j=-\infty}^{\infty}$, which consists of a doubly indexed collection $\{T_{ij}\}$ of operators, is said to be upper when $T_{ij} = 0$ for $i > j$. We consider the case where the T_{ij} are finite matrices and the operator T is bounded, and such that the T_{ij} are generated by a strictly stable, non-stationary but linear dynamical state space model or colligation. For such a model, we consider *model reduction*, which is a procedure to obtain optimal approximating models of lower system order. Our approximation theory uses a norm which generalizes the Hankel norm of classical stationary linear dynamical systems. We obtain a parametrization of all solutions of the model order reduction problem in terms of a fractional representation based on a non-stationary J-unitary operator constructed from the data. In addition, we derive a state space model for the so-called maximum entropy approximant. In the stationary case, the problem was solved by Adamyan, Arov and Krein in their paper on Schur-Takagi interpolation. Our approach extends that theory to cover general, non-Toeplitz upper operators.

1. INTRODUCTION

Approximating a matrix with one of low complexity is an important problem in linear algebra. In one special case where it has been approached successfully, the matrix —say A— is close to a matrix of low rank. A singular value decomposition (SVD) of A will yield a diagonal matrix of singular values, many of which are close to zero and can be neglected, *i.e.,* set equal to zero. One can show (see *e.g.,* [1]) that the so obtained approximation is optimal, both in Euclidean operator norm and in Frobenius norm. The SVD has been used by Adamjan, Arov and Krein (AAK)[2] to obtain another kind of approximation in the context of complex function theory, in relation with the approximation of infinite-size Hankel matrices. This problem arises as follows. In classical model reduction theory, one is given a transfer function $T(z)$ belonging to H_∞ of the unit

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'Integral Equations and Operator Theory', 17(1):1-45, 1993.

circle, $T(z) = t_0 + t_1 z + t_2 z^2 + \cdots$. Associated to $T(z)$ are its transfer operator,

$$T = \begin{bmatrix} \ddots & \vdots & & & \vdots & \\ & t_0 & t_1 & t_2 & t_3 & \cdots \\ & & t_0 & t_1 & t_2 & \\ & & & t_0 & t_1 & \\ & \mathbf{0} & & & t_0 & \cdots \\ & & & & & \ddots \end{bmatrix}$$

(a Toeplitz matrix), which maps an ℓ_2-sequence u boundedly to an output sequence y via $y = uT$, and its Hankel operator

$$H_T = \begin{bmatrix} t_1 & t_2 & t_3 & \cdots \\ t_2 & t_3 & & \\ t_3 & & \ddots & \\ \vdots & & & \end{bmatrix} .$$

(For historical reasons, we use mostly a prefix notation such as uT for the application of a function or operator T to an argument u.) Kronecker [3] has shown that the *model order* of $T(z)$ —the minimal number of states needed in a state space realization of T, or the number of poles of $T(z)$— is equal to the rank of H_T and finite if and only if the system has a rational transfer function. However, the rank of a matrix or operator is not a well conditioned quantity in the presence of numerical inaccuracies. An SVD on the Hankel operator (if it exists) will determine the so-called 'numerical rank'. The approximating matrix (in Hilbert-Schmidt norm) resulting from setting the neglectable singular values equal to zero is not of Hankel-type anymore: the approximant does not correspond to a linear time-invariant system. AAK showed that there is a Hankel matrix of low rank nearby, namely such that the Euclidean norm difference between the original Hankel operator and the approximant is equal to the value of the largest neglected singular value. This approximation can be called the *optimal reduced system* in Hankel norm.

Corresponding to the approximating Hankel operator there is a transfer function $\hat{T}(z)$ — often called its symbol — with degree equal to the number of singular values (multiplicities counted) that have not been neglected. $\hat{T}(z)$ approximates $T(z)$ in a certain sense. If γ is the value of the largest neglected singular value, then it is known that $\|T(z) - \hat{T}(z)\|_2 \le \gamma$. However, there is a much stronger result. One may introduce a norm on transfer functions in H_∞ via the Hankel operator and write

$$\| T \|_H \; = \; \| H_T \| \, .$$

The approximant $\hat{T}$ then surely has the property $\| T - \hat{T} \|_H \le \gamma$. The Hankel norm is considerably stronger than the L_2-norm [4]. Nehari's theorem [5] provides the connection between the Hankel norm and the space L_∞ on the unit circle of $\mathbb{C}$. Let $h(z)$ be a function on the unit circle belonging to L_∞ and such that its Fourier coefficients with non-negative index vanish (a strictly conjugate analytic function), then it is not hard to see that

$$\| T(z) + h(z) \|_\infty \; \ge \; \| T \|_H \, .$$

Nehari's theorem asserts that the infimum of $\| T(z) + h(z) \|_\infty$ over all qualifying $h(z)$ is precisely $\| T \|_H$.

Related to the Hankel approximation problem, and discussed in [2], is the Schur-Takagi interpolation problem. Suppose that a number of complex values are given at a set of points in the interior of the unit disc of the complex plane, then this problem consists in finding a complex function (a) which interpolates these values at the given points (multiplicities counted), (b) which is meromorphic with at most k poles inside the unit disc, and (c) whose restriction to the unit circle (if necessary via a limiting procedure from inside the unit disc) belongs to L_∞ with minimal norm. It turns out that the Schur-Takagi problem can be seen as an extension problem whereby the 'conjugate-analytic' or anti-causal part of a function is given, and it is desired to extend it to a function which is meromorphic inside the unit disc with at most k poles, and belongs to L_∞ with minimal norm.

It was remarked in Bultheel-Dewilde[6] and subsequently worked out by a number of authors (Glover[7], Kung-Lin[8], Genin-Kung[4]) that the procedure of AAK could be utilized to solve the problem of optimal model-order reduction of a dynamical system, as outlined above. The computational problem with the general theory is that it involves an operator which maps a Hilbert space of (input) sequences to a Hilbert space of (output) sequences, and which is thus intrinsically non-finite. In [6] it was shown that the computations are finite if one puts oneself in the context of a system of finite (but possibly large) degree, *i.e.,* an approximant to the original system of high order. It turns out that the resulting computations involve only the realization matrices $\{A, B, C, D\}$ of the approximating system and can be done with classical matrix calculus. They can also be done in a recursive fashion, see [9] as a pioneering paper in this respect. The recursive method is based on interpolation theory of Schur-Takagi type.

In [10, 11, 12, 13, 14, 15, 16], the one-port and multiport lossless inverse scattering (LIS) problem was considered and a mathematical machinery involving reproducing kernel Hilbert spaces to solve it was set up. The connection with interpolation theory both in the global and the recursive variety was firmly established and the monograph [14] devoted to this aspect of the problem. In a parallel development, the state space theory for the interpolation problem was extensively studied in the book [17]. The great interest in this type of problems was kindled by one of its many applications: the robust control problem formulated by Zames in [18] and brought in the context of scattering and interpolation theory by Helton [19]. (We only give the very early references here, an immense literature exists in the field.) A special mention is due to the broadband matching problem [20] which provided the link between the circuit and system theory problems and the mathematical techniques around interpolation, reproducing kernels and lifting of a contractive operator.

But then, what about approximating matrices the same way as system transfer functions? Or, put differently, is there an algebraic analogue to the analytic theories? In a recent series of papers [21, 22, 15, 16, 23, 24], such a theory was developed. The cornerstone of the theory is the definition of the W-transform (originally in [15]) which proves to be a perfect analogue to the classical z-transform. It turns out that classical interpolation problems of Schur or Nevanlinna-Pick type carry over almost effortlessly in the new algebraic context, provided the 'point-evaluation' concepts forced by the W-transform are used. The W-transform has also been called the diagonal transform, because it treats diagonals of matrices as if they were scalars. A comprehensive treatment can be found in [23]. We shall adopt the notation of that paper.

In the present paper, the aim is to extend the model reduction theory to the time-varying context, by considering bounded upper ℓ_2-operators with matrix representation

$$T = \begin{bmatrix} \ddots & \vdots & & \vdots & \\ & \boxed{T_{00}} & T_{01} & T_{02} & \cdots \\ & & T_{11} & T_{12} & \\ & \mathbf{0} & & T_{22} & \cdots \\ & & & & \ddots \end{bmatrix} \tag{1.1}$$

which are now no longer taken to be Toeplitz. The 00-entry in the matrix representation is distinguished by a surrounding square. T maps ℓ_2-sequences $u = [\cdots \ \boxed{u_0} \ u_1 \ u_2 \ \cdots]$ into ℓ_2-sequences y via $y = uT$, and is thus seen to be a *causal* operator: an entry y_i only depends on entries u_k for $k \leq i$. The rows of T can be viewed as the impulse responses of the system. T_{ij} is the transfer of the entry u_i in an input sequence u to entry y_j of the corresponding output sequence.

The approximation theory in this paper draws heavily onto realization theory for such operators. This theory is an extension of time-invariant (Ho-Kalman) realization theory [25] and has been developed since the 1950s. While most of the early work is on time-continuous linear systems and differential equations with time-varying coefficients (see *e.g.*, [26] for a 1960 survey), time-discrete systems have gradually come into favor and are applicable to our context. Some important more recent approaches are the monograph by Feintuch/Saeks [27], in which a Hilbert resolution space setting is taken, and recent work by Kamen, Poolla and Khargonekar [28, 29, 30]. The realization theory as used in the present paper can be found in [24]. Some results are summarized below.

We will be interested in systems T that admit a realization in the form of the recursion

$$\begin{aligned} x_{k+1} &= x_k A_k + u_k B_k \\ y_k &= x_k C_k + u_k D_k \end{aligned} \qquad \mathbf{T}_k = \begin{bmatrix} A_k & C_k \\ B_k & D_k \end{bmatrix} \tag{1.2}$$

in which we will require the matrices $\{A_k, B_k, C_k, D_k\}$ to have finite (but not necessarily fixed) dimensions. Let A_k be of size $d_k \times d_{k+1}$, then the size of x_k, *i.e.*, the *system order* at point k, is equal to d_k. See figure $1(a)$. The collection $\{A_k, B_k, C_k, D_k\}_{-\infty}^{\infty}$ is a realization of T if its entries T_{ij} are given by

$$T_{ij} = \begin{cases} 0\,, & i > j \\ D_i\,, & i = j \\ B_i A_{i+1} \cdots A_{j-1} C_j\,, & i < j\,. \end{cases} \tag{1.3}$$

Figure 1. (*a*) Time-varying state realization, (*b*) Hankel matrices are (mirrored) submatrices of T. H_0 is shaded.

Define a sequence of operators $\{H_k\}_{-\infty}^{\infty}$ with matrix representations

$$H_k = \begin{bmatrix} T_{k-1,k} & T_{k-1,k+1} & T_{k-1,k+2} & \cdots \\ T_{k-2,k} & T_{k-2,k+1} & & \\ T_{k-3,k} & & \ddots & \\ \vdots & & & \end{bmatrix} . \tag{1.4}$$

We will call the H_k time-varying Hankel matrices of T, although they do not have the traditional Hankel structure unless T is a Toeplitz operator. Their matrix representations are mirrored submatrices of T (see figure 1(b)). Although we have lost the traditional anti-diagonal Hankel structure, a number of important properties are retained:

1. If $\{A_k, B_k, C_k, D_k\}$ is a realization of T, then H_k has a factorization

$$H_k = \begin{bmatrix} B_{k-1} \\ B_{k-2}A_{k-1} \\ B_{k-3}A_{k-2}A_{k-1} \\ \vdots \end{bmatrix} [C_k \quad A_kC_{k+1} \quad A_kA_{k+1}C_{k+2} \quad \cdots] =: \mathcal{C}_k\mathcal{O}_k . \tag{1.5}$$

   $\mathcal{C}_k$ and $\mathcal{O}_k$ can be regarded as time-varying controllability and observability operators. If the realization is minimal, then the rank of H_k is equal to the system order at time k.

2. *H_k has shift-invariant properties.* Denote by $H_k^{\leftarrow}$ the operator H_k, with its first column deleted. Then

$$H_k^{\leftarrow} = \begin{bmatrix} B_{k-1} \\ B_{k-2}A_{k-1} \\ B_{k-3}A_{k-2}A_{k-1} \\ \vdots \end{bmatrix} A_k\,[C_{k+1} \quad A_{k+1}C_{k+2} \quad A_{k+1}A_{k+2}C_{k+3} \quad \cdots] \;=\; \mathcal{C}_kA_k\mathcal{O}_{k+1} . \tag{1.6}$$

The shift-invariance property states that the row space of $H_k^{\leftarrow}$ is contained in the row space of H_{k+1}.

These two properties are sufficient to derive a minimal realization for T [24]. The construction consists mainly of the factorization of each H_k into minimal rank factors $\mathcal{C}_k$ and $\mathcal{O}_k$. Once these have been determined, then the B_k and C_k are equal to the first (block)-column of $\mathcal{C}_{k+1}$ and first row of $\mathcal{O}_k$, whereas A_k can be determined by using the shift-invariance property. We will assume, throughout, that all H_k have finite rank (we will call such systems *locally finite*), so that the factorization is well defined. In this case, one can show that *the rank of H_k is equal to the system order at point k of any minimal realization of T.*

Because the minimal system order of any realization is at each point k given by the rank of the Hankel matrix H_k at that point, a possible approximation scheme is to approximate each H_k by one that is of lower rank (this could be done using the SVD). The approximation error could then very well be defined in terms of the individual Hankel matrix approximations as the supremum over these approximations. Because the Hankel matrices have many entries in common, it is not clear at once that such an approximation scheme is feasible: replacing one Hankel matrix by one of lower rank in a certain norm might make it impossible for the next Hankel matrix to find an optimal (in that norm) approximant such that the part that it has in common with the previous Hankel matrix will be approximated by the same matrix. This situation parallels what already occurs for linear time-invariant systems.

The Hankel norm of an operator T can be defined at present as

$$\| T \|_H = \sup_k \| H_k \| . \tag{1.7}$$

(The definition which we will use in the paper appears in equation (2.5) below.) This definition is a generalization of the time-invariant Hankel norm and reduces to it if all H_k are the same. Let $\Gamma = \operatorname{diag}(\gamma_i)$ be an acceptable approximation tolerance, with $\gamma_i > 0$. If an operator T_a is such that $\| \Gamma^{-1}(T - T_a) \|_H \leq 1$, then T_a is called a *Hankel norm approximant* of T, parameterized by Γ. We are interested in Hankel norm approximants of minimal system order.

In this respect, we will prove the following theorem:

THEOREM 1.1. *Let T be a bounded operator which is strictly upper, strictly stable and locally finite, and let Γ be an invertible Hermitian diagonal operator. Let H_k be the Hankel matrix of $\Gamma^{-1}T$ at time instant k. Suppose that the singular values of each H_k decompose into two sets $\sigma_{-,k}$ and $\sigma_{+,k}$, with lower bound of all $\sigma_{-,k}$ larger than 1, and upper bound of all $\sigma_{+,k}$ smaller than 1. Let N_k be equal to the number of singular values of H_k which are larger than 1.*

Then there exists a strictly upper locally finite operator T_a of system order at most N_k at point k, such that

$$\| \Gamma^{-1}(T - T_a) \|_H \leq 1 .$$

(The notion of strict stability is defined in section 2.1.) In fact, there is a collection of such T_a. We will obtain a state space realization of a particular T_a as well (theorem 7.5). Theorem 8.8 gives a parametrization of all solutions. It is shown that no Hankel norm approximants of order lower than N_k exist. Finally, if all singular values of all H_k are smaller than 1, Nehari's theorem for time-varying systems is recovered. Such extensions have been described by Gohberg, Kaashoek and Woerdeman [31, 32].

2. PRELIMINARIES

2.1. Spaces

The basic elements of our theory are generalizations of ℓ_2-series to sequences of which the components have non-uniform dimensions [24]. Let $\{N_i \in \mathbb{N}, i \in \mathbb{Z}\}$ be an indexed collection of natural numbers* which we will always take finite. The sequence

$$N = [\,N_i\,]_{-\infty}^{\infty} \;=\; [\cdots \quad N_{-1} \quad \boxed{N_0} \quad N_1 \quad N_2 \quad \cdots] \;\in\; \mathbb{N}^{\mathbb{Z}}$$

is called an index sequence. Using N, signals $u = [\cdots \; u_{-1} \; \boxed{u_0} \; u_1 \; u_2 \; \cdots]$ live in the space of non-uniform sequences which is the Carthesian product of the $\mathcal{N}_i$:

$$\mathcal{N} = \cdots \times \mathcal{N}_{-1} \times \boxed{\mathcal{N}_0} \times \mathcal{N}_1 \times \mathcal{N}_2 \times \cdots \;\in\; \mathbb{C}^{\mathbb{N}} \;,$$

where $\mathcal{N}_i \in \mathbb{C}^{N_i}$ so that N_i is the dimension of $\mathcal{N}_i$. Some of these components may have zero dimension: we define $\mathbb{C}^0 = \emptyset$. In this way, finite dimensional vectors are also incorporated in the space of non-uniform sequences, by putting $N_i = 0$ for i outside a finite interval. We will write $N = \#(\mathcal{N})$ to indicate the sequence of dimensions N of the sequence of spaces $\mathcal{N}$.

The inner product of two non-uniform sequences f, g in $\mathcal{N}$ is defined in terms of the usual inner product of (row)-vectors in $\mathcal{N}_i$ as $(\,f,\,g) \;=\; \sum_i (\,f_i,\,g_i)$ where $(\,f_i,\,g_i) = f_i g_i^*$ is defined to be 0 if $N_i = 0$. † The norm of a non-uniform sequence is the standard 2-norm (vector norm) defined on this inner product:

$$u = [\,u_i\,]_{-\infty}^{\infty} \;:\quad \|\,u\,\|_2^2 = (u,\,u) \;=\; \sum_{-\infty}^{\infty} \|\,u_i\,\|_2^2\,.$$

The space of non-uniform sequences with index sequence N and with finite 2-norm is denoted by $\ell_2^{\mathcal{N}}$ ($\mathcal{N} \in \mathbb{C}^N$, with $N \in \mathbb{N}^{\mathbb{Z}}$). $\ell_2^{\mathcal{N}}$ is a Hilbert space.

Let $\mathcal{M}$ and $\mathcal{N}$ be sequences of spaces corresponding to sequences of indices M, N. We denote by $\mathcal{X}(\mathcal{M},\mathcal{N})$ the space of *bounded* linear operators $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}$: an operator T is in $\mathcal{X}(\mathcal{M},\mathcal{N})$ if and only if for each $u \in \ell_2^{\mathcal{M}}$, the result $y = uT$ is in $\ell_2^{\mathcal{N}}$, in which case the induced operator norm of T,

$$\|\,T\,\| = \sup_{\|u\|_2 \le 1} \|\,uT\,\|_2\,,$$

is bounded. $T \in \mathcal{X}(\mathcal{M},\mathcal{N})$ has a block matrix representation $[T_{ij}]_{i,j=-\infty}^{\infty}$ such that

$$y = uT \quad \Leftrightarrow \quad y_j = \sum_i u_i T_{ij}\,. \tag{2.1}$$

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*0 is included in $\mathbb{N}$.

†More generally, we define the product of an $n \times 0$ matrix with a $0 \times m$ matrix to be the zero matrix of dimensions $n \times m$.

Consequently, we will identify T with its matrix representation and write

$$T = [T_{ij}]_{i,j=-\infty}^{\infty} = \begin{bmatrix} \ddots & & \vdots & & ⋰ \\ & T_{-1,-1} & T_{-1,0} & T_{-1,1} & \\ \cdots & T_{0,-1} & \boxed{T_{00}} & T_{01} & \cdots \\ & T_{1,-1} & T_{10} & T_{11} & \\ ⋰ & & \vdots & & \ddots \end{bmatrix} \tag{2.2}$$

(where the square identifies the 00-entry) such that it fits the usual vector-matrix multiplication rules. The block entry T_{ij} is an $M_i \times N_j$ matrix.

As explained in [24] and [23], operators T in $\mathcal{X}$ have an upper part and a lower part: all entries T_{ij} above the main (0-th) diagonal and including this diagonal form the upper part, while all entries below the diagonal, including the diagonal, form the lower part. When T is a bounded operator $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}$, then its upper part need not represent a bounded operator. The situation generalizes what already happens with Toeplitz operators: if the symbol of such an operator belongs to L_∞ of the unit circle, then its analytic part need not belong to L_∞. Nonetheless, each diagonal of a bounded operator, taken by itself, is again a bounded operator with norm not exceeding the norm of the original operator. Concerning the diagonal calculus on operators between non-uniformly indexed spaces, we adopt the notation of [24] and [23]:

- $\mathcal{M}^{(k)}$ — the k-th shift rightwards in the series of spaces as in $\mathcal{M}^{(1)} = [\cdots \ \mathcal{M}_{-2} \ \boxed{\mathcal{M}_{-1}} \ \mathcal{M}_0 \ \cdots]$.
- Z — the shift operator: $[\cdots \ x_{-1} \ \boxed{x_0} \ x_1 \ \cdots] Z = [\cdots \ x_{-2} \ \boxed{x_{-1}} \ x_0 \ \cdots]$. Notice that $Z : \ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{M}^{(1)}}$.
- $Z^{[k]}$ — the product of k shifts. It is an operator $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{M}^{(k)}}$.
- $\mathcal{X}(\mathcal{M},\mathcal{N})$ — the space of bounded operators $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}$.
- $\mathcal{U}(\mathcal{M},\mathcal{N})$ — the space of bounded, upper triangular operators $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}$.
- $\mathcal{L}(\mathcal{M},\mathcal{N})$ — the space of bounded, lower triangular operators $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}$.
- $\mathcal{D}(\mathcal{M},\mathcal{N})$ — the space of diagonal operators $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}$. The norm of a member of $\mathcal{D}$ will be the supremum over the norms of its components.

Let $T \in \mathcal{U}$, then we can formally decompose T into a sum of shifted diagonal operators as in

$$T = \sum_{k=0}^{\infty} Z^{[k]} T_{[k]}\,, \tag{2.3}$$

where $T_{[k]} \in \mathcal{D}(\mathcal{M}^{(k)},\mathcal{N})$ is the k-th diagonal above the main (0-th) diagonal. Given an operator A, we can define its k-th shift in the South-East direction as

$$A^{(k)} = (Z^{[k]})^* A Z^{[k]}\,. \tag{2.4}$$

We will often encounter products $(AZ)^n$, where $A \in \mathcal{X}(\mathcal{N}, \mathcal{N}^{(-1)})$. These evaluate as

$$
\begin{aligned}
(AZ)^n &= (AZ)\,(AZ)\,\cdots(AZ) \\
&= Z^{[n]} A^{(n)} A^{(n-1)} \cdots A^{(1)} \\
&=: Z^{[n]} A^{\{n\}}
\end{aligned}
$$

where $A^{\{n\}}$ is defined as

$$
\begin{aligned}
A^{\{0\}} &= I\,, \\
A^{\{n\}} &= A^{(n)} A^{\{n-1\}} = A^{(n)} A^{(n-1)} \cdots A^{(1)}\,.
\end{aligned}
$$

The spectral radius $r(AZ) = \lim_{n\to\infty} \parallel (AZ)^n \parallel^{1/n} = \lim_{n\to\infty} \parallel A^{\{n\}} \parallel^{1/n}$ of AZ will be of considerable interest and is denoted by ℓ_A.

Besides the spaces $\mathcal{X}$, $\mathcal{U}$, $\mathcal{L}$, $\mathcal{D}$ in which the operator norm reigns, we shall need Hilbert-Schmidt spaces $\mathcal{X}_2$, $\mathcal{U}_2$, $\mathcal{L}_2$, $\mathcal{D}_2$ which consist of elements of $\mathcal{X}$, $\mathcal{U}$, $\mathcal{L}$, $\mathcal{D}$ respectively, and for whom the norms of the entries are square summable. These spaces are Hilbert spaces for the usual Hilbert-Schmidt inner product. They will often be considered to be input or output spaces for our system operators. Indeed, if T is a bounded operator $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}$, then it may be extended as a bounded operator $\mathcal{X}_2 \to \mathcal{X}_2$ by stacking sequences in ℓ_2 to form elements of $\mathcal{X}_2$. This leads for example to the expression $y = uT$, where $u \in \mathcal{X}_2(\mathbb{C}^{\mathbb{Z}}, \mathcal{M})$ and $y \in \mathcal{X}_2(\mathbb{C}^{\mathbb{Z}}, \mathcal{N})$ [23]. We will use the shorthand $\mathcal{X}_2^{\mathcal{M}}$ for $\mathcal{X}_2(\mathbb{C}^{\mathbb{Z}}, \mathcal{M})$, but continue to write $\mathcal{X}_2$ if the precise form of $\mathcal{M}$ is not of interest.

We define $\mathbf{P}$ as the projection operator of $\mathcal{X}_2$ on $\mathcal{U}_2$, $\mathbf{P}_0$ as the projection operator of $\mathcal{X}_2$ on $\mathcal{D}_2$, and $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}$ as the projection operator of $\mathcal{X}_2$ on $\mathcal{L}_2 Z^{-1}$.

2.2. Left D-invariant subspaces

We say that a subspace $\mathcal{H}$ of $\mathcal{X}_2$ is *left D-invariant* if $A \in \mathcal{H} \Rightarrow DA \in \mathcal{H}$ for all $D \in \mathcal{D}$.

Let $\Delta_i = \operatorname{diag}[\cdots\ 0\ \ 0\ \ I\ \ 0\ \ 0\ \cdots]$, where the unit operator appears at the i-th position, and let $\mathcal{H}$ be a left D-invariant subspace. Define $\mathcal{H}_i = \Delta_i \mathcal{H}$, then $\mathcal{H}_i$ is also left D-invariant, and $\mathcal{H}_i \subset \mathcal{H}$. If $i \neq j$, then $\mathcal{H}_i \perp \mathcal{H}_j$ (where orthogonality is with respect to the Hilbert-Schmidt inner product). It follows that $\mathcal{H} = \oplus_i \mathcal{H}_i$.

A left D-invariant subspace is said to be *locally finite* if, for all i, $\dim \mathcal{H}_i$ is finite. In that case, there exists a *local basis* for $\mathcal{H}$, where each basisvector is itself a basisvector of some $\mathcal{H}_i$. The conjunction of the basisvectors of all $\mathcal{H}_i$ span $\mathcal{H}$. With $d_i = \dim \mathcal{H}_i$, we will call the sequence $[\cdots\ d_0\ \ d_1\ \ d_2\ \cdots]$ the sequence of dimensions of $\mathcal{H}$, and denote this as s-dim $\mathcal{H}$.

We list some properties of D-invariant subspaces. If $\mathcal{A} \subset \mathcal{X}_2$ is left D-invariant, then so is $\mathcal{A}^\perp$. If $\mathcal{A}, \mathcal{B} \subset \mathcal{X}_2$ are subspaces, then let $\mathbf{P}_{\mathcal{A}}(\mathcal{B})$ indicate the projection of $\mathcal{B}$ onto $\mathcal{A}$, obtained by projecting vectors of $\mathcal{B}$ onto $\mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are left D-invariant, then so are $\mathbf{P}_{\mathcal{A}}(\mathcal{B})$ and $\mathbf{P}_{\mathcal{A}^\perp}(\mathcal{B})$. If $\mathcal{A}$ or $\mathcal{B}$ is locally finite, then so is $\mathbf{P}_{\mathcal{A}}(\mathcal{B})$.

If two linearly independent subspaces $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{X}_2$ are locally finite, then so is their direct sum $\mathcal{A} + \mathcal{B}$, so that a local basis of $\mathcal{A} + \mathcal{B}$ can be obtained from bases of $\mathcal{A}_i + \mathcal{B}_i$, for all i. If $\mathcal{A}$ is a left D-invariant subspace and B is a linear operator, then $\mathcal{A}B$ is also left D-invariant.

2.3. Hankel operators and state spaces

Let $T \in \mathcal{X}$ be a bounded operator. An abstract version of the Hankel operator maps inputs in $\mathcal{L}_2 Z^{-1}$ to outputs restricted to $\mathcal{U}_2$: the Hankel operator H_T connected to T is the map $u \in \mathcal{L}_2 Z^{-1} \mapsto \mathbf{P}(uT)$. Note that only the strictly upper part of T plays a role in this definition. The operators H_k of equation (1.4) are 'snapshots' of it: H_k can be obtained from H_T by considering a further restriction to inputs $\Delta_k u$ of which only the k-th row is non-zero: the operator $(\Delta_k \cdot)H_T$ is isomorphic to H_k. [The isomorphism consists of removing zero rows of $\Delta_k u$ and $y = (\Delta_k u)H_T$, and writing the resulting ℓ_2-sequences as one-sided sequences.] The canonical realization theory in [24] is based on distinguishing characteristic spaces in $\mathcal{L}_2 Z^{-1}$ and $\mathcal{U}_2$, which are the range and kernel of H_T and $(H_T)^*$:

- the natural input state space $\mathcal{H}(T) = \text{ran }(H_T^*) = \{\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(yT^*) : y \in \mathcal{U}_2\} \subset \mathcal{L}_2 Z^{-1}$,
- the natural output state space $\mathcal{H}_0(T) = \text{ran }(H_T) = \{\mathbf{P}(uT) : u \in \mathcal{L}_2 Z^{-1}\} \subset \mathcal{U}_2$.

These spaces are left D-invariant: $D\mathcal{H} \subset \mathcal{H}$, $D\mathcal{H}_0 \subset \mathcal{H}_0$. $\mathcal{H}$ and $\mathcal{H}_0$ are not necessarily closed; their closures $\overline{\mathcal{H}}$ and $\overline{\mathcal{H}}_0$ are left D-invariant subspaces, which have shift-invariance properties as are explained in the cited paper. Throughout the paper, it will be assumed that T is such that $\overline{\mathcal{H}}(T)$ and $\overline{\mathcal{H}}_0(T)$ are locally finite subspaces. Such T will be called *locally finite* transfer operators.

2.4. Hankel norm

Let the Hankel norm of T be defined as the operator norm of its Hankel operator:

$$\| T \|_H = \| H_T \| . \tag{2.5}$$

This definition is equivalent to the definition in (1.7). It is a norm on $Z\mathcal{U}$, but a seminorm on $\mathcal{X}$. We will also employ another norm, the *diagonal 2-norm*. Let T_i be the i-th row of a block matrix representation $T \in \mathcal{X}$, then

$$\begin{aligned} D \in \mathcal{D}: &\ \| D \|_{\mathcal{D}2} = \sup_i \| D_i \|, \\ T \in \mathcal{X}: &\ \| T \|_{\mathcal{D}2}^2 = \| \mathbf{P}_0(TT^*) \|_{\mathcal{D}2} = \sup_i \| T_i T_i^* \| . \end{aligned}$$

For diagonals, it is equal to the operator norm, but for more general operators, it is the supremum over the 'vector' 2-norms of each row of T.

PROPOSITION 2.1. *The Hankel norm satisfies the following inequalities:*

$$T \in \mathcal{X}: \qquad \| T \|_H \le \| T \| \tag{2.6}$$

$$T \in Z\mathcal{U}: \qquad \| T \|_{\mathcal{D}2} \le \| T \|_H . \tag{2.7}$$

PROOF The first norm inequality is proven by

$$\begin{aligned} \| T \|_H &= \sup_{u \in \mathcal{L}_2 Z^{-1}, \| u \|_{HS} \le 1} \| \mathbf{P}(uT) \|_{HS} \\ &\le \sup_{u \in \mathcal{L}_2 Z^{-1}, \| u \|_{HS} \le 1} \| uT \|_{HS} \\ &\le \sup_{u \in \mathcal{X}_2, \| u \|_{HS} \le 1} \| uT \|_{HS} = \| T \| . \end{aligned}$$

For the second norm inequality, we first prove $\| T \|_{\mathcal{D}2}^2 \le \sup_{D\in\mathcal{D}_2,\, \| D \|_{HS}\le 1} \;\; \| DTT^*D^* \|_{HS}$. Indeed,

$$\begin{array}{rcll} \| T \|_{\mathcal{D}2}^2 & = & \| \mathbf{P}_0(TT^*) \|_{\mathcal{D}2}^2 & \\ & = & \sup_{D\in\mathcal{D}_2,\, \| D \|_{\mathcal{D}2}\le 1} & \| D\mathbf{P}_0(TT^*)D^* \|_{\mathcal{D}2} \\ & = & \sup_{D\in\mathcal{D}_2,\, \| D \|_{HS}\le 1} & \| D\mathbf{P}_0(TT^*)D^* \|_{HS} \\ & \le & \sup_{D\in\mathcal{D}_2,\, \| D \|_{HS}\le 1} & \| DTT^*D^* \|_{HS}\,. \end{array}$$

Then (2.7) follows, with use of the fact that $T \in Z\mathcal{U}$, since the latter term is majorized by $\| T \|_H^2$:

$$\begin{array}{rcll} \| T \|_{\mathcal{D}2}^2 & \le & \sup_{D\in\mathcal{D}_2,\, \| D \|_{HS}\le 1} & \| DTT^*D^* \|_{HS} \\ & = & \sup_{D\in\mathcal{D}_2,\, \| D \|_{HS}\le 1} & \| DZ^*TT^*ZD^* \|_{HS} \\ & = & \sup_{D\in\mathcal{D}_2,\, \| D \|_{HS}\le 1} & \| \mathbf{P}(DZ^*T)\, [\mathbf{P}(DZ^*T)]^* \|_{HS} \\ & \le & \sup_{u\in\mathcal{L}_2 Z^{-1},\, \| u \|_{HS}\le 1} & \| \mathbf{P}(uT)\, [\mathbf{P}(uT)]^* \|_{HS} \quad = \quad \| T \|_H^2\,. \end{array}$$

□

We see that the Hankel norm is not as strong as the operator norm, but is stronger than the row-wise uniform least square norm.

2.5. Realizations

If the natural input state space and the natural output state space have a locally finite basis then realizations of type (1.2) can be derived. Turning to this type of realizations, we can assemble the matrices $\{A_k\}$, $\{B_k\}$ etc. as operators on spaces of sequences of appropriate dimensions, by defining $A = \mathrm{diag}(A_k)$, $B = \mathrm{diag}(B_k)$, $C = \mathrm{diag}(C_k)$ and $D = \mathrm{diag}(D_k)$. Let $\ell_2^{\mathcal{M}}$ be the space of input sequences, $\ell_2^{\mathcal{N}}$ the space of output sequences, and let us define $\mathcal{B} = \{\mathcal{B}_k :\; k \in \mathbb{Z}\}$ as the sequences of spaces to which the state $x = [\cdots\; x_0\;\; x_1\;\; x_2\;\; \cdots]$ belongs. If all operators $\{A_k\}$, $\{B_k\}$, etc. are uniformly bounded over k, then A, B, etc. may be viewed as bounded diagonal operators

$$\begin{array}{rclcrcl} A & \in & \mathcal{D}(\mathcal{B}, \mathcal{B}^{(-1)})\,, & \qquad & C & \in & \mathcal{D}(\mathcal{B}, \mathcal{N})\,, \\ B & \in & \mathcal{D}(\mathcal{M}, \mathcal{B}^{(-1)})\,, & & D & \in & \mathcal{D}(\mathcal{M}, \mathcal{N})\,, \end{array} \tag{2.8}$$

which together define the dynamical equations

$$\begin{array}{rcl} xZ^{-1} & = & xA + uB \\ y & = & xC + uD \end{array} \qquad \mathbf{T} = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \tag{2.9}$$

With ℓ_A the spectral radius of the operator AZ, we shall say that the realization (2.9) is *strictly stable* if $\ell_A < 1$. In that case, the operator $(I - AZ)^{-1}$ exists as a bounded operator and elimination of x in (2.9) leads to

$$x = uBZ(I - AZ)^{-1} \tag{2.10}$$

so that T can be written in terms of $\{A, B, C, D\}$ as $T = D + BZ(I - AZ)^{-1}C$.

The realization (2.9) can be generalized further, by considering inputs and outputs in $\mathcal{X}_2^{\mathcal{M}}$ and $\mathcal{X}_2^{\mathcal{N}}$, respectively, for which again the same relations hold:

$$\begin{array}{rcl} xZ^{-1} & = & xA + uB \\ y & = & xC + uD \end{array} \qquad \mathbf{T} = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \tag{2.11}$$

If $\ell_A < 1$, then also $x = uBZ(I - AZ)^{-1} \in \mathcal{X}_2^{\mathcal{B}}$. By projecting both equations in (2.11) onto the k-th diagonal, and using the fact that A, B, C, D are diagonal operators, a generalization of the recursive realization (1.2) is obtained as

$$\begin{array}{rcl} x_{[k+1]}^{(-1)} & = & x_{[k]}A + u_{[k]}B \\ y_{[k]} & = & x_{[k]}C + u_{[k]}D \end{array} \tag{2.12}$$

(see figure 2(b).) Note the diagonal shift in $x_{[k+1]}^{(-1)}$. Two more representations for T can be derived from (2.12). For $u \in \mathcal{X}_2$, we define the past and future signals, with respect to the 0-th diagonal of u, by the projection of u onto $\mathcal{L}_2 Z^{-1}$ and $\mathcal{U}_2$ respectively, so that $u = u_p + u_f \in \mathcal{L}_2 Z^{-1} \oplus \mathcal{U}_2$, with $u_p = \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(u) = Z^{-1}u_{[-1]} + Z^{-2}u_{[-2]} + \cdots$ and $u_f = \mathbf{P}(u) = u_{[0]} + Zu_{[1]} + Z^2 u_{[2]} + \cdots$. If the same is done to $y \in \mathcal{X}_2$, then $y = uT$ is equivalent to $y = y_p + y_f$, with

$$\left\{ \begin{array}{rcl} y_p & = & u_p K_T \\ y_f & = & u_p H_T \;+\; u_f E_T \end{array} \right. \qquad \text{where} \qquad \left\{ \begin{array}{rcl} K_T & = & \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\,\cdot\, T)\,|_{\mathcal{L}_2 Z^{-1}} \\ E_T & = & \mathbf{P}(\,\cdot\, T)\,|_{\mathcal{U}_2} \end{array} \right. \tag{2.13}$$

and where H_T is the Hankel operator defined before (see figure 2(c)). If $u_f = 0$, then $y_f = u_p H_T$. The following construction shows that the existence of a realization implies that H_T can be factored into two operators. According to (2.12), and assuming $\ell_A < 1$, $x_{[0]}$ is equal to

$$\begin{array}{rcl} x_{[0]} = \mathbf{P}_0(x) & = & \mathbf{P}_0(\,u\,BZ(I-AZ)^{-1}\,) \\ & = & \mathbf{P}_0(\,u_p\,BZ(I-AZ)^{-1}\,) \;+\; \mathbf{P}_0(\,u_f\,BZ(I-AZ)^{-1}\,) \\ & = & \mathbf{P}_0(\,u_p\,BZ(I-AZ)^{-1}\,)\,. \end{array} \tag{2.14}$$

If $u_f = 0$ then, for $k \geq 0$, $y_{[k]} = x_{[k]}C = x_{[0]}^{(k)} A^{\{k\}} C$, so that

$$\begin{array}{rcl} y_f & = & \sum_0^\infty \; Z^{[k]} y_{[k]} \\ & = & \sum_0^\infty \; Z^{[k]} x_{[0]}^{(k)} A^{\{k\}} C \\ & = & \sum_0^\infty \; x_{[0]} (AZ)^k C \\ & = & x_{[0]} (I - AZ)^{-1} C\,. \end{array} \tag{2.15}$$

Hence H_T has the factorization $H_T = \mathbf{P}_0(\,\cdot\,\mathbf{F}^*)\,\mathbf{G}$, with $\mathbf{F}^* = BZ(I-AZ)^{-1}$ and $\mathbf{G} = (I-AZ)^{-1}C$. Representation (2.13) is thus equivalent to the equations

$$\left\{ \begin{array}{l} \begin{bmatrix} x_{[0]} & y_p \end{bmatrix} \;=\; u_p T_p \\ y_f \;=\; [\,x_{[0]} \quad u_f\,] T_f \end{array} \right. \qquad \text{with} \qquad \left\{ \begin{array}{rcl} T_p & = & [\,\mathbf{P}_0(\,\cdot\,\mathbf{F}^*) \quad K_T\,] \\ T_f & = & \begin{bmatrix} \mathbf{G} \\ E_T \end{bmatrix} \end{array} \right. \tag{2.16}$$

See figure 2(d). We call the decomposition of T into the operators T_p and T_f a *state splitting* of T.

2.6. Controllability and observability operators

An important aspect of the factorization $H_T = \mathbf{P}_0(\,\cdot\,\mathbf{F}^*)\,\mathbf{G}$ is its minimality, since this will imply the minimality of the sequence of dimensions of x and thus the minimality of the realization. A realization is said to be *controllable* if the range of $\mathbf{P}_0(\,\cdot\,\mathbf{F}^*)|_{\mathcal{L}_2 Z^{-1}}$ is dense in $\mathcal{D}_2^{\mathcal{B}}$, and

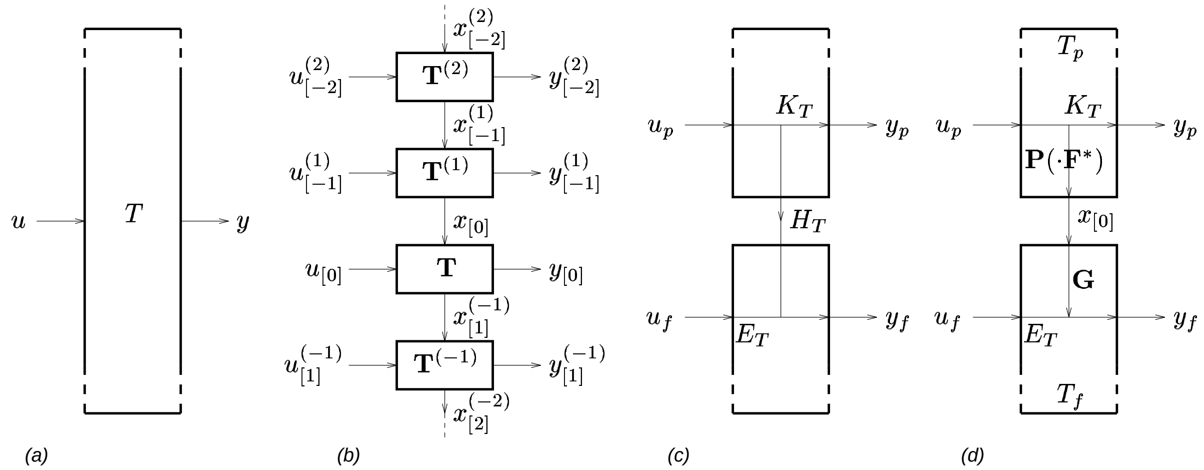

Figure 2. (a) Causal transfer operator T, (b) realization $\mathbf{T}$, (c) splitting into past and future signals, (d) representation by T_p and T_f.

uniformly controllable if its range is all of $\mathcal{D}_2^{\mathcal{B}}$, that is, if $\mathbf{P}_0(\mathcal{L}_2 Z^{-1}\,\mathbf{F}^*) = \mathcal{D}_2$. If $\mathbf{P}_0(\,\cdot\,\mathbf{F}^*)$ is regarded as an operator from $\mathcal{L}_2 Z^{-1} \to \mathcal{D}_2$, then its adjoint is $\cdot\,\mathbf{F}$ with domain $\mathcal{D}_2$, and the realization is controllable if $D\mathbf{F} = 0 \Rightarrow D = 0$ $(D \in \mathcal{D}_2)$, that is, if the gramian $\mathbf{P}_0(\mathbf{F}\mathbf{F}^*) > 0$. The realization is uniformly controllable if $\mathbf{P}_0(\mathbf{F}\mathbf{F}^*)$ is uniformly positive, by which we mean $\exists\, \epsilon > 0 \,:\, \mathbf{P}_0(\mathbf{F}\mathbf{F}^*) \geq \epsilon I$. Observability is defined in much the same way. A realization is observable if $\mathbf{P}_0(\mathcal{U}_2\mathbf{G}^*)$ is dense in $\mathcal{D}_2^{\mathcal{B}}$, which is equivalent to $\mathbf{P}_0(\mathbf{G}\mathbf{G}^*) > 0$, and uniformly observable if $\mathbf{P}_0(\mathcal{U}_2\mathbf{G}^*) = \mathcal{D}_2^{\mathcal{B}}$, *i.e.*, $\mathbf{P}_0(\mathbf{G}\mathbf{G}^*)$ is uniformly positive. A realization which is both controllable and observable is said to be minimal. If a realization has $\ell_A < 1$ and is uniformly observable, then the input state space is $\mathcal{H}(T) = \mathcal{D}_2^{\mathcal{B}}\mathbf{F} = \mathcal{D}_2^{\mathcal{B}}(I - Z^*A^*)^{-1}Z^*B^*$. Similarly, for a uniformly controllable realization the output state space is $\mathcal{H}_0(T) = \mathcal{D}_2^{\mathcal{B}}\mathbf{G} = \mathcal{D}_2^{\mathcal{B}}(I - AZ)^{-1}C$. This shows that if the realization is minimal, then s-dim $\mathcal{H}$ = s-dim $\mathcal{H}_0$ is equal to the sequence of dimensions of $\mathcal{B}$, the space of state sequences. It is possible to prove the converse, *i.e.*, to show that if s-dim $\mathcal{H}(T)$ = s-dim $\mathcal{H}_0(T) = [\cdots\ d_0\ \ d_1\ \ d_2\ \cdots]$ is a uniformly bounded sequence of dimensions, then there exist realizations of T with $d_k = \dim \mathcal{H}_k$ equal to the system order at point k [24]. This number is equal to the rank of H_k. We will call the sequence the *minimal system order* of T. If T admits a locally finite realization, then it is always possible to choose this realization to be either uniformly controllable or uniformly observable, although it may not be possible to have both (this is typically the case if the range of H_T is not closed).

The action of the operators $\mathbf{P}_0(\,\cdot\,\mathbf{F}^*)$ and $\mathbf{G}$ can be made more 'visual' by considering a representation as sequences of diagonals [24]. From (2.14) and (2.15) we obtain

$$
\begin{aligned}
x_{[0]} &= \mathbf{P}_0\left(u_p BZ(I-AZ)^{-1}\right) \\
&= \mathbf{P}_0\left([Z^{-1}u_{[-1]} + Z^{-2}u_{[-2]} + \cdots][BZ + BZ\,AZ + BZ\,AZ\,AZ + \cdots]\right) \\
&= \mathbf{P}_0\left([u_{[-1]}^{(1)}Z^{-1} + u_{[-2]}^{(2)}Z^{-2} + \cdots][ZB^{(1)} + Z^2B^{(2)}A^{(1)} + Z^3B^{(3)}A^{(2)}A^{(1)} + \cdots]\right) \\
&= \begin{bmatrix} u_{[-1]}^{(1)} & u_{[-2]}^{(2)} & u_{[-3]}^{(3)} \cdots \end{bmatrix} \begin{bmatrix} B^{(1)} \\ B^{(2)}A^{(1)} \\ B^{(3)}A^{(2)}A^{(1)} \\ \vdots \end{bmatrix}
\end{aligned}
$$

and

$$
[\, y_{[0]} \quad y_{[1]}^{(-1)} \quad y_{[2]}^{(-2)} \quad \cdots] = x_{[0]} \begin{bmatrix} C & AC^{(-1)} & AA^{(-1)}C^{(-2)} & \cdots \end{bmatrix}.
$$

The operators

$$
\mathcal{C} := \begin{bmatrix} B^{(1)} \\ B^{(2)}A^{(1)} \\ B^{(3)}A^{(2)}A^{(1)} \\ \vdots \end{bmatrix} \qquad \mathcal{O} := \begin{bmatrix} C & AC^{(-1)} & AA^{(-1)}C^{(-2)} & \cdots \end{bmatrix}
$$

are representations of $\mathbf{P}_0(\,\cdot\mathbf{F}^*)$ and $\mathbf{G}$ in 'diagonal sequence' spaces $\ell_2(\mathcal{D})$ which are isomorphic to $\mathcal{L}_2Z^{-1}$ and $\mathcal{U}_2$. $\mathcal{C}_k$ and $\mathcal{O}_k$ as defined in the introduction (equation (1.5)) are obtained from $\mathcal{C}$ and $\mathcal{O}$ by taking the k-th entry along each diagonal of $\mathcal{C}$ and $\mathcal{O}$. The controllability gramian $\mathbf{P}_0(\mathbf{FF}^*)$ is equal to $\mathcal{C}^*\mathcal{C}$. Likewise, the observability gramian is $\mathbf{P}_0(\mathbf{GG}^*) = \mathcal{O}\mathcal{O}^*$.

2.7. Lyapunov Equations

Another notion that we shall need is that of state transformations. If $\{A, B, C, D\}$ is a strictly stable realization of a system with transfer operator T, then an equivalent strictly stable realization is found by applying a state transformation $\hat{x} = xR$ on the state sequence x of the system with a bounded and boundedly invertible diagonal operator R. The transition operator $\mathbf{T}$ is then transformed to

$$
\mathbf{T}' = \begin{bmatrix} R & \\ & I \end{bmatrix} \begin{bmatrix} A & C \\ B & D \end{bmatrix} \begin{bmatrix} (R^{(-1)})^{-1} & \\ & I \end{bmatrix}.
$$

(Note the diagonal shift in $(R^{(-1)})^{-1}$). It is easy to see that $\ell_{RA(R^{(-1)})^{-1}} = \ell_A$, hence that strict stability is preserved under the transformation. State transformations are often used to bring a transition operator into some desirable form. This then leads to equations of the famous Lyapunov or Lyapunov-Stein type. For example, the Lyapunov equation

$$
M^{(-1)} = A^*MA + B^*B\,, \qquad M \in \mathcal{D}(\mathcal{B}, \mathcal{B}) \tag{2.17}
$$

arises in the transformation of a strictly stable and uniformly controllable realization to input normal form: one for which $A^*A + B^*B = I$. If the original realization is uniformly controllable, then a boundedly invertible state transformator R can be found such that $A_1 = RA(R^{(-1)})^{-1}$, $B_1 = B(R^{(-1)})^{-1}$ and $A_1^*A_1 + B_1^*B_1 = I$. Substitution leads to equation (2.17), with $M = R^*R$, and hence it suffices to solve this equation for M and to verify that M is boundedly invertible, in which case a factor R is boundedly invertible too. Equation (2.17) will have the unique solution

$$M \;=\; \left\{ \sum_{k=0}^{\infty} (A^{\{k\}})^* \, (B^*B)^{(k)} \, A^{\{k\}} \right\}^{(1)}$$

provided $\ell_A < 1$, in which case the sum converges in operator norm. By taking the k-th entry of each diagonal which appears in (2.17), this equation leads to $M_{k+1} = A_k^* M_k A_k + B_k^* B_k$, which can be solved recursively if an initial value for some M_k is known. Finally, if $\mathcal{C}$ is the controllability operator of the given realization, then $M = \mathcal{C}^*\mathcal{C}$ is the solution of (2.17), which shows that M is boundedly invertible if the realization is uniformly controllable. Likewise, if the realization is strictly stable and uniformly observable ($\mathcal{O}$ is such that $Q = \mathcal{O}\mathcal{O}^*$ is boundedly invertible), then Q is the unique bounded solution of the Lyapunov equation

$$Q = AQ^{(-1)}A^* + CC^* \tag{2.18}$$

and with the factoring of $Q = RR^*$ this yields a boundedly invertible state transformation R such that $A_1 = R^{-1}AR^{(-1)}$, $B_1 = BR^{(-1)}$, $C_1 = R^{-1}C$, and $A_1A_1^* + C_1C_1^* = I$. The resulting $\{A_1, B_1, C_1, D\}$ then form an output normal realization for the operator. In section 6 we shall assume that the operator to be approximated is indeed specified by a realization in output normal form. If T is locally finite, then it is always possible to obtain a realization in output normal form by constructing it from an orthonormal basis of the output state space $\mathcal{H}_0(T)$ [24]. For a general upper operator T, one other way to obtain such a realization is by breaking off the diagonal series representation of T at a sufficiently high order, $T \approx \sum_0^N \, Z^{[k]}T_{[k]}$, in which case we can take as a trivial (possibly non-minimal) realization

$$\begin{bmatrix} A_k & C_k \\ B_k & D_k \end{bmatrix} = \left[\begin{array}{ccccc|c} 0 & 1 & & & 0 & 0 \\ & 0 & 1 & & & \vdots \\ & & \ddots & \ddots & & \vdots \\ & & & 0 & 1 & 0 \\ 0 & & & & 0 & 1 \\ \hline T_{k,k+N} & & \cdots & & T_{k,k+1} & T_{k,k} \end{array}\right]$$

In this case, $\ell_A = 0$ (A is nilpotent) and the realization is strictly stable, while it is also in output normal form: $AA^* + CC^* = I$. This realization is a high-order approximating model of T, which can subsequently be reduced to a low-order Hankel-norm approximant by application of theorem 1.1.

3. J-UNITARY OPERATORS

In the sequel, we shall be dependent on the properties of certain unitary and J-unitary operators. These are well-known for their time-invariant analogs, and some of the generalizations have already been derived for the context of upper operators which act on 'constant size' sequences of spaces in [16, 23]. The present section considers the more general case of non-uniform sequences of spaces.

3.1. J-unitary operators and J-unitary realizations

If an operator is at the same time unitary and upper, we shall call it an *inner* operator. In this paper we shall make extensive use of operators Θ that consist of 2×2 block entries which are upper operators such that Θ is J-unitary in a generalized sense. To introduce this notion properly, we must define a splitting of the sequence of input spaces of Θ into two sequences $\mathcal{M}_1$ and $\mathcal{N}_1$, a splitting of the sequence of output space sequences into two sequences $\mathcal{M}_2$ and $\mathcal{N}_2$, and define corresponding signature sequences J_1 and J_2:

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}, \qquad J_1 = \begin{bmatrix} I_{\mathcal{M}_1} & \\ & -I_{\mathcal{N}_1} \end{bmatrix}, \qquad J_2 = \begin{bmatrix} I_{\mathcal{M}_2} & \\ & -I_{\mathcal{N}_2} \end{bmatrix}. \tag{3.1}$$

Θ decomposes in four blocks, mapping $\ell_2^{\mathcal{M}_1} \oplus \ell_2^{\mathcal{N}_1}$ to $\ell_2^{\mathcal{M}_2} \oplus \ell_2^{\mathcal{N}_2}$. If each of these maps is upper, we say that Θ is block-upper. Θ will be called J-unitary relative to this splitting in blocks, when

$$\Theta^* J_1 \Theta = J_2 \quad \text{and} \quad \Theta J_2 \Theta^* = J_1 \,.$$

We will *not* say that such a Θ is J-inner unless additional constraints which go beyond the scope of the present paper are satisfied (see also [23]). We will now show that a state realization $\mathbf{\Theta}$ which is J-unitary in a certain sense leads to a J-unitary transfer operator Θ. Let $\mathcal{B}$ be the sequence of spaces of the state of $\mathbf{\Theta}$, and let $\mathcal{B} = \mathcal{B}_+ \oplus \mathcal{B}_-$ be a decomposition of $\mathcal{B}$ into two sequences of spaces. Let

$$J_{\mathcal{B}} = \begin{bmatrix} I_{\mathcal{B}_+} & \\ & -I_{\mathcal{B}_-} \end{bmatrix} \tag{3.2}$$

be a corresponding signature matrix, which we call in this context the state signature sequence of $\mathbf{\Theta}$. We have the following connection between a realization matrix $\mathbf{\Theta}$ and the transfer operator Θ.

THEOREM 3.1. *Let J_1, J_2 and $J_{\mathcal{B}}$ be signature sequences with dimensions as given in (3.1), (3.2). If a state realization operator $\mathbf{\Theta} = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$ is strictly stable and satisfies*

$$\mathbf{\Theta}^* \begin{bmatrix} J_{\mathcal{B}} & \\ & J_1 \end{bmatrix} \mathbf{\Theta} = \begin{bmatrix} J_{\mathcal{B}}^{(-1)} & \\ & J_2 \end{bmatrix} \tag{3.3}$$

$$\mathbf{\Theta} \begin{bmatrix} J_{\mathcal{B}}^{(-1)} & \\ & J_2 \end{bmatrix} \mathbf{\Theta}^* = \begin{bmatrix} J_{\mathcal{B}} & \\ & J_1 \end{bmatrix} \tag{3.4}$$

then the corresponding transfer operator $\Theta = \delta + \beta Z(I - \alpha Z)^{-1}\gamma$ will be J-unitary in the sense that

$$\Theta^* J_1 \Theta = J_2 \,, \qquad \Theta J_2 \Theta^* = J_1 \,. \tag{3.5}$$

With '#' indicating the sequence of dimensions of a space sequence, we will have in addition that

$$\begin{array}{rcl} \#\mathcal{B}_+ + \#\mathcal{M}_1 & = & \#\mathcal{B}_+^{(-1)} + \#\mathcal{M}_2 \\ \#\mathcal{B}_- + \#\mathcal{N}_1 & = & \#\mathcal{B}_-^{(-1)} + \#\mathcal{N}_2 \,. \end{array} \tag{3.6}$$

PROOF The theorem is readily verified by evaluating $J_2 - \Theta^* J_1 \Theta$ and $J_1 - \Theta J_2 \Theta^*$, *e.g.*,

$$\begin{array}{rcl} J_2 - \Theta^* J_1 \Theta & = & J_2 - \delta^* J_1 \delta \,+\, \gamma^*(I - Z^*\alpha^*)^{-1} Z^*\alpha^* J_{\mathcal{B}}\gamma \,+\, \gamma^* J_{\mathcal{B}}\alpha Z(I-\alpha Z)^{-1}\gamma \,+ \\ & & -\, \gamma^*(I - Z^*\alpha^*)^{-1} Z^*\{J_{\mathcal{B}}^{(-1)} - \alpha^* J_{\mathcal{B}}\alpha\} Z(I-\alpha Z)^{-1}\gamma \\ & = & \gamma^* J_{\mathcal{B}}\gamma \,+\, \gamma^*(I - Z^*\alpha^*)^{-1}\{Z^*\alpha^* J_{\mathcal{B}} + J_{\mathcal{B}}\alpha Z - J_{\mathcal{B}} - Z^*\alpha^* J_{\mathcal{B}}\alpha Z\}(I-\alpha Z)^{-1}\gamma \end{array}$$

since $\beta^* J_1 \delta = -\alpha^* J_{\mathcal{B}}\gamma$, $\beta^* J_1 \beta = J_{\mathcal{B}}^{(-1)} - \alpha^* J_{\mathcal{B}}\alpha$ and $J_2 - \delta^* J_1 \delta = \gamma^* J_{\mathcal{B}}\gamma$, and hence

$$\begin{array}{rcl} J_2 - \Theta^* J_1 \Theta & = & \gamma^*(I - Z^*\alpha^*)^{-1}\{(I - Z^*\alpha^*) J_{\mathcal{B}}(I - \alpha Z) \,+ \\ & & +\, Z^*\alpha^* J_{\mathcal{B}} + J_{\mathcal{B}}\alpha Z - J_{\mathcal{B}} - Z^*\alpha^* J_{\mathcal{B}}\alpha Z\}(I-\alpha Z)^{-1}\gamma \\ & = & 0\,. \end{array}$$

The second equality of (3.5) follows by an analogous procedure.

Equations (3.3) and (3.4) show that $\boldsymbol{\Theta}$ is invertible, so that for each time instant k, the total positive signature at the left hand side of each equation is equal to the total positive signature at the right hand side (the so-called inertia theorem). A similar observation holds for the total negative signature at each point k. This is expressed in (3.6). □

3.2. Interpolation properties of J-unitary operators

A J-unitary upper operator has the special property that it maps its input state space to $[\mathcal{U}_2 \;\; \mathcal{U}_2]$.

PROPOSITION 3.2. *If $\{\alpha, \beta, \gamma, \delta\}$ is a J-unitary state realization for a J-unitary block-upper operator Θ, then*

$$(I - Z^*\alpha^*)^{-1} Z^* \beta^* J_1\, \Theta \;\in\; [\mathcal{U} \;\; \mathcal{U}] \tag{3.7}$$

that is, $(I - Z^*\alpha^*)^{-1} Z^*\beta^* J_1$, which consists of two strictly lower blocks, is mapped by Θ to block upper.

Note that the input state space of Θ is $\mathcal{H}(\Theta) = \mathcal{D}_2(I - Z^*\alpha^*)^{-1} Z^*\beta^* \in [\mathcal{L}_2 Z^{-1} \;\; \mathcal{L}_2 Z^{-1}]$.

PROOF Evaluation of (3.7) and using (3.3) reveals that

$$\begin{array}{l} (I - Z^*\alpha^*)^{-1} Z^*\beta^* J_1 \;\; \{\delta + \beta Z(I-\alpha Z)^{-1}\gamma\} \\ \quad = (I - Z^*\alpha^*)^{-1} Z^* \left\{-\alpha^* J_{\mathcal{B}} + (J_{\mathcal{B}}^{(-1)} - \alpha^* J_{\mathcal{B}}\alpha) Z(I-\alpha Z)^{-1}\right\}\gamma \\ \quad = (Z - \alpha^*)^{-1}\left\{-\alpha^* J_{\mathcal{B}}(I - \alpha Z) + J_{\mathcal{B}}^{(-1)} Z - \alpha^* J_{\mathcal{B}} Z\right\}(I-\alpha Z)^{-1}\gamma \\ \quad = J_{\mathcal{B}}(I-\alpha Z)^{-1}\gamma \quad \in \; [\mathcal{U} \;\; \mathcal{U}]\,. \end{array} \tag{3.8}$$

□

This property has an 'interpolation' interpretation which is explained in detail (for the less general context of uniform sequences of spaces) in [23]. The interpolation principle will provide us with the necessary factorizations and will be used in section 6.

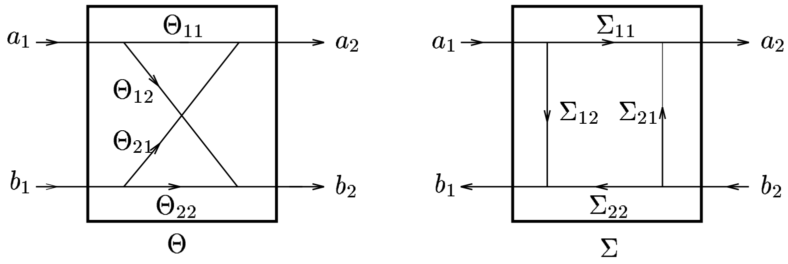

Figure 3. Relation between a J-unitary operator Θ and its corresponding unitary operator Σ.

3.3. Scattering operators

Two of the relations in the expressions $\Theta^* J_1 \Theta = J_2$, $\Theta J_2 \Theta^* = J_1$ are $\Theta_{22}^*\Theta_{22} = I + \Theta_{12}^*\Theta_{12}$ and $\Theta_{22}\Theta_{22}^* = I + \Theta_{21}\Theta_{21}^*$. Hence Θ_{22} is a one-to-one map of $\ell_2^{\mathcal{N}_1}$ onto $\ell_2^{\mathcal{N}_2}$, which ensures that it is boundedly invertible [16, lemma 5.2]. Associated to Θ is an operator Σ,

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

such that $[a_1 \;\; b_2]\Sigma = [a_2 \;\; b_1] \;\Leftrightarrow\; [a_1 \;\; b_1]\Theta = [a_2 \;\; b_2]$, (see figure 3). Σ can be evaluated in terms of the block-entries of Θ as

$$\Sigma = \begin{bmatrix} I & -\Theta_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} \Theta_{11} & 0 \\ 0 & \Theta_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ \Theta_{21} & I \end{bmatrix} = \begin{bmatrix} \Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{21} & -\Theta_{12}\Theta_{22}^{-1} \\ \Theta_{22}^{-1}\Theta_{21} & \Theta_{22}^{-1} \end{bmatrix} \tag{3.9}$$

It is well-known and straightforward to prove that from the J-unitarity of Θ it follows that Σ is unitary. Σ is known as a *scattering operator*, while Θ is called a *chain scattering operator*. Σ and Θ constitute the same linear relations between the quantities a_1, a_2, b_1, b_2. However, Σ has the connotation of visualizing the 'positive energy flow' between these quantities: $a_1a_1^* + b_2b_2^* = a_2a_2^* + b_1b_1^*$, whereas for Θ, $a_1a_1^* - b_1b_1^* = a_2a_2^* - b_2b_2^*$. In engineering literature, a_1 and b_2 are known as incident waves, whereas a_2 and b_1 are reflected waves. One fact which will be essential for the approximation theory in the later chapters is that, although Θ has block-entries which are upper, Θ_{22}^{-1} need not be upper but can be of mixed causality, so that the block-entries of Σ are in general not upper.

3.4. Realization for Σ

The realization $\boldsymbol{\Theta}$ for Θ satisfies (3.3) and (3.4). The state space sequence $\mathcal{B}$ decomposes into two complementary space sequences $\mathcal{B} = \mathcal{B}_+ \oplus \mathcal{B}_-$ of locally finite dimensions. Let any state sequence $x \in \mathcal{X}_2^{\mathcal{B}}$ and $\boldsymbol{\Theta}$ be partitioned accordingly into $x = [x_+ \;\; x_-]$ with $x_+ \in \mathcal{X}_2^{\mathcal{B}+}$ and $x_- \in \mathcal{X}_2^{\mathcal{B}-}$ and

$$[x_+ \quad x_- \quad a_1 \quad b_1]\,\boldsymbol{\Theta} \;=\; [x_+Z^{-1} \quad x_-Z^{-1} \quad a_2 \quad b_2]\,. \tag{3.10}$$

$\mathbf{\Theta}$ has a corresponding partitioning into

$$\mathbf{\Theta} \; = \; \begin{array}{c} \\ x_+ \\ x_- \\ a_1 \\ b_1 \end{array} \begin{array}{c} \begin{array}{cccc} x_+Z^{-1} & x_-Z^{-1} & a_2 & b_2 \end{array} \\ \left[\begin{array}{cc|cc} \alpha_{11} & \alpha_{12} & \gamma_{11} & \gamma_{12} \\ \alpha_{21} & \alpha_{22} & \gamma_{21} & \gamma_{22} \\ \hline \beta_{11} & \beta_{12} & \delta_{11} & \delta_{12} \\ \beta_{21} & \beta_{22} & \delta_{21} & \delta_{22} \end{array}\right] \end{array} . \tag{3.11}$$

A reordering of rows and columns with respect to their signatures converts $\mathbf{\Theta}$ into a genuine square-block J-unitary operator, *i.e.*, each matrix

$$\left[\begin{array}{cc|cc} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} \\ \hline \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ \beta_{21} & \delta_{21} & \beta_{22} & \delta_{22} \end{array}\right]_k$$

is a square and J-unitary matrix with signature $[I_{(\mathcal{B}_+)_k \oplus (\mathcal{M}_1)_k} \;\dot{+}\; -I_{(\mathcal{B}_-)_k \oplus (\mathcal{N}_1)_k}]$. In particular, each submatrix

$$\begin{bmatrix} \alpha_{22} & \gamma_{22} \\ \beta_{22} & \delta_{22} \end{bmatrix}_k$$

of $\mathbf{\Theta}_k$ is square and invertible, and because $\mathbf{\Theta}$ is J-unitary, the block-diagonal operator constructed from these submatrices is boundedly invertible as well. It follows that the following block-diagonal operators are well-defined (*cf.* equation (3.9)):

$$\begin{aligned} \begin{bmatrix} F_{11} & H_{11} \\ G_{11} & K_{11} \end{bmatrix} &= \begin{bmatrix} \alpha_{11} & \gamma_{11} \\ \beta_{11} & \delta_{11} \end{bmatrix} - \begin{bmatrix} \alpha_{12} & \gamma_{12} \\ \beta_{12} & \delta_{12} \end{bmatrix} \begin{bmatrix} \alpha_{22} & \gamma_{22} \\ \beta_{22} & \delta_{22} \end{bmatrix}^{-1} \begin{bmatrix} \alpha_{21} & \gamma_{21} \\ \beta_{21} & \delta_{21} \end{bmatrix} \\ \begin{bmatrix} F_{12} & H_{12} \\ G_{12} & K_{12} \end{bmatrix} &= - \begin{bmatrix} \alpha_{12} & \gamma_{12} \\ \beta_{12} & \delta_{12} \end{bmatrix} \begin{bmatrix} \alpha_{22} & \gamma_{22} \\ \beta_{22} & \delta_{22} \end{bmatrix}^{-1} \\ \begin{bmatrix} F_{21} & H_{21} \\ G_{21} & K_{21} \end{bmatrix} &= \begin{bmatrix} \alpha_{22} & \gamma_{22} \\ \beta_{22} & \delta_{22} \end{bmatrix}^{-1} \begin{bmatrix} \alpha_{21} & \gamma_{21} \\ \beta_{21} & \delta_{21} \end{bmatrix} \\ \begin{bmatrix} F_{22} & H_{22} \\ G_{22} & K_{22} \end{bmatrix} &= \begin{bmatrix} \alpha_{22} & \gamma_{22} \\ \beta_{22} & \delta_{22} \end{bmatrix}^{-1} \end{aligned} \tag{3.12}$$

and we obtain the relation

$$[x_+ \quad x_-Z^{-1} \quad a_1 \quad b_2]\,\mathbf{\Sigma} \;=\; [x_+Z^{-1} \quad x_- \quad a_2 \quad b_1] \tag{3.13}$$

where

$$\mathbf{\Sigma} = \begin{array}{c} \\ x_+ \\ x_-Z^{-1} \\ a_1 \\ b_2 \end{array} \begin{array}{c} \begin{array}{cccc} x_+Z^{-1} & x_- & a_2 & b_1 \end{array} \\ \left[\begin{array}{cc|cc} F_{11} & F_{12} & H_{11} & H_{12} \\ F_{21} & F_{22} & H_{21} & H_{22} \\ \hline G_{11} & G_{12} & K_{11} & K_{12} \\ G_{21} & G_{22} & K_{21} & K_{22} \end{array}\right] \end{array} . \tag{3.14}$$

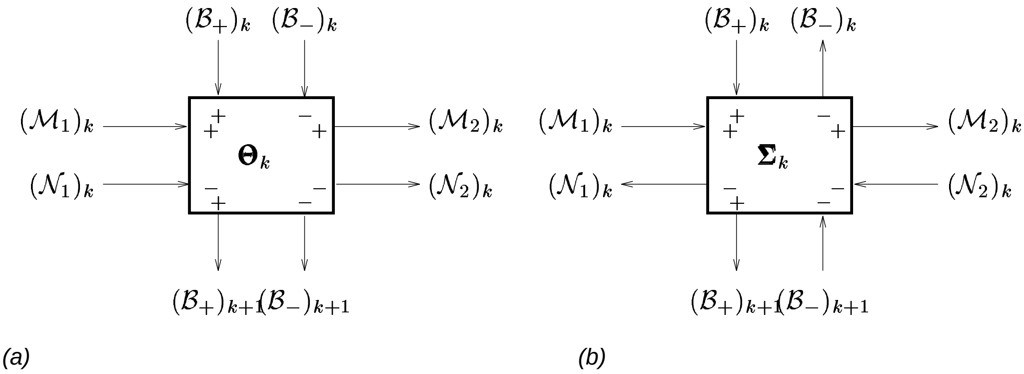

Figure 4. (a) The spaces connected with a realization for a J-unitary block-upper operator Θ which transfers $\ell_2^{\mathcal{M}_1} \times \ell_2^{\mathcal{N}_1}$ to $\ell_2^{\mathcal{M}_2} \times \ell_2^{\mathcal{N}_2}$. The state transition operator is marked as $\mathbf{\Theta}$. $\mathbf{\Theta}$ transfers, at stage k, $(\mathcal{B}_+)_k \times (\mathcal{B}_-)_k \times (\mathcal{M}_1)_k \times (\mathcal{N}_1)_k$ to $(\mathcal{B}_+)_{k+1} \times (\mathcal{B}_-)_{k+1} \times (\mathcal{M}_2)_k \times (\mathcal{N}_2)_k$. ($b$) The corresponding scattering situation.

See figure 4. $\mathbf{\Sigma}$ is a kind of generalized or implicit realization for Σ, which can be obtained after elimination of x_- and x_+. It can be interpreted as a realization having an 'upward' state sequence x_- and a downward state sequence x_+, as depicted in figure 5. Be that as it may, the important point is the existence of $\mathbf{\Sigma}$ and its unitarity:

$$\mathbf{\Sigma}\mathbf{\Sigma}^* = I\,; \quad \mathbf{\Sigma}^*\mathbf{\Sigma} = I$$

which is easily derived from the J-unitarity of $\mathbf{\Theta}$.

3.5. Existence of Σ_p and Σ_f

In section 2, we defined for a signal $u \in \mathcal{X}_2$ the decomposition $u = u_p + u_f$, where $u_p = \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(u) \in \mathcal{L}_2 Z^{-1}$ is the 'past' part of the signal (with reference to its 0-th diagonal), and $u_f = \mathbf{P}(u) \in \mathcal{U}_2$ is its 'future' part. We also showed how a causal operator T with state realization $\mathbf{T}$ can be split into a past operator T_p which maps u_p to $[x_{[0]} \;\; y_p]$ and a future operator T_f which maps $[x_{[0]} \;\; u_f]$ to y_f. In the present context, let the signals a_1, b_1, a_2, b_2 and the state sequences x_+, x_- be in $\mathcal{X}_2$ and be related by $\mathbf{\Theta}$ as in (3.10). With the partitioning of a_1, etc., into a past and a future part, Θ can be split into operators $(\cdot)\Theta_p$ and $(\cdot)\Theta_f$ via

$$\begin{array}{rcl} [a_{1p} \quad b_{1p}]\,\Theta_p &=& [x_{+[0]} \quad x_{-[0]} \quad a_{2p} \quad b_{2p}] \\ {[x_{+[0]} \quad x_{-[0]} \quad a_{1f} \quad b_{1f}]}\,\Theta_f &=& [a_{2f} \quad b_{2f}]\,. \end{array} \tag{3.15}$$

We wish to define operators $(\cdot)\Sigma_p$, $(\cdot)\Sigma_f$:

$$\begin{array}{rcl} [x_{-[0]} \quad a_{1p} \quad b_{2p}]\,\Sigma_p &=& [x_{+[0]} \quad a_{2p} \quad b_{1p}] \\ {[x_{+[0]} \quad a_{1f} \quad b_{2f}]}\,\Sigma_f &=& [x_{-[0]} \quad a_{2f} \quad b_{1f}] \end{array}$$

which are the (non-causal) scattering operators corresponding to Θ_p and Θ_f, respectively. (See figure 5(b).) The existence of Σ_p and Σ_f is asserted in the following proposition.

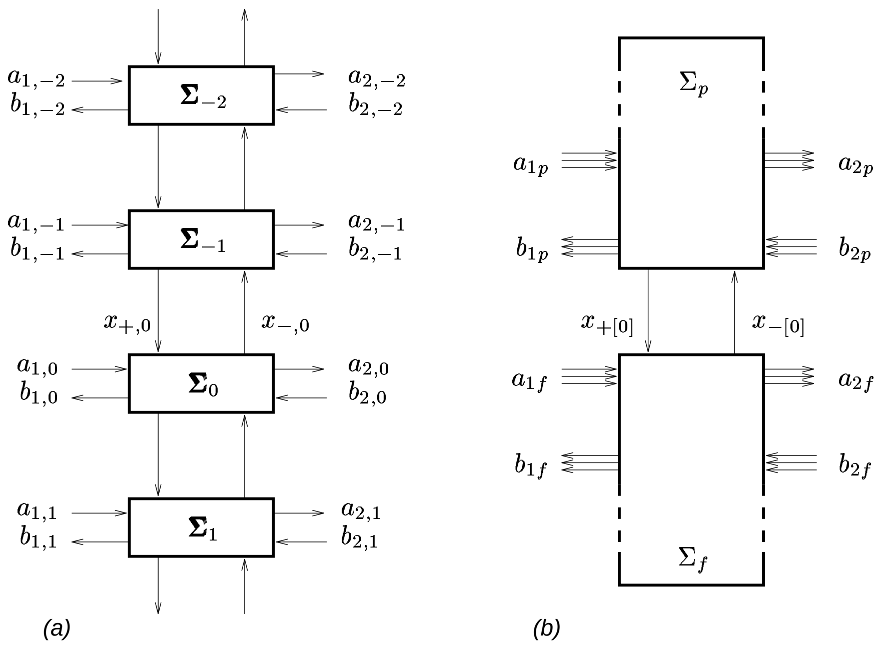

Figure 5. (a) The state transition scheme for Σ, with ℓ_2-sequences as inputs. (b) The decomposition of Σ into a past operator Σ_p and a future operator Σ_f linked by the state $[x_{+[0]} \quad x_{-[0]}]$. This summarizes the figure on the left for all time.

PROPOSITION 3.3. *Let $\Theta \in \mathcal{U}$ be a locally finite J-unitary operator with strictly stable J-unitary realization $\boldsymbol{\Theta}$. Then the associated scattering operators Σ_p and Σ_f are well defined and unitary.*

PROOF The proof follows an argument that parallels the proof of existence of Σ from Θ, but now applied to restrictions of Θ_p and Θ_f to inputs and outputs for which only the k-th row is non-zero, for all k in turn (*cf.* the connection of H_k with H_T). In this proof, the operator Θ is considered as an operator from $\ell_2^{\mathcal{M}_1} \oplus \ell_2^{\mathcal{N}_1}$ to $\ell_2^{\mathcal{M}_2} \oplus \ell_2^{\mathcal{N}_2}$. Fix some time point k, and let $\mathcal{M}_{1p,k}$ be the restriction of $\mathcal{M}_1$ to the interval $(-\infty, k-1)$, and likewise for $\mathcal{N}_{1p,k}$. Let the operator Θ_{pk} define the state splitting at time point k, mapping signals belonging to $\ell_2^{\mathcal{M}_{1p,k}} \oplus \ell_2^{\mathcal{N}_{1p,k}}$ to $\mathcal{B}_{+,k} \oplus \mathcal{B}_{-,k} \oplus \ell_2^{\mathcal{M}_{2p,k}} \oplus \ell_2^{\mathcal{N}_{2p,k}}$:

$$\Theta_{pk} : \quad [a_{1p,k} \;\; b_{1p,k}] \mapsto [x_{+,k} \;\; x_{-,k} \;\; a_{2p,k} \;\; b_{2p,k}] \,.$$

It has a state realization, $\{\alpha_i, \beta_i, \gamma_i, \delta_i\}_{-\infty}^{\infty}$ say, which coincides with the state realization of Θ up to the index $k-2$. For the index $k-1$, the computed $x_{+,k}$ and $x_{-,k}$ have to be considered as outputs (not as states), and hence the realization at that point has vanishing $\alpha_{k-1}, \beta_{k-1}$, leaving a square J-unitary pair

$$\begin{bmatrix} \gamma_{k-1} \\ \delta_{k-1} \end{bmatrix} = \boldsymbol{\Theta}_{k-1} \,.$$

From point k on, all state matrices of Θ_{pk} have vanishing dimensions. Because Θ_{pk} now has a J-unitary realization, it is a J-unitary operator itself (by theorem 3.1), and the corresponding scattering map Σ_{pk} exists as well as a bounded, unitary operator linking

$$\Sigma_{pk}: \quad [x_{-,k} \;\; a_{1p,k} \;\; b_{2p,k}] \mapsto [x_{+,k} \;\; a_{2p,k} \;\; b_{1p,k}]$$

Finally, returning from ℓ_2 to the $\mathcal{X}_2$ context, Σ_p is obtained by piecing the Σ_{pk} together into one global operator. A parallel, dual reasoning holds for Σ_f. □

4. SUMMARY OF THE APPROXIMATION PROCEDURE

In the present section we shall outline the overall procedure to obtain a reduced-order approximant, and put the various facts needed in perspective. Details are proven in subsequent sections.

Let $\mathcal{M}$ and $\mathcal{N}$ be sequences of spaces, and let $T \in \mathcal{U}(\mathcal{M}, \mathcal{N})$ be a given bounded, locally finite, strictly upper operator which has a strictly stable realization $\{A, B, C, 0\}$. We assume, without loss of generality, that this realization is uniformly observable. Let Γ be a diagonal and hermitian operator belonging to $\mathcal{D}(\mathcal{M}, \mathcal{M})$. We shall use Γ as a measure for the local accuracy of the reduced order model. It will also parametrize the solutions. We will look for a bounded operator T' such that

$$\| \Gamma^{-1}(T - T') \| \;\leq\; 1\,, \tag{4.1}$$

and such that the strictly upper part of T' has state space dimensions of low order — as low as possible for a given Γ. Let T_a be the strictly causal part of T'. Proposition 2.1 shows that

$$\begin{aligned} \| \Gamma^{-1}(T - T_a) \|_H &= \| \Gamma^{-1}(T - T') \|_H \\ &\leq \| \Gamma^{-1}(T - T') \| \;\leq\; 1\,, \end{aligned} \tag{4.2}$$

so that T_a is a Hankel-norm approximant when T' is an operator-norm approximant. A generalization of Nehari's theorem to the present setting would state that $\inf \| \Gamma^{-1}(T - T') \|$ over all possible extensions T' of T_a actually equals $\| \Gamma^{-1}(T - T_a) \|_H$ (see section 9).

The construction of an operator T' satisfying (4.1) consists of three steps. We start by determining a (minimal) factorization of T in the form

$$T = \Delta^* U \tag{4.3}$$

where Δ and U are upper operators which have finite state space dimensions of the same size as that of T, and U is inner. We call this factorization an *external factorization* of T, and show in section 5 that it always exists if the realization $\{A, B, C, 0\}$ for T is strictly stable and is chosen to be in output normal form, *i.e.,* such that $AA^* + CC^* = I$, a condition that is always possible to achieve by a state transformation starting from an initial realization which is uniformly observable (see the Lyapunov equation (2.18)). It will follow from the construction that a state space realization for U shares A and C with T.

Next, we look (in section 6) for a locally finite J-unitary operator Θ with 2×2 block-upper entries chosen such that

$$[U^* \quad -T^*\Gamma^{-1}]\,\Theta \;=\; [A' \quad -B'] \tag{4.4}$$

consists of two upper operators. We will show that a solution to this problem exists if certain conditions on a Lyapunov equation associated to $\Gamma^{-1}T$ are satisfied (this can always be the case for judiciously chosen Γ). Θ will again be locally finite. There is an underlying generalized interpolation problem leading to Θ, which will also be elucidated in section 6. The procedure is an extension of the method used in [23] to solve the time-varying Nevanlinna-Pick problem.

With a block-decomposition of Θ as in (3.1), it is known that Θ_{22} is boundedly invertible so that $\Sigma_{12} = -\Theta_{12}\Theta_{22}^{-1}$ exists and is contractive. From (4.4) we have $B' = -U^*\Theta_{12} + T^*\Gamma^{-1}\Theta_{22}$. As the third step in the construction of the approximant T_a, define

$$T'^* = B'\Theta_{22}^{-1}\Gamma\,. \tag{4.5}$$

Substitution leads to

$$\begin{aligned} T'^*\Gamma^{-1} &= T^*\Gamma^{-1} - U^*\Theta_{12}\Theta_{22}^{-1} \\ &= T^*\Gamma^{-1} - U^*\Sigma_{12} \end{aligned} \tag{4.6}$$

and it follows that $(T^* - T'^*)\Gamma^{-1} = -U^*\Sigma_{12}$. Because Σ_{12} is contractive and U unitary, we infer that

$$\begin{aligned} \|\,(T^* - T'^*)\Gamma^{-1}\,\| &= \|\,-U^*\Sigma_{12}\,\| \\ &= \|\,\Sigma_{12}\,\| \;\le\; 1\,, \end{aligned} \tag{4.7}$$

so that $T' = \left(B'\Theta_{22}^{-1}\Gamma\right)^*$ is indeed an approximant with an admissible modeling error. In view of the target theorem 1.1, it remains to show that the strictly causal part of T' has the stated number of states and to verify the relation with the Hankel singular values of $\Gamma^{-1}T$. This will done in section 6 and 7. The definition of T' in (4.5) can be generalized by the introduction of a contractive operator S_L which parameterizes the possible approximants. This is the subject of section 8. We first show that T has indeed a factorization $T = \Delta^*U$, and derive expressions for Θ satisfying the interpolation condition (4.4).

5. EXTERNAL FACTORIZATION FOR T

THEOREM 5.1. *Let T be an upper operator which has a strictly stable locally finite and uniformly observable state space realization $\{A, B, C, D\}$. Then there exists a factorization of T as*

$$T = \Delta^*U\,, \tag{5.1}$$

where Δ and U are upper operators, again locally finite and strictly stable, and U is inner, i.e., upper and unitary.

PROOF We start from a realization of T in output normal form, such that

$$AA^* + CC^* = I\,, \tag{5.2}$$

i.e., at each time point k the equation $A_kA_k^* + C_kC_k^* = I$ is satisfied. We will assume that A is a diagonal operator mapping $\ell_2^{\mathcal{B}}$ to $\ell_2^{\mathcal{B}^{(-1)}}$ and T is an operator from $\ell_2^{\mathcal{M}}$ to $\ell_2^{\mathcal{N}}$. For each time instant

k, augment the state transition matrices $[A_k \quad C_k]$ of T with as many extra rows as needed to yield a unitary (hence square) matrix $\mathbf{U}_k$:

$$\mathbf{U}_k = \begin{array}{c} \\ \mathcal{B}_k \\ (\mathcal{M}_U)_k \end{array} \begin{array}{c} \begin{array}{cc} \mathcal{B}_{k+1} & \mathcal{N}_k \end{array} \\ \begin{bmatrix} A_k & C_k \\ (B_U)_k & (D_U)_k \end{bmatrix} \end{array} . \tag{5.3}$$

The added rows introduce a space $(\mathcal{M}_U)_k$ with the property $\#\mathcal{B}_k + \#(\mathcal{M}_U)_k = \#\mathcal{B}_{k+1} + \#\mathcal{N}_k$. From $A_kA_k^* + C_kC_k^* = I$ it follows that $\#\mathcal{B}_{k+1} + \#\mathcal{N}_k \geq \#\mathcal{B}_k$, hence $\#(\mathcal{M}_U)_k \geq 0$. Assemble the individual matrices $\{A_k, (B_U)_k, C_k, (D_U)_k\}$ in diagonal operators $\{A, B_U, C, D_U\}$, and define U by taking the corresponding operator $\mathbf{U}$ as a state space realization for U; $U = D_U + B_U Z(I - AZ)^{-1}C$. U is well-defined and upper because $\ell_A < 1$, and it is unitary because it has a unitary realization (this fact is a specialization of theorem 3.1).

It remains to show that $\Delta = UT^*$ is upper. This follows by direct computation of Δ, in which we make use of the relations $AA^* + CC^* = I$, $B_UA^* + D_UC^* = 0$:

$$\begin{aligned} \Delta &= UT^* = [D_U + B_UZ(I - AZ)^{-1}C]\,[D^* + C^*(I - Z^*A^*)^{-1}Z^*B^*] \\ &= [D_U + B_UZ(I - AZ)^{-1}C]\,D^* + \underline{D_UC^*}(I - Z^*A^*)^{-1}Z^*B^* + \\ &\quad + B_UZ(I - AZ)^{-1}\underline{CC^*}(I - Z^*A^*)^{-1}Z^*B^* \\ &= [D_U + B_UZ(I - AZ)^{-1}C]\,D^* - B_UA^*(I - Z^*A^*)^{-1}Z^*B^* + \\ &\quad + B_UZ(I - AZ)^{-1}\,(I - AA^*)\,(I - Z^*A^*)^{-1}Z^*B^* . \end{aligned}$$

Now, we make use of the relation

$$Z(I - AZ)^{-1}(I - AA^*)(I - Z^*A^*)^{-1}Z^* = (I - ZA)^{-1} + A^*(I - Z^*A^*)^{-1}Z^*$$

which is easily verified by pre- and postmultiplying with $(I - ZA)$ and $Z(I - Z^*A^*)$, respectively. Plugging this relation into the expression for Δ, it is seen that the anti-causal parts of the expression cancel, and we obtain

$$\begin{aligned} \Delta &= [D_U + B_UZ(I - AZ)^{-1}C]\,D^* + B_U(I - ZA)^{-1}B^* \\ &= D_UD^* + B_UB^* + B_U(I - ZA)^{-1}Z(AB^* + CD^*) . \end{aligned}$$

□

Underlying theorem 5.1 is the interpolation property 3.2, specialized to inner operators. For a minimal realization of T, the space $\mathcal{D}_2^{\mathcal{B}}(I - AZ)^{-1}C$ may be seen as the natural output state space $\mathcal{H}_0 = \mathcal{H}_0(T)$ of T. The procedure in the theorem amounts to finding a unitary U with an output state space that at least contains the output state space of T. The interpolation property then ensures $\mathcal{H}(U)\,U \supset \mathcal{H}_0$, *i.e.*,

$$\mathcal{H}_0U^* \subset \mathcal{H}(U)$$

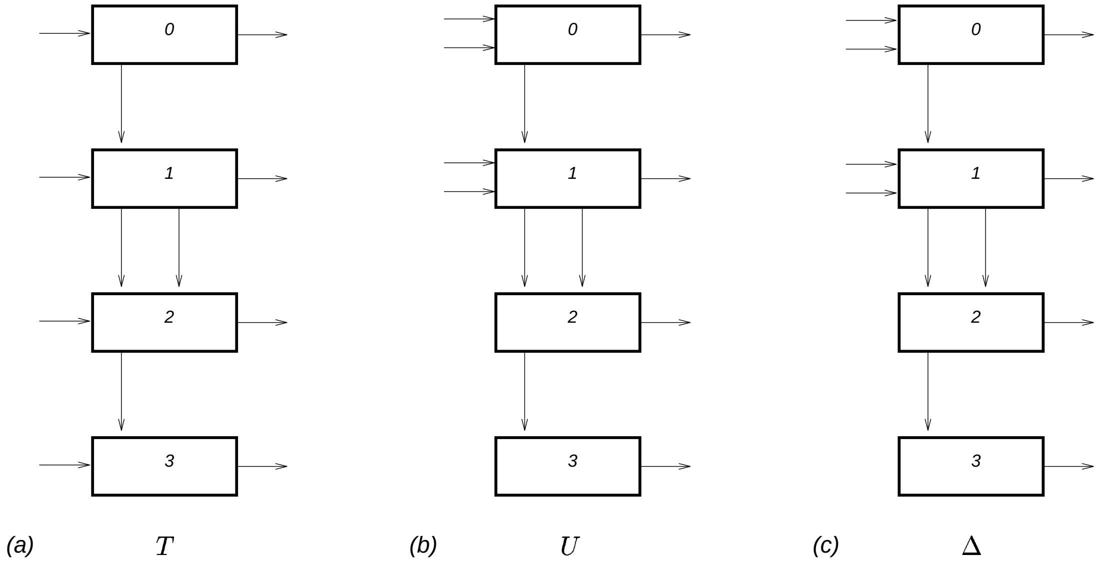

Figure 6. (a) The state space structure of an example T, (b) The structure of the corresponding inner factor U and (c) of Δ.

and it follows that $\Delta^* = TU^*$ must be lower triangular: $\mathbf{P}(\mathcal{L}_2 Z^{-1}\Delta^*) = 0$. The latter is checked as follows:

$$\begin{aligned}
\mathbf{P}(\mathcal{L}_2 Z^{-1}\,\Delta^*) &= \mathbf{P}(\mathcal{L}_2 Z^{-1} T U^*) \\
&= \mathbf{P}(\mathcal{H} T U^*) && [\text{since } U^* \in \mathcal{L}] \\
&= \mathbf{P}\,[\,\mathbf{P}(\mathcal{H}T)U^* + \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{H}T)U^*] \\
&= \mathbf{P}\,[\,\mathbf{P}(\mathcal{H}T)U^*] \\
&= \mathbf{P}\,[\mathcal{H}_0 U^*] \;\subset\; \mathbf{P}(\mathcal{H}(U)) \;=\; 0\,.
\end{aligned}$$

Because the A_k are not necessarily square matrices, the dimension of the state space may vary in time. A consequence of this will be that the number of inputs of U will vary in time for an inner U having minimal state dimension. The varying number of inputs of U will of course be matched by a varying number of outputs of Δ^*. Figure 6 illustrates this point.

6. DETERMINATION OF Θ

6.1. Construction of Θ

In this section we shall show how, under satisfaction of a condition of Lyapunov type, equation (4.4) can be satisfied with a J-unitary transfer operator Θ. We use the fact that an operator is J-unitary if its state realization is J-unitary in the sense of theorem 3.1. Let $\{A, B, C, 0\}$ be the realization for T used in section 5 (it is in output normal form), and let $\{A, B_U, C, D_U\}$ be the unitary realization for the inner factor U of T as derived in that section. Let $\mathcal{M}$ be the input space sequence of T, and $\mathcal{M}_U$ for U. The strategy consists in constructing a J-unitary (for an appropriate set $\{J_{\mathcal{B}}, J_1, J_2\}$) state space realization $\{\alpha, \beta, \gamma, \delta\}$ for Θ which is such that the input state space $\mathcal{H}(\Theta)$ is generated by

$$\mathcal{H}(\Theta) \;=\; \mathcal{D}_2^{\mathcal{B}}\,(I - Z^*A^*)^{-1}Z^*\,\left[\,B_U^* \quad B^*\Gamma^{-1}\,\right] \tag{6.1}$$

It will be shown in the proof of theorem 6.1 that this definition will ensure that $[U^* \quad -T^*\Gamma^{-1}]$ is mapped by Θ to block upper. The definition (6.1) specifies $\{\alpha,\, \beta\}$ of the realization $\boldsymbol{\Theta}$ of Θ up to a state transformation X, which must be used to ensure that the realization is J-unitary in the sense of equation (3.3): in particular $\alpha^* J_{\mathcal{B}}\alpha + \beta^* J_1 \beta = J_{\mathcal{B}}^{(-1)}$ for some signature operator $J_{\mathcal{B}}$ and $J_1 = [I_{\mathcal{M}_U} \;\dot{+}\; -I_{\mathcal{M}}]$. This condition leads to a Lyapunov-Stein type equation which will play a key role. Indeed, we try to find a boundedly invertible diagonal operator $X \in \mathcal{D}(\mathcal{B},\mathcal{B})$ such that

$$\left[\begin{array}{c} \alpha \\ \beta \end{array}\right] \;=\; \left[\begin{array}{c} XA(X^{(-1)})^{-1} \\ B_U(X^{(-1)})^{-1} \\ \Gamma^{-1}B(X^{(-1)})^{-1} \end{array}\right] \tag{6.2}$$

is J-isometric, *i.e.*, such that

$$\begin{array}{c}(X^{(-1)})^{-*}A^*X^*\,J_{\mathcal{B}}XA(X^{(-1)})^{-1} \;+\; (X^{(-1)})^{-*}B_U^*B_U(X^{(-1)})^{-1} \;+ \\ -\,(X^{(-1)})^{-*}B^*\Gamma^{-2}B(X^{(-1)})^{-1} \;=\; J_{\mathcal{B}}^{(-1)}\,. \end{array} \tag{6.3}$$

Writing $\Lambda = X^*J_{\mathcal{B}}X$, this produces

$$A^*\Lambda A \;+\; B_U^*B_U - B^*\Gamma^{-2}B \;=\; \Lambda^{(-1)}\,. \tag{6.4}$$

Since $\ell_A < 1$, this equation will always have a solution Λ (see section 2), and the signature of Λ will determine $J_{\mathcal{B}}$. For X to be boundedly invertible, it is sufficient to require Λ to be boundedly invertible. Equation (6.4) may be rewritten in terms of the original data by using $B_U^*B_U = I - A^*A$ to obtain the equation

$$A^*MA + B^*\Gamma^{-2}B = M^{(-1)}\,, \qquad M = I - \Lambda\,. \tag{6.5}$$

M is the solution of one of the Lyapunov equations associated to $\Gamma^{-1}T$ (viz. equation (2.17)), and hence can be given in closed form as

$$M = \left\{\, \sum_{k=0}^{\infty} (A^{\{k\}})^*(B^*\Gamma^{-2}B)^{(k)}A^{\{k\}} \right\}^{(1)}\,. \tag{6.6}$$

We shall see later that M is closely related to the Hankel operator of $\Gamma^{-1}T$, and in particular that the singular values of this Hankel operator determine the signature of Λ, and hence $J_{\mathcal{B}}$.

THEOREM 6.1. *Let T be a strictly upper locally finite operator mapping $\ell_2^{\mathcal{M}}$ to $\ell_2^{\mathcal{N}}$, with output normal realization $\{A,B,C,0\}$ such that $\ell_A < 1$, and let Γ be a Hermitian diagonal operator. Also let U be the inner factor of a factorization (5.1) of T. If the solution M of the Lyapunov equation*

$$A^*MA + B^*\Gamma^{-2}B = M^{(-1)} \tag{6.7}$$

is such that $\Lambda = I - M$ is boundedly invertible, then there exists a J-unitary block upper operator Θ such that

$$[U^* \quad -T^*\Gamma^{-1}]\Theta \;\in\; [\mathcal{U}\;\;\mathcal{U}]\,.$$

PROOF The condition insures that there exists a state transformation X such that (6.2) is J-isometric, *i.e.*, such that $\alpha = XA(X^{(-1)})^{-1}$, $\beta_1 = B_U(X^{(-1)})^{-1}$, $\beta_2 = \Gamma^{-1}B(X^{(-1)})^{-1}$ satisfies

$$\alpha^* J_{\mathcal{B}} \alpha \;+\; \beta_1^* \beta_1 \;-\; \beta_2^* \beta_2 \;=\; J_{\mathcal{B}}^{(-1)} . \tag{6.8}$$

X is obtained by solving the Lyapunov equation (6.7) for M, putting $\Lambda = I - M$, and factoring Λ into $\Lambda = X^* J_{\mathcal{B}} X$. This also determines the signature operator $J_{\mathcal{B}}$ and thus the space sequence decomposition $\mathcal{B} = \mathcal{B}_+ \oplus \mathcal{B}_-$. We proceed with the construction of a realization $\mathbf{\Theta}$ of the form

$$\mathbf{\Theta} = \left[\begin{array}{c|c} X & \\ \hline & I \end{array}\right] \left[\begin{array}{c|cc} A & C_1 & C_2 \\ \hline B_U & D_{11} & D_{12} \\ \Gamma^{-1}B & D_{21} & D_{22} \end{array}\right] \left[\begin{array}{c|c} (X^{(-1)})^{-1} & \\ \hline & I \end{array}\right] =: \left[\begin{array}{c|cc} \alpha & \gamma_1 & \gamma_2 \\ \hline \beta_1 & \delta_{11} & \delta_{12} \\ \beta_2 & \delta_{21} & \delta_{22} \end{array}\right] = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \tag{6.9}$$

which is a square matrix at each point k, and where the C_i, D_{ij} are yet to be determined. $\mathbf{\Theta}$ is to satisfy (3.3) for

$$J_1 = \begin{bmatrix} I_{\mathcal{M}_U} & \\ & -I_{\mathcal{M}} \end{bmatrix}, \qquad J_2 := \begin{bmatrix} I_{\mathcal{M}_2} & \\ & -I_{\mathcal{N}_2} \end{bmatrix} \tag{6.10}$$

where J_2 is still to be determined, and with it the dimensionality of the sequences of output spaces $\mathcal{M}_2$ and $\mathcal{N}_2$. However, since all other signatures are known at this point, these follow from theorem 3.1, equation (3.6) as

$$\begin{array}{rcl} \#\mathcal{M}_2 &=& \#\mathcal{B}_+ - \#\mathcal{B}_+^{(-1)} + \#\mathcal{M}_U \;\geq 0 \\ \#\mathcal{N}_2 &=& \#\mathcal{B}_- - \#\mathcal{B}_-^{(-1)} + \#\mathcal{M} \;\geq 0 . \end{array}$$

Finally, to obtain $\mathbf{\Theta}$, it remains to show that $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ can be completed to form $\mathbf{\Theta}$ in (6.9), in such a way that the whole operator is now J-unitary in the sense of (3.3). This completion can be achieved at the local level: it is for each time instant k an independent problem of matrix algebra. For each time instant k, α_k and β_k are known and satisfy

$$[\alpha_k^* \;\; \beta_k^*] \begin{bmatrix} (J_{\mathcal{B}})_k & \\ & (J_1)_k \end{bmatrix} \begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} = (J_{\mathcal{B}})_{k+1} .$$

Because $(J_{\mathcal{B}})_{k+1}$ is non-singular, the $\#\mathcal{B}_k$ columns of $\begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix}$ are linearly independent. Choose a matrix $\begin{bmatrix} c_k \\ d_k \end{bmatrix}$ with $\#(\mathcal{M}_2)_k + \#(\mathcal{N}_2)_k$ independent columns such that

$$[\alpha_k^* \;\; \beta_k^*] \begin{bmatrix} (J_{\mathcal{B}})_k & \\ & (J_1)_k \end{bmatrix} \begin{bmatrix} c_k \\ d_k \end{bmatrix} \;=\; 0 . \tag{6.11}$$

We claim that the square matrix $\begin{bmatrix} \alpha_k & c_k \\ \beta_k & d_k \end{bmatrix}$ is invertible. To prove this, it is enough to show that its null space is zero. Suppose that

$$\begin{bmatrix} \alpha_k & c_k \\ \beta_k & d_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

then

$$[\alpha_k^* \ \ \beta_k^*] \begin{bmatrix} (J_{\mathcal{B}})_k & \\ & (J_1)_k \end{bmatrix} \begin{bmatrix} \alpha_k & c_k \\ \beta_k & d_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (J_{\mathcal{B}})_{k+1} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence $x_1 = 0$ and $\begin{bmatrix} c_k \\ d_k \end{bmatrix} x_2 = 0$. But the columns of $\begin{bmatrix} c_k \\ d_k \end{bmatrix}$ are linearly independent, so that $x_2 = 0$. Thus

$$\begin{bmatrix} \alpha_k^* & \beta_k^* \\ c_k & d_k^* \end{bmatrix} \begin{bmatrix} (J_{\mathcal{B}})_k & \\ & (J_1)_k \end{bmatrix} \begin{bmatrix} \alpha_k & c_k \\ \beta_k & d_k \end{bmatrix} = \begin{bmatrix} (J_{\mathcal{B}})_{k+1} & \\ & N_k \end{bmatrix}$$

where N_k is a square invertible matrix. By the usual inertia argument, the signature of N_k is equal to $(J_2)_k$, and hence N_k has a factorization $N_k = R_k^* (J_2)_k R_k$, where R_k is invertible. Thus putting

$$\begin{bmatrix} \gamma_k \\ \delta_k \end{bmatrix} = \begin{bmatrix} c_k \\ d_k \end{bmatrix} R_k^{-1}, \qquad \mathbf{\Theta}_k = \begin{bmatrix} \alpha_k & \gamma_k \\ \beta_k & \delta_k \end{bmatrix}$$

ensures that $\mathbf{\Theta}$ is J-unitary as required.

To conclude the proof, we have to show that $[U^* \quad -T^*\Gamma^{-1}]\,\Theta$ is block upper. We have

$$[U^* \quad -T^*\Gamma^{-1}] \ = \ [D_U^* \quad -D^*\Gamma^{-1}] + C^*(I - Z^*A^*)^{-1} Z^* [B_U^* \quad -B^*\Gamma^{-1}] \tag{6.12}$$

and it will be enough to show that

$$\mathcal{D}_2 (I - Z^*A^*)^{-1} Z^* [B_U^* \quad -B^*\Gamma^{-1}]\,\Theta \tag{6.13}$$

is block upper. With entries as in equation (6.9), and using the state equivalence transformation defined by X, this is equivalent to showing that $\mathcal{D}_2 X (I - Z^*\alpha^*)^{-1} Z^* \beta^* J_1 \Theta$ is block-upper. That this is indeed the case follows directly from proposition 3.2—see equation (3.7). □

For later use, we evaluate $[U^* \quad -T^*\Gamma^{-1}]\Theta$. Equation (3.8) gives

$$\begin{aligned} C^*(I - Z^*A^*)^{-1} Z^* [B_U^* \quad -B^*\Gamma^{-1}]\,\Theta &= C^* X (I - Z^*\alpha^*)^{-1} Z^* \beta^* J_1 \Theta \\ &= C^* X J_{\mathcal{B}} (I - \alpha Z)^{-1} \gamma \\ &= C^* \Lambda (I - AZ)^{-1} [C_1 \ \ C_2]\,. \end{aligned}$$

Consequently,

$$\begin{aligned} [U^* \quad -T^*\Gamma^{-1}]\Theta &= [D_U^* \quad 0]\,\{\delta + [B_U^* \quad B^*\Gamma^{-1}]^* Z (I - AZ)^{-1} [C_1 \quad C_2]\} + C^*\Lambda (I - AZ)^{-1} [C_1 \quad C_2] \\ &= \{[D_U^* \quad 0]\delta + C^*\Lambda [C_1 \quad C_2]\} + C^*(\Lambda - I) A Z (I - AZ)^{-1} [C_1 \quad C_2] \end{aligned}$$

(in which we used $C^*A + D_U^* B_U = 0$). Since this expression is equal to $[A' \quad -B']$, we obtain a state space model for B' as

$$B' = \{-D_U^* D_{12} + C^*(I - M) C_2\} + C^* M A Z (I - AZ)^{-1} C_2\,. \tag{6.14}$$

6.2. Connection with the Hankel operator

We conclude this section by establishing the link between the Lyapunov equation and the Hankel operator of $\Gamma^{-1}T$.

THEOREM 6.2. *Let T be a locally finite upper operator which has a strictly stable realization $\{A, B, C, 0\}$ which is in output normal form. Let H_k be the Hankel matrix of $\Gamma^{-1}T$ at time instant k, and suppose that the singular values of each H_k decompose into two sets $\sigma_{-,k}$ and $\sigma_{+,k}$, with lower bound of all $\sigma_{-,k}$ larger than 1, and upper bound of all $\sigma_{+,k}$ smaller than 1. Let N_k be equal to the number of singular values of H_k which are larger than 1. Then the solution M of the Lyapunov equation*

$$A^*MA + B^*\Gamma^{-2}B = M^{(-1)}$$

is such that $\Lambda = I - M$ is boundedly invertible and has a signature operator $J_{\mathcal{B}}$ having N_k negative entries at point k.

PROOF The solutions of the two Lyapunov equations associated to $\Gamma^{-1}T$,

$$\begin{array}{lcl} M^{(-1)} & = & A^*MA + B^*\Gamma^{-2}B \\ Q & = & AQ^{(-1)}A^* + CC^* \end{array}$$

may be expressed in terms of the controllability and observability operators of $\Gamma^{-1}T$,

$$\mathcal{C} := \begin{bmatrix} (\Gamma^{-1}B)^{(+1)} \\ (\Gamma^{-1}B)^{(+2)}A^{(+1)} \\ (\Gamma^{-1}B)^{(+3)}A^{(+2)}A^{(+1)} \\ \vdots \end{bmatrix} \qquad \mathcal{O} := \begin{bmatrix} C & AC^{(-1)} & AA^{(-1)}C^{(-2)} & \cdots \end{bmatrix}$$

as $M = \mathcal{C}^*\mathcal{C}$, $Q = \mathcal{O}\mathcal{O}^*$. The Hankel operator H_k of $\Gamma^{-1}T$ at time instant k satisfies the decomposition $H_k = \mathcal{C}_k\mathcal{O}_k$. Hence

$$H_kH_k^* = \mathcal{C}_k\mathcal{O}_k\mathcal{O}_k^*\mathcal{C}_k^* .$$

The state realization of T is assumed to be in output normal form: $Q = \mathcal{O}\mathcal{O}^* = I$. With the current finiteness assumption, the non-zero eigenvalues of $H_kH_k^* = \mathcal{C}_k\mathcal{C}_k^*$ will be the same as those of $\mathcal{C}_k^*\mathcal{C}_k = M_k$. In particular, the number of singular values of H_k that are larger than 1 is equal to the number of eigenvalues of M_k that are larger than 1. Writing $\Lambda_k = I - M_k$, this is in turn equal to the number of negative eigenvalues of Λ_k. □

Figure 7 shows a simple instance of the application of the theory developed in this section, emphasizing the dimensions of the input, output and state sequence spaces related to the Θ-operator. We assume in the figure that one singular value of the Hankel operator of $\Gamma^{-1}T$ at time 1 is larger than 1, so that the state signature $J_{\mathcal{B}}$ of Θ has one negative entry in total. We known from equation (3.13) that the negative entries of $J_{\mathcal{B}}$ determine the number of upward arrows in the diagram of the unitary scattering operator $\mathbf{\Sigma}$. We will show, in the next section, that this number also determines the number of states of the Hankel-norm approximant T_a of T.

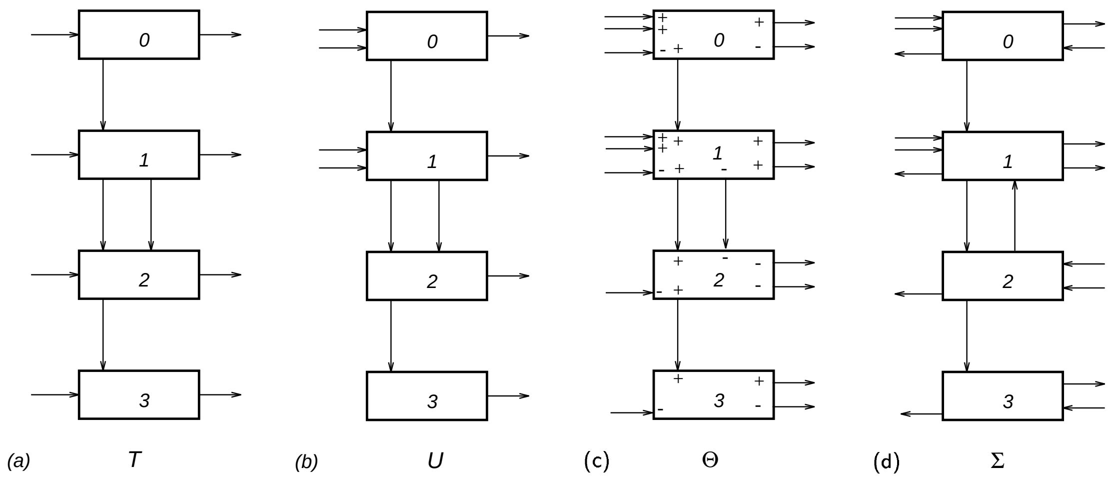

Figure 7. (a) State space realization scheme for T and (b) for U. (c) State space realization scheme for a possible Θ, where it is assumed that one singular value of the Hankel operator of $\Gamma^{-1}T$ at time 1 is larger than 1, and (d) for the corresponding scattering operator Σ.

7. COMPLEXITY AND STATE REALIZATION OF THE APPROXIMANT

At this point we have proven the first part of theorem 1.1: we have constructed a J-unitary operator Θ and from it an operator T_a which is a Hankel-norm approximant of T. It remains to verify the complexity assertion, which stated that the dimension of the state space of T_a is at most equal to N: the number of Hankel singular values of $\Gamma^{-1}T$ that are larger than one. In view of theorems 6.1 and 6.2, N is equal to the number of negative entries in the state signature $J_{\mathcal{B}}$ of Θ. We will now show that a state model for T_a can be derived from the model of Θ, and that its complexity is indeed given by $\#_-(J_{\mathcal{B}})$. The construction is, again, based on the determination of the natural input state space for T_a, which can be derived in terms of the realization of a scattering operator that is connected to Θ.

Throughout section 7, we take signals a_1, a_2, b_1, b_2 to be elements of $\mathcal{X}_2$, generically related by

$$[a_1 \quad b_1]\,\Theta \;=\; [a_2 \quad b_2]$$

where Θ is as constructed in the previous sections. In particular, Θ is a bounded operator, and Θ_{22}^{-1} exists and is bounded. In section 6 we showed that Θ has a (block-diagonal) realization $\mathbf{\Theta}$ which is J-unitary with state signature matrix $J_{\mathcal{B}}$ — see (6.9) and following. $\mathbf{\Theta}$ is bounded by construction (due to the assumption that none of the Hankel singular values of $\Gamma^{-1}T$ are equal or 'asymptotically close' to 1), and is strictly stable. Any realization $\mathbf{T} = \{A, B, C, D\}$ with $\ell_A < 1$ defines a bounded map from input $u \in \mathcal{X}_2^{\mathcal{M}}$ to state $x \in \mathcal{X}_2^{\mathcal{B}}$, $x \;=\; u\,BZ(I-AZ)^{-1}$, which we will call the state evolution of $\mathbf{T}$. The diagonal $x_{[0]}$ in the state evolution defined by the realization $\mathbf{\Theta}$ will play an important role in the definition of the state evolution for T_a.

Associated to the transfer operator Θ, there is the scattering operator Σ which relates

$$[a_1 \quad b_2]\,\Sigma \;=\; [a_2 \quad b_1]\,.$$

We have derived in section 3 a representation $\mathbf{\Sigma} = \{F, G, H, K\}$ in terms of entries $\{\alpha, \beta, \gamma, \delta\}$ in $\mathbf{\Theta}$.

7.1. State dimension of T_a

Suppose that the conditions of theorem 6.1 are fulfilled so that Θ satisfies

$$[U^* \quad -T^*\Gamma^{-1}]\,\Theta \;=\; [A' \quad -B']$$

with A', $B' \in \mathcal{U}$. Let $T'^*\Gamma^{-1} = B'\Theta_{22}^{-1}$ as in section 4. The approximating transfer function T_a is, in principle, given by the strictly upper part of T' (see section 4 for the summary of the procedure). It might not be a bounded operator, since operators in $\mathcal{X}$ are not necessarily decomposable into an upper and lower part in $\mathcal{X}$. However, its extension T' is bounded, and hence its Hankel map $H_{T_a} = H_{T'}$ is well-defined and bounded. We have

PROPOSITION 7.1. *Under the conditions of theorem 6.1, the natural input state space of* $\Gamma^{-1}T_a$ *satisfies*

$$\mathcal{H}(\Gamma^{-1}T_a) \;\subset\; \mathcal{H}(\Theta_{22}^{-*})\,. \tag{7.1}$$

PROOF From the definition of $\mathcal{H}$ and the operators we have

$$\begin{array}{rcll}
\mathcal{H}(\Gamma^{-1}T_a) & = & \mathbf{P}_{\mathcal{L}_2Z^{-1}}(\,\mathcal{U}_2T_a^*\Gamma^{-1}\,) & \\
& = & \mathbf{P}_{\mathcal{L}_2Z^{-1}}(\,\mathcal{U}_2T'^*\Gamma^{-1}\,) & \\
& = & \mathbf{P}_{\mathcal{L}_2Z^{-1}}(\,\mathcal{U}_2B'\Theta_{22}^{-1}\,) & \\
& \subset & \mathbf{P}_{\mathcal{L}_2Z^{-1}}(\,\mathcal{U}_2\Theta_{22}^{-1}\,) & \qquad [\text{since } B' \in \mathcal{U}] \\
& = & \mathcal{H}(\Theta_{22}^{-*})\,. &
\end{array}$$

$\square$

Hence the dimension sequence of $\mathcal{H}(\Theta_{22}^{-*})$ is of interest. Define the 'conjugate-Hankel' operator

$$H' := H'_{\Theta_{22}^{-1}} = \mathbf{P}_{\mathcal{L}_2Z^{-1}}(\,\cdot\,\Theta_{22}^{-1})|_{\mathcal{U}_2}\,. \tag{7.2}$$

Then $\mathcal{H}(\Theta_{22}^{-*}) = \text{ran}\,(H')$. Let the signals a_1, b_1, a_2, b_2 and the state sequences x_+, x_- be in $\mathcal{X}_2$ and be related by $\mathbf{\Theta}$ as in (3.10). As discussed in section 3, Θ can be split into operators $(\cdot)\Theta_p$ and $(\cdot)\Theta_f$ via

$$\begin{array}{rcl}
[a_{1p} \quad b_{1p}]\,\Theta_p & = & [x_{+[0]} \quad x_{-[0]} \quad a_{2p} \quad b_{2p}] \\
{[x_{+[0]} \quad x_{-[0]} \quad a_{1f} \quad b_{1f}]\,\Theta_f} & = & [a_{2f} \quad b_{2f}]\,.
\end{array} \tag{7.3}$$

and according to proposition 3.3, the associated scattering operators Σ_p and Σ_f are well defined by

$$\begin{array}{rcl}
[x_{-[0]} \quad a_{1p} \quad b_{2p}]\,\Sigma_p & = & [x_{+[0]} \quad a_{2p} \quad b_{1p}] \\
{[x_{+[0]} \quad a_{1f} \quad b_{2f}]\,\Sigma_f} & = & [x_{-[0]} \quad a_{2f} \quad b_{1f}]
\end{array} \tag{7.4}$$

and constitute the same relations as in (7.3). (See figure 5(b).) Because $\Theta_{22}^{-1} = \Sigma_{22}$, the conjugate-Hankel operator H' defined in (7.2) is a restriction of the partial map $\Sigma_{22} : b_2 \mapsto b_1$, that is,

$H' : b_{2f} \mapsto b_{1p}$ is such that b_{2p} and b_{1p} satisfy the input-output relations defined by Σ under the conditions $a_1 = 0$ and $b_{2p} = 0$. H' can be factored as $H' = \sigma\tau$, where the operators

$$\begin{array}{rrcl} \sigma : & b_{2f} & \mapsto & x_{-[0]} \\ \tau : & x_{-[0]} & \mapsto & b_{1p} \end{array}$$

can be derived from Σ_f and Σ_p by elimination of $x_{+[0]}$, again taking $a_1 = 0$ and $b_{2p} = 0$. We will show, in proposition 7.2, that the operator σ is 'onto' while τ is 'one-to-one', so that the factorization of H' into these operators is minimal. It is even *uniformly* minimal: the state $x_{-[0]}$ is uniformly controllable by b_{2f} (*i.e.*, the range of σ spans $\mathcal{D}_2$), and $x_{-[0]}$ as input of τ is uniformly observable. It follows, in proposition 7.3, that the dimension of $x_{-[0]}$ at each point in time determines the local dimension of the subspace $\mathcal{H}(\Theta_{22}^{-*})$ at that point.

PROPOSITION 7.2. *Let $\Theta \in \mathcal{U}$ be a locally finite J-unitary operator with strictly stable J-unitary realization $\boldsymbol{\Theta}$. Let $x_{+[0]}, x_{-[0]}, a_1, b_1, a_2, b_2 \in \mathcal{X}_2$ satisfy (7.3) and (7.4).*

1. *If $a_{1p} = 0$ and $b_{2p} = 0$, then the map $\tau : x_{-[0]} \mapsto b_{1p}$ is one-to-one and boundedly invertible on its range, i.e.,*

$$\exists\, \epsilon > 0 : \quad \| b_{1p} \| \;\geq\; \epsilon \,\| x_{-[0]} \| \,. \tag{7.5}$$

2. *The relations*

$$\begin{array}{rcll} x_{-[0]} S & = & x_{+[0]} & \textit{when } a_{1p} = 0,\, b_{2p} = 0 \\ x_{+[0]} R & = & x_{-[0]} & \textit{when } a_{1f} = 0,\, b_{2f} = 0\,, \end{array} \tag{7.6}$$

   define bounded maps S, R which are strictly contractive: $\| S \| < 1$, $\| R \| < 1$.

3. *The map $\sigma : b_{2f} \mapsto x_{-[0]}$ is onto, and moreover, there exists $M < \infty$ such that for any $x_{-[0]}$ there is a b_{2f} in its pre-image such that*

$$\| b_{2f} \| \;\leq\; M \| x_{-[0]} \| \,.$$

PROOF

1. *The map $\tau : x_{-[0]} \mapsto b_{1p}$ is one-to-one.* Put $a_{1p} = 0$ and $b_{2p} = 0$. Equation (7.4) gives $[x_{-[0]} \;\; 0 \;\; 0]\Sigma_p = [x_{+[0]} \;\; a_{2p} \;\; b_{1p}]$, that is, we have for some $x_{+[0]}$ and a_{2p}

$$[0 \quad b_{1p}]\Theta_p \;=\; [x_{+[0]} \quad x_{-[0]} \quad a_{2p} \quad 0]\,. \tag{7.7}$$

   Since Θ_p is bounded, $\| b_{1p} \| < 1 \Rightarrow \| x_{-[0]} \| < M$ and hence, with $\epsilon = 1/M$: $\| x_{-[0]} \| \geq 1 \Rightarrow \| b_{1p} \| \geq \epsilon$. It follows that $x_{-[0]} \mapsto b_{1p}$ is one-to-one as claimed, and that (7.5) holds.

2. *S exists as partial map of Σ_p* (taking $a_{1p} = 0$, $b_{2p} = 0$). Referring to (7.7), we have

$$\| x_{-[0]} \|^2 \;=\; \| x_{+[0]} \|^2 + \| b_{1p} \|^2 + \| a_{2p} \|^2$$

   and since $\| b_{1p} \|^2 \geq \epsilon^2 \| x_{-[0]} \|^2$ for some ϵ, $0 < \epsilon \leq 1$, we have

$$\| x_{-[0]} \|^2 \;\geq\; \| x_{+[0]} \|^2 + \epsilon^2 \| x_{-[0]} \|^2$$

   and hence there exists a constant μ $(0 \leq \mu < 1)$ such that $\| x_{+[0]} \|^2 \leq \mu^2 \| x_{-[0]} \|^2$ (take $\mu = \sqrt{1 - \epsilon^2}$). This shows that $\| S \| < 1$. A similar argument holds for R.

3. *The map $\sigma : b_{2f} \mapsto x_{-[0]}$ is onto.* Let be given any $x_{-[0]}$. We have to show that there is a b_{2f} that via Σ_f can generate this state. First, with $a_{1p} = b_{2p} = 0$, Σ_p associates a unique b_{1p} and $x_{+[0]}$ to $x_{-[0]}$. Put also $a_{1f} = b_{1f} = 0$, then Θ generates a corresponding b_{2f} as $b_{2f} = b_1\Theta_{22}$. Because Σ_f is well-defined, application of Σ_f to $[x_{+[0]} \;\; 0 \;\; b_{2f}]$ gives again a state $x'_{-[0]}$; but this must be equal to $x_{-[0]}$ because they both generate the same b_{1p} and the map $x_{-[0]} \mapsto b_{1p}$ is one-to-one. Hence this b_{2f} generates the given state $x_{-[0]}$. In addition, we have from $\| b_{1p} \| \leq \| x_{-[0]} \|$ and $\| \Theta \| \leq M < \infty$ that

$$\begin{aligned} \| b_{2f} \| &\leq \| \Theta_{22} \| \, \| b_{1p} \| \\ &\leq M \| x_{-[0]} \| \,. \end{aligned}$$

This means that the state $x_{-[0]}$ is uniformly controllable by b_{2f} as well. □

PROPOSITION 7.3. *The s-dimension of the input state space $\mathcal{H}(\Theta_{22}^{-*})$ is equal to $N = \#(\mathcal{B}_-)$, i.e., the number of negative entries in the state signature sequence of Θ.*

PROOF $\mathcal{H}(\Theta_{22}^{-*}) = \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2\Theta_{22}^{-1}) = \{\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(b_{2f}\,\Theta_{22}^{-1}) : \; b_2 \in \mathcal{U}_2\}$.
Put $a_1 = 0$ and $b_{2p} = 0$ so that $b_{1p} = \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(b_{2f}\Theta_{22}^{-1})$. The space $\mathcal{H}(\Theta_{22}^{-*}) = \{b_{1p} : \; b_2 \in \mathcal{U}_2\}$ is generated by the map $H' : \; b_{2f} \mapsto b_{1p}$. But this map can be split into $\sigma : \; b_{2f} \mapsto x_{-[0]}$ and $\tau : \; x_{-[0]} \mapsto b_{1p}$. Because $[x_{-[0]} \;\; 0 \;\; 0]\Sigma_p = [x_{+[0]} \;\; a_{2p} \;\; b_{1p}]$, the signal $x_{-[0]}$ determines b_{1p} completely. In proposition 7.2 we have shown that $x_{-[0]} \mapsto b_{1p}$ is one-to-one and that $b_{2f} \mapsto x_{-[0]}$ is onto. Hence, the state $x_{-[0]}$ is both uniformly observable in b_{1p} and uniformly controllable by b_{2f}, *i.e.,* its state dimension sequence for the map $b_{2f} \mapsto b_{1p}$ is minimal at each point in time. Since the number of state variables in $x_{-[0]}$ is given by $N = \#_-(J_\mathcal{B})$, it follows that

$$\text{s-dim}\; \mathcal{H}(\Theta_{22}^{-*}) = \#(\mathcal{B}_-)\,.$$

□

Proposition 7.3 completes the proof of theorem 1.1.

PROOF of theorem 1.1. Under the conditions mentioned in the theorem, T has an inner factor U of an external factorization (5.1), and there exists, by theorem 6.1, a J-unitary operator Θ such that $[U^* \;\; -T^*\Gamma^{-1}]\Theta = [A' \;\; -B']$ is block upper. It was established in equation (4.2) that T_a, defined as the strictly upper part of $\Gamma^{-1}\Theta_{22}^{-*}B'^*$, is a Hankel-norm approximant of T: $\| \Gamma^{-1}(T - T_a) \|_H \leq 1$. Propositions 7.1 and 7.3 claimed that T_a has an input state space whose dimensionality does not exceed that of $\mathcal{H}(\Theta_{22}^{-*})$, and that the latter dimension sequence is precisely equal to the sequence of the number of negative entries in the state signature sequence $J_\mathcal{B}$ of Θ. In turn, it was shown in theorem 6.2 that this sequence is equal to N, the sequence of the number of Hankel singular values of $\Gamma^{-1}T$ that are larger than 1. □

In the remainder of the section, we shall derive explicit formulas for a realization of the approximant T_a.

7.2. State realization for T_a

In order to obtain a state realization for T_a, we will first determine a model for the strictly upper part of Θ_{22}^{-*} from the representation $\mathbf{\Sigma} = \{F, G, H, K\}$. It will be given in terms of the operators S and R defined in equation (7.6) which can be obtained from $\mathbf{\Sigma}$ in terms of two recursive equations. S is for example obtained as the input scattering matrix of a ladder network consisting of a semi-infinite chain of contractive (*i.e.*, lossy) scattering matrices F_{ij}. It is well-known that such ladder or continuous fraction descriptions converge. An early, independent proof of this fact can be found in [33]. See also [34].

PROPOSITION 7.4. *The operators S and R defined in (7.6) are determined in terms of $\mathbf{\Sigma}$ with block decomposition as in (3.14) by the following recursions:*

$$\begin{aligned} S &= \left(F_{21} + F_{22}(I - SF_{12})^{-1} SF_{11} \right)^{(+1)} \\ R &= F_{12} + F_{11}(I - R^{(-1)}F_{21})^{-1} R^{(-1)} F_{22} \,. \end{aligned} \tag{7.8}$$

A state space model $\{A_a, B_a, C_r\}$ of the strictly upper part of Θ_{22}^{-*} is given in terms of S, R by

$$\begin{aligned} A_a &= \left(F_{22}(I - SF_{12})^{-1} \right)^* \\ B_a &= \left(H_{22} + F_{22}(I - SF_{12})^{-1} SH_{12} \right)^* \\ C_r &= (I - SR)^{-*} \left[G_{22} + G_{21}(I - R^{(-1)}F_{21})^{-1} R^{(-1)} F_{22} \right]^* \,. \end{aligned} \tag{7.9}$$

PROOF The existence and contractivity of $S \in \mathcal{D}$ and $R \in \mathcal{D}$ have already been determined. First observe that although S satisfies by definition $x_{-[0]}S = x_{+[0]}$ ($a_{1p} = b_{2p} = 0$), it also satisfies $x_{-[1]}S = x_{+[1]}$ ($a_{1p} = b_{2p} = 0$ and $a_{1[0]} = b_{2[0]} = 0$), etc. This is readily obtained by applying inputs $Z^{-1}a_1$, etc., so that we get states $Z^{-1}x_+$ and $Z^{-1}x_-$. If $(Z^{-1}a_1)_p = Z^{-1}a_{1p} + Z^{-1}a_{1[0]} = 0$, then $(Z^{-1}x_-)_{[0]}S = (Z^{-1}x_+)_{[0]}$. But $(Z^{-1}x_-)_{[0]} = x_{-[1]}$, and likewise $(Z^{-1}x_+)_{[0]} = x_{+[1]}$. Hence $x_{-[1]}S = x_{+[1]}$.

To determine a state realization for the strictly upper part of $\Sigma_{22}^* = \Theta_{22}^{-*}$, we start from the definition of $\mathbf{\Sigma}$ (3.13), and specialize to the 0-th diagonal to obtain

$$[x_{+[0]} \quad x_{-[1]}^{(-1)} \quad a_{1[0]} \quad b_{2[0]}]\, \mathbf{\Sigma} = [x_{+[1]}^{(-1)} \quad x_{-[0]} \quad a_{2[0]} \quad b_{1[0]}] \,.$$

Taking $a_1 = 0$ throughout this proof, inserting the partitioning of $\mathbf{\Sigma}$ in (3.14) gives

$$\left\{ \begin{array}{lcl} x_{+[1]}^{(-1)} &=& x_{+[0]}F_{11} + x_{-[1]}^{(-1)}F_{21} + b_{2[0]}G_{21} \\ x_{-[0]} &=& x_{+[0]}F_{12} + x_{-[1]}^{(-1)}F_{22} + b_{2[0]}G_{22} \\ b_{1[0]} &=& x_{+[0]}H_{12} + x_{-[1]}^{(-1)}H_{22} + b_{2[0]}K_{21} \end{array} \right. \tag{7.10}$$

With $b_{2p} = 0$ and $b_{2[0]} = 0$, these equations yield an expression for $S^{(-1)}$:

$$\left\{ \begin{array}{lcl} x_{+[1]}^{(-1)} = x_{-[1]}^{(-1)}S^{(-1)} &=& x_{-[0]}SF_{11} + x_{-[1]}^{(-1)}F_{21} \\ x_{-[0]} &=& x_{-[0]}SF_{12} + x_{-[1]}^{(-1)}F_{22} \end{array} \right. \tag{7.11}$$

$$\Leftrightarrow$$

$$\left\{ \begin{array}{lcl} x_{-[0]} &=& x_{-[1]}^{(-1)}F_{22}(I - SF_{12})^{-1} \\ x_{-[1]}^{(-1)}S^{(-1)} &=& x_{-[1]}^{(-1)}\{F_{22}(I - SF_{12})^{-1}SF_{11} + F_{21}\} \end{array} \right. \tag{7.12}$$

(note that $(I - SF_{12})^{-1}$ is bounded because $\| S \| < 1$ and $\| F_{12} \| \le 1$), and hence S satisfies the indicated recursive relations (see also figure 8). The recursion for R is determined likewise.

Take x_- as state for a realization of the strictly upper part of Θ_{22}^{-*}, and let $\{A_a, B_a, C_r\}$ be a corresponding state realization, so that the strictly lower part of Θ_{22}^{-1} has an anti-causal state realization

$$\left\{ \begin{array}{lcl} x_{-[0]} & = & x_{-[1]}^{(-1)} A_a^* \; + \; b_{2[0]} C_r^* \\ b_{1[0]} & = & x_{-[1]}^{(-1)} B_a^* \end{array} \right. .$$

The unknowns A_a, B_a and C_r can be expressed in terms of F, G, H by substitution in equations (7.10), and using S and R as intermediate quantities. Doing so with $b_2 = 0$, the first equation in (7.12) yields the expression for A_a in (7.9) and B_a can be determined in terms of S from the last equation in (7.10).

Finally, C_r^* is obtained as the transfer $b_{2[0]} \mapsto x_{-[0]}$ for $a_1 = 0$ and $b_2 = b_{2[0]} \in \mathcal{D}_2$, so that $x_{-[0]} S = x_{+[0]}$ and $x_{-[1]}^{(-1)} = x_{+[1]}^{(-1)} R^{(-1)}$. Inserting the latter expression into the first equation of (7.10) twice yields

$$x_{-[1]}^{(-1)} \;=\; x_{+[0]} F_{11} (I - R^{(-1)} F_{21})^{-1} R^{(-1)} + b_{2[0]} G_{21} (I - R^{(-1)} F_{21})^{-1} R^{(-1)} .$$

Inserting this in the second equation of (7.10), and using $x_{+[0]} = x_{-[0]} S$ results in

$$\begin{array}{rcl} x_{-[0]} & = & x_{-[0]} S F_{12} \; + \; x_{-[0]} S F_{11} (I - R^{(-1)} F_{21})^{-1} R^{(-1)} F_{22} \\ & & + \; b_{2[0]} G_{21} (I - R^{(-1)} F_{21})^{-1} R^{(-1)} F_{22} \; + \; b_{2[0]} G_{22} \\ \Rightarrow & & \\ x_{-[0]} (I - SR) & = & b_{2[0]} (G_{22} + G_{21} (I - R^{(-1)} F_{21})^{-1} R^{(-1)} F_{22}) \end{array}$$

which gives the expression for C_r. □

We are now in a position to determine a state realization for T_a, when also $\ell_{A_a} < 1$ is satisfied.

THEOREM 7.5. *Let T, Γ, U and Θ be as in theorem 6.1, so that $[U^* \;\; -T^*\Gamma^{-1}]\Theta = [A' \;\; -B']$. Let $\{A, B, C, 0\}$ be an output normal strictly stable state realization for T, let M be defined by the recursion in (6.7), and let $\{A, B_U, C, D_U\}$ be a realization for U. Let $\mathbf{\Theta}$ have a partitioning as in (6.9), and $\mathbf{\Sigma}$ corresponding to $\mathbf{\Theta}$ have partitioning (3.14). Let S, R, C_r, A_a, $B_a \in \mathcal{D}$ be defined by the relations (7.8) and (7.9), and suppose that $\ell_{A_a} < 1$.*

Then the approximant T_a, defined as the strictly upper part of $T' = \Gamma \Theta_{22}^{-*} B'^*$, has a state realization $\{A_a, \Gamma B_a, C_a, 0\}$, where C_a is given by

$$C_a = C_r \left[-D_{12}^* D_U + C_2^* (I - M) C \right] + A_a Y^{(-1)} A^* M C ,$$

and $Y \in \mathcal{D}$ satisfies the recursion $Y = A_a Y^{(-1)} A^* + C_r C_2^*$.

PROOF The state realization for T_a will be obtained by multiplying a model for B' by the model $\{A_a, B_a, C_r\}$ of the strictly upper part of Θ_{22}^{-*} as obtained in proposition 7.4. A model for B' has already been obtained in equation (6.14). With $D' := -D_U^* D_{12} + C^*(I - M) C_2$, T_a is

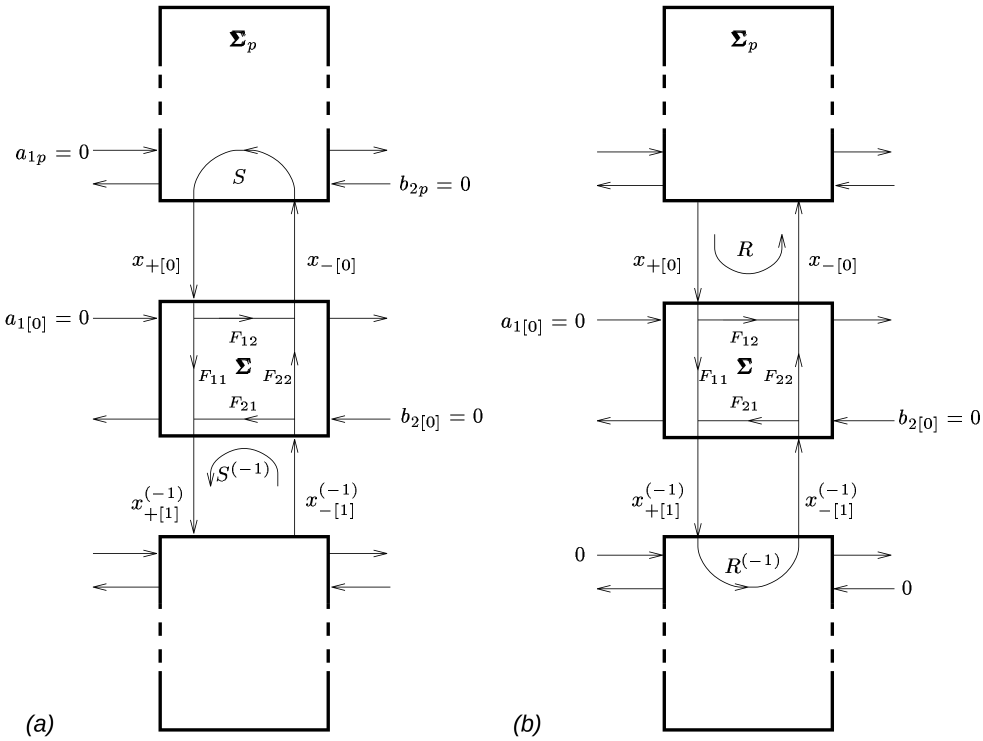

Figure 8. (a) The propagation of S, (b) the propagation of R.

given by the strictly upper part of

$$\begin{aligned}
&\Gamma\left\{B_a Z(I-A_a Z)^{-1} C_r\right\} B'^* \\
&= \Gamma\left\{B_a Z(I-A_a Z)^{-1} C_r\right\} \cdot \left\{C_2^*(I-Z^*A^*)^{-1} Z^*A^*MC + D'^*\right\} \\
&= \Gamma B_a Z(I-A_a Z)^{-1} C_r D'^* + \Gamma B_a \left\{Z(I-A_a Z)^{-1} C_r C_2^* (I-Z^*A^*)^{-1}\right\} Z^*A^*MC\,.
\end{aligned}$$

The computation of the strictly upper part of this expression requires a partial fraction decomposition of the expression $Z(I-A_aZ)^{-1}C_rC_2^*(I-Z^*A^*)^{-1}$. We seek operators $X, Y \in \mathcal{D}$ such that

$$Z(I-A_aZ)^{-1}C_rC_2^*(I-Z^*A^*)^{-1} = Z(I-A_aZ)^{-1}Y + X(I-Z^*A^*)^{-1}\,.$$

Pre- and postmultiplying with $(Z^* - A_a)$ and $(I - Z^*A^*)$, respectively, we obtain the equations

$$\left\{\begin{array}{lcl} C_rC_2^* & = & Y - A_aX \\ 0 & = & -Y^{(-1)}A^* + X \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{lcl} X & = & Y^{(-1)}A \\ Y & = & A_aY^{(-1)}A^* + C_rC_2^* \end{array}\right.$$

Hence Y is determined by the above recursive equation. Via $Z(I-A_aZ)^{-1}YZ^* = Y^{(-1)} + Z(I-A_aZ)^{-1}A_aY^{(-1)}$ we obtain

$$T_a = \Gamma B_a Z(I-A_aZ)^{-1}\left\{C_rD'^* + A_aY^{(-1)}A^*MC\right\}\,,$$

that is, $C_a = C_r\left\{-D_{12}^*D_U + C_2^*(I-M)C\right\} + A_aY^{(-1)}A^*MC\,.$ □

A check on the dimensions of A_a reveals that this state realization for T_a has indeed state dimension sequence given by $N = \#(\mathcal{B}_-)$: at each point in time it is equal to the number of Hankel singular values of T at that point that are larger than 1. The realization is given in terms of four recursions: two for M and S that run forward in time, the other two for R and Y that run backward in time and depend on S. One implication of this is that it is not possible to compute part of an optimal approximant of T if the model of T is known only partly, say up to time instant k.

8. PARAMETRIZATION OF ALL APPROXIMANTS

Section 8 is devoted to the description of all possible solutions to the Hankel norm approximation problem of order smaller than or equal to N, where N = s-dim $\mathcal{H}(\Theta_{22}^{-*})$ is the sequence of dimensions of the input state space of Θ_{22}^{-*}. We shall determine all possible bounded operators of mixed type T' for which it is true that

(1) $\quad \| \Gamma^{-1}(T - T') \| = \| S^* U \| \leq 1\,,$

and (2) $\quad$ the state dimension sequence of T_a = (upper part of T') is at most equal to N.

It turns out that there are no Hankel norm approximants with state dimension lower than N. Notice that we do not assume boundedness of T_a. The result is that all solutions are obtained by a linear fractional transform (chain scattering transformation) of Θ with an upper and contractive parameter S_L. That this procedure effectively generates all approximants of locally finite type of s-degree at most equal to the sequence N can be seen from the fact that if $\| \Gamma^{-1}(T - T_a) \|_H \leq 1$, then an extension T' of T_a must exist such that $\| \Gamma^{-1}(T - T') \| \leq 1$.

The notation is as in the previous sections. We started out with an operator $T \in Z\mathcal{U}$, and we assumed it to be locally finite, with a state realization in output normal form and related factorization $T = \Delta^* U$, where $\Delta \in \mathcal{U}$ and $U \in \mathcal{U}$, unitary and locally finite. Then we proceeded to solve the interpolation problem $[U^* \quad -T^*\Gamma^{-1}]\,\Theta = [A' \quad -B'] \in [\mathcal{U} \;\; \mathcal{U}]$, and we saw that the problem was solvable provided a related Lyapunov-Stein equation had a boundedly invertible solution. The solution was given in terms of an operator $T' = \Gamma^{-1}\Theta_{22}^{-*}B'^*$ in $\mathcal{X}$ of mixed causality type, and the approximant T_a of low order was given by the strictly upper part of T'. In the present section we shall first show that a large class of Hankel-norm approximants can be given in terms of the same J-unitary operator Θ and an arbitrary upper, contractive parameter S_L. Our previous result is the special case $S_L = 0$. Then we move on to show that all approximants of s-degree at most N are obtained in this way.

We first derive a number of preliminary results which will allow us to determine the state dimension sequence of a product of certain matrices.

8.1. Preliminary facts

PROPOSITION 8.1. *Let $B = I - X$, where $X \in \mathcal{X}$ and $\| X \| < 1$. Then $\mathbf{P}(\cdot B)|_{\mathcal{U}_2}$ and $\mathbf{P}(\cdot B^{-1})|_{\mathcal{U}_2}$ are Hilbert space isomorphisms on $\mathcal{U}_2$. Likewise, $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\cdot B)|_{\mathcal{L}_2 Z^{-1}}$ and $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\cdot B^{-1})|_{\mathcal{L}_2 Z^{-1}}$ are isomorphisms on $\mathcal{L}_2 Z^{-1}$.*

PROOF For $B_f = \mathbf{P}(\cdot B)|_{\mathcal{U}_2}$ and $B_p = \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\cdot B)|_{\mathcal{L}_2 Z^{-1}}$ the proof is almost trivial, since $B_f = I - X_f$, where $\| X_f \| < 1$ on $\mathcal{U}_2$, is invertible in $\mathcal{U}_2$. For B^{-1} the proof follows in a similar fashion if one remarks that $B^{-1} = (I - \|X\|)^{-1}(I - Y)$ for some Y with $\| Y \| < 1$. This follows in turn from the fact that $\| (I - X)^{-1} \| \leq (I - \|X\|)^{-1}$, since $\| X \| < 1$. □

Proposition 8.1 allows to conclude, in particular, that if $\mathcal{A}$ is a subspace in $\mathcal{U}_2$, then

$$\text{s-dim } \mathbf{P}(\mathcal{A} B^{-1}) \;=\; \text{s-dim } \mathcal{A}$$

and if $\mathcal{B}$ is another subspace in $\mathcal{U}_2$, then $\mathcal{B} \subset \mathcal{A} \;\Leftrightarrow\; \mathbf{P}(\mathcal{B} B^{-1}) \subset \mathbf{P}(\mathcal{A} B^{-1})$.

PROPOSITION 8.2. *If $B = I - X$, $X \in \mathcal{X}$ and $\| X \| < 1$, and if $\mathcal{B} = \mathbf{P}(\mathcal{L}_2 Z^{-1} B)$, then*

$$\mathbf{P}(\mathcal{B} B^{-1}) = \mathbf{P}(\mathcal{L}_2 Z^{-1} B^{-1}).$$

PROOF We show mutual inclusion.

(1) $\mathbf{P}(\mathcal{B} B^{-1}) \subset \mathbf{P}(\mathcal{L}_2 Z^{-1} B^{-1})$. Let $y \in \mathbf{P}(\mathcal{B} B^{-1})$. Then there exists $u \in \mathcal{L}_2 Z^{-1}$ and $u_1 \in \mathcal{L}_2 Z^{-1}$ such that $y = \mathbf{P}\left((uB + u_1) B^{-1} \right) = \mathbf{P}(u_1 B^{-1})$. Hence $y \in \mathbf{P}(\mathcal{L}_2 Z^{-1} B^{-1})$.

(2) $\mathbf{P}(\mathcal{L}_2 Z^{-1} B^{-1}) \subset \mathbf{P}(\mathcal{B} B^{-1})$. Assume: $y = \mathbf{P}(u_1 B^{-1})$ for some $u_1 \in \mathcal{L}_2 Z^{-1}$. Since $B_p = \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\cdot B)|_{\mathcal{L}_2 Z^{-1}}$ is an isomorphism (proposition 8.1), there exists a $u \in \mathcal{L}_2 Z^{-1}$ such that $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(uB) = -u_1$. It follows that

$$\begin{aligned} y &= \mathbf{P}(u_1 B) \;=\; \mathbf{P}\left((uB + u_1) B^{-1} \right) \\ &= \mathbf{P}\left((uB - \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(uB))\, B^{-1} \right) \\ &= \mathbf{P}\left(\mathbf{P}(uB) B^{-1} \right) \;\; \in \mathbf{P}(\mathcal{B} B^{-1}). \end{aligned}$$

□

PROPOSITION 8.3. *If $A \in \mathcal{L}$ and $A^{-1} \in \mathcal{X}$ and if $\mathcal{A} = \mathbf{P}(\mathcal{L}_2 Z^{-1} A^{-1})$, then*

$$\mathcal{L}_2 Z^{-1} A^{-1} \;=\; \overline{\mathcal{A}} \oplus \mathcal{L}_2 Z^{-1}.$$

PROOF The left-to-right inclusion is obvious. To show the right-to-left inclusion, we show that $\mathcal{L}_2 Z^{-1} \subset \mathcal{L}_2 Z^{-1} A^{-1}$. Assume that $u \in \mathcal{L}_2 Z^{-1}$. Then $u = (uA) A^{-1}$. But since $A \in \mathcal{L}$, we have $uA \in \mathcal{L}_2 Z^{-1}$, and $u \in \mathcal{L}_2 Z^{-1} A^{-1}$. The fact that $\overline{\mathcal{A}}$ is also in the image follows by complementation. □

THEOREM 8.4. *Let $A \in \mathcal{L}$, $A^{-1} \in \mathcal{X}$, and suppose that the space $\mathcal{A} = \mathbf{P}(\mathcal{L}_2 Z^{-1} A^{-1})$ is locally finite of s-dimension N. Let $B = I - X$ with $X \in \mathcal{X}$ and $\| X \| < 1$. Then*

$$\text{s-dim } \mathbf{P}(\mathcal{L}_2 Z^{-1} A^{-1} B^{-1}) = N + p \quad \Rightarrow \quad \text{s-dim } \mathbf{P}(\mathcal{L}_2 Z^{-1} BA) = p.$$

PROOF

$$
\begin{aligned}
\mathbf{P}(\mathcal{L}_2 Z^{-1} A^{-1} B^{-1}) &= \mathbf{P}\left((\mathcal{L}_2 Z^{-1} \oplus \overline{\mathcal{A}})\, B^{-1}\right) && \text{[Prop. 8.3]} \\
&= \mathbf{P}(\mathcal{L}_2 Z^{-1} B^{-1}) \;+\; \mathbf{P}(\overline{\mathcal{A}} B^{-1}) && \text{[linearity]} \\
&= \mathbf{P}(\mathcal{B} B^{-1}) \;+\; \mathbf{P}(\overline{\mathcal{A}} B^{-1}) && \text{[Prop. 8.2]}
\end{aligned}
$$

where $\mathcal{B} = \mathbf{P}(\mathcal{L}_2 Z^{-1} B)$.

In the sequel of the proof, we will use the following two properties. The closure of a D-invariant locally finite linear manifold $\mathcal{H}$ yields a locally finite D-invariant subspace $\overline{\mathcal{H}}$ with the same s-dim . Secondly, let $\mathcal{M}$ be another locally finite D-invariant subspace and let X be a bounded operator on $\mathcal{X}_2$, then $\mathcal{H}X = [\mathbf{P}_{\mathcal{M}}(\mathcal{H})]\, X$ if $\mathcal{M}^{\perp} X = \emptyset$.

Since $\overline{\mathcal{A}}$ and $\mathcal{B}$ are spaces in $\mathcal{U}_2$, and since according to proposition 8.1, $\mathbf{P}(\,\cdot\, B^{-1})|_{\mathcal{U}_2}$ is an isomorphism mapping $\overline{\mathcal{A}}$ and $\mathcal{B}$ to $\mathbf{P}(\overline{\mathcal{A}} B^{-1})$ and $\mathbf{P}(\mathcal{B} B^{-1})$, respectively, we obtain that s-dim $(\overline{\mathcal{A}} + \mathcal{B}) = N + p$. With $\mathcal{A}^{\perp} = \mathcal{U}_2 \ominus \overline{\mathcal{A}}$, it follows that $\mathbf{P}_{\mathcal{A}^{\perp}}(\mathcal{B})$ has s-dim equal to p, because s-dim $\overline{\mathcal{A}} = N$. The proof terminates by showing that

(1) $\mathbf{P}(\mathcal{L}_2 Z^{-1} BA) = \mathbf{P}(\,\mathbf{P}_{\mathcal{A}^{\perp}}(\mathcal{B}) A\,)$, for

$$
\begin{aligned}
\mathbf{P}(\mathcal{L}_2 Z^{-1} BA) &= \mathbf{P}(\,\mathbf{P}(\mathcal{L}_2 Z^{-1} B) A) \\
&= \mathbf{P}(\mathcal{B} A) \\
&= \mathbf{P}\,(\,\mathbf{P}_{\mathcal{A}^{\perp}}(\mathcal{B}) A\,)\ ,
\end{aligned}
$$

because $\overline{\mathcal{A}} A \subset \mathcal{L}_2 Z^{-1}$.

(2) $\mathbf{P}(\mathbf{P}_{\mathcal{A}^{\perp}}(\mathcal{B}) A)$ is isomorphic to $\mathbf{P}_{\mathcal{A}^{\perp}}(\mathcal{B})$, which follows from the fact that the map $\mathbf{P}(\,\cdot\, A)\,|_{\mathcal{A}^{\perp}}$ is one-to-one, for $\mathbf{P}(xA) = 0 \Rightarrow x \in \overline{\mathcal{A}} \oplus \mathcal{L}_2 Z^{-1}$, and the kernel of $\mathbf{P}(\,\cdot\, A)\,|_{\mathcal{A}^{\perp}}$ is thus just $\{0\}$.

Consequently, s-dim $\mathbf{P}(\mathcal{L}_2 Z^{-1} BA) =$ s-dim $\mathbf{P}\,(\mathbf{P}_{\mathcal{A}^{\perp}}(\mathcal{B}) A) =$ s-dim $\mathbf{P}_{\mathcal{A}^{\perp}}(\mathcal{B}) = p$. □

In the above proposition, we had $A \in \mathcal{L}$. A comparable result for $A \in \mathcal{U}$ follows directly by considering a duality property, and results in the proposition below.

COROLLARY 8.5. *Let $A \in \mathcal{U}$, $X \in \mathcal{X}$, $B = I - X$ and $\| X \| < 1$, and let A be invertible in $\mathcal{X}$. Suppose that $\mathcal{A} = \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 A^{-1})$ has s-dimension N. Then*

$$
\text{s-dim } \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\,\mathcal{U}_2 B^{-1} A^{-1}\,) = N + p \qquad \Rightarrow \qquad \text{s-dim } \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\,\mathcal{U}_2 AB\,) = p\,.
$$

PROOF For any bounded operator it is true that the dimension of its range is equal to the dimension of its corange. This property extends to the present case, where we can claim that for $T \in \mathcal{X}$ it is true that s-dim ran $(H_T) =$ s-dim ran (H_T^*), or

$$
\text{s-dim } \mathbf{P}(\,\mathcal{L}_2 Z^{-1} T\,) \;=\; \text{s-dim } \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\,\mathcal{U}_2 T^*\,)\,.
$$

This can be verified for each entry in the sequence of dimensions. Indeed, let $H_i = \Delta_i \mathbf{P}(\,\cdot\, T)|_{\Delta_i \mathcal{L}_2 Z^{-1}}$, where $\Delta_i = \mathrm{diag}[\cdots 0 \;\; 0 \;\; I \;\; 0 \;\; \cdots]$ (with the identity operator appearing at the i-th entry) is the projection operator onto the i-th row. Then $H_i^* = \Delta_i \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\,\cdot\, T^*)|_{\Delta_i \mathcal{U}_2}$ and the dimensions of the ranges of H_i and H_i^* will match. The sequence of these dimensions make up the respective spaces. □

8.2. Generating new solutions of the interpolation problem

Throughout the remainder of the section we use the notion of *causal state dimension sequence* of an operator $T \in \mathcal{X}$ as the s-dimension N of the space $\mathcal{H}(T) = \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 T^*)$. N is thus a sequence of numbers $\{N_i : i \in \mathbb{Z}\}$ where all N_i in our case will be finite. Dually, the s-dimension of $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 T)$ will be described as the *anti-causal state dimension sequence.* We will use the following lemma.

LEMMA 8.6. *Let T, Γ and Θ be as in theorem 6.1, such that $T = \Delta^* U$ is a factorization of T with $\Delta \in \mathcal{U}$ and $U \in \mathcal{U}$ inner, and Θ is the J-unitary operator with input state space given by (6.1) and defined by the realization (6.9). Then*

$$\begin{array}{lcl} [U^* \;\; 0]\,\Theta & \in & [\mathcal{L} \;\; \mathcal{L}] \\ [-\Delta^* \;\; \Gamma]\,\Theta & \in & [\mathcal{L} \;\; \mathcal{L}]\,. \end{array}$$

PROOF We will prove this by brute-force calculations on the realizations of U and Θ, as used *e.g.*, in theorem 6.1.

$$\begin{aligned} [U^* \;\; 0]\Theta &= \{D_U^* + C^*(I - Z^*A^*)^{-1}Z^*B_U^*\}\,\{[D_{11} \;\; D_{12}] + B_U Z(I - AZ)^{-1}[C_1 \;\; C_2]\} \\ &= D_U^*[D_{11} \;\; D_{12}] + D_U^* B_U Z\,(I - AZ)^{-1}[C_1 \;\; C_2] + \\ &\quad + C^*(I - Z^*A^*)^{-1}Z^*B_U^*\,[D_{11} \;\; D_{12}] + \\ &\quad + C^*\underline{(I - Z^*A^*)^{-1}Z^*B_U^*B_U Z\,(I - AZ)^{-1}}[C_1 \;\; C_2]\,. \end{aligned}$$

Upon using the identities $D_U^* B_U + C^*A = 0$, $B_U^* B_U + A^*A = I$, and

$$(I - Z^*A^*)^{-1}Z^*\,(I - A^*A)\,Z(I - AZ)^{-1} = AZ(I - AZ)^{-1} + (I - Z^*A^*)^{-1}$$

it is seen that the terms with $(I - AZ)^{-1}$ cancel each other, so that

$$\begin{aligned} [U^* \;\; 0]\,\Theta &= D_U^*\,[D_{11} \;\; D_{12}] + C^*[C_1 \;\; C_2] + \\ &\qquad + C^*(I - Z^*A^*)^{-1}Z^*\,\{A^*[C_1 \;\; C_2] + B_U^*\,[D_{11} \;\; D_{12}]\} \\ &\in [\mathcal{L} \;\; \mathcal{L}] \end{aligned}$$

In much the same way,

$$\begin{aligned} [-\Delta^* \;\; \Gamma]\,\Theta &= [\{-DD_U^* - BB_U^* - (DC^* + BA^*)Z^*\,(I - A^*Z^*)^{-1}B_U^*\} \;\; \Gamma] \times \\ &\quad \times \left\{ \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} B_U \\ \Gamma^{-1}B \end{bmatrix} Z(I - AZ)^{-1}[C_1 \;\; C_2] \right\} \\ &= \text{(lower)} + \{(-DD_U^* - BB_U^*)\,B_U + B\}\,Z\,(I - AZ)^{-1}[C_1 \;\; C_2] + \\ &\quad + (-DC^* - BA^*)\,\underline{Z^*(I - A^*Z^*)^{-1}B_U^*B_U Z\,(I - AZ)^{-1}}[C_1 \;\; C_2] \\ &= \text{(lower)} + \{-DD_U^*B_U - BB_U^*B_U + B - DC^*A - BA^*A\}\,Z(I - AZ)^{-1}[C_1 \;\; C_2] \\ &= \text{(lower)} + \{DC^*A - B + BA^*A + B - DC^*A - BA^*A\}\,Z(I - AZ)^{-1}[C_1 \;\; C_2] \\ &= \text{(lower)} + 0\,. \end{aligned}$$

□

THEOREM 8.7. *Let $T \in \mathcal{ZU}$ be a locally finite operator with strictly stable output normal realization $\{A, B, C, 0\}$, let Γ be an invertible Hermitian diagonal operator. Let H_k be the Hankel operator of $\Gamma^{-1}T$ at time point k, and suppose that there exists $\epsilon > 0$ such that, for each k, none of the singular values of H_k are in the interval $[1-\epsilon, 1+\epsilon]$. Let N be the sequence of the numbers N_k of singular values of H_k that are larger than 1. Let U be the inner factor of a factorization (5.1), with unitary realization $\{A, B_U, C, D_U\}$, and let Θ be a J-unitary block-upper operator such that its input state space $\mathcal{H}(\Theta)$ is given by (6.1).*

(1) *If $S_L \in \mathcal{U}$ is contractive, then $\Theta_{22} - \Theta_{21}S_L$ is boundedly invertible, and*

$$S = (\Theta_{11}S_L - \Theta_{12})(\Theta_{22} - \Theta_{21}S_L)^{-1}$$

is contractive.

(2) *Let, furthermore, $T' = T + \Gamma S^* U$. Then*

(*a*) $\|\, \Gamma^{-1}(T - T')\,\| = \|\, S^*U \,\| \le 1\,,$

(*b*) *the causal state dimension sequence of T_a = (upper part of T') is precisely equal to N.*

That is, T_a is a Hankel norm approximant of T.

Proof (1) By the J-unitarity of Θ, Θ_{22} is boundedly invertible and $\|\,\Theta_{22}^{-1}\Theta_{21}\,\| < 1$, whence $\Theta_{22} - \Theta_{21}S_L = \Theta_{22}(I - \Theta_{22}^{-1}\Theta_{21}S_L)$ is boundedly invertible. Hence S exists. Its contractivity follows by the usual direct calculation (see *e.g.*, [23]).

(2*a*) follows immediately since $\Gamma^{-1}(T - T') = S^*U$ and U is unitary.

(2*b*) The proof uses the following equality:

$$\begin{aligned} T'^*\Gamma^{-1} &= [U^* \;\; -T^*\Gamma^{-1}] \begin{bmatrix} S \\ -I \end{bmatrix} \\ &= [U^* \;\; -T^*\Gamma^{-1}] \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} S_L \\ -I \end{bmatrix} (\Theta_{22} - \Theta_{21}S_L)^{-1} \\ &= [A' \;\; -B'] \begin{bmatrix} S_L \\ -I \end{bmatrix} (\Theta_{22} - \Theta_{21}S_L)^{-1} \\ &= (A'S_L + B')\,(\Theta_{22} - \Theta_{21}S_L)^{-1}\,. \end{aligned}$$

Since $(A'S_L + B') \in \mathcal{U}$, the anti-causal state dimension sequence of T'^* is at each point in time *at most* equal to the number of anti-causal states of $(\Theta_{22} - \Theta_{21}S_L)^{-1}$ at that point. Because the latter expression is equal to $(I - \Theta_{22}^{-1}\Theta_{21}S_L)^{-1}\Theta_{22}^{-1}$, and $\|\Theta_{22}^{-1}\Theta_{21}S_L\| < 1$, application of corollary 8.5 with $A = \Theta_{22}$ and $B = I - \Theta_{22}^{-1}\Theta_{21}S_L$ shows that this sequence is equal to the anti-causal state dimension sequence of Θ_{22}^{-1}, *i.e.*, equal to N. Hence s-dim $\mathcal{H}(T') \le N$ (pointwise).

The proof terminates by showing that also s-dim $\mathcal{H}(T') \ge N$, so that in fact s-dim $\mathcal{H}(T') = N$. Define

$$\begin{cases} G_2 &= (\Theta_{22} - \Theta_{21}S_L)^{-1} \\ G_1 &= S_L G_2 \end{cases}$$

so that

$$\begin{bmatrix} S \\ -I \end{bmatrix} = \Theta \begin{bmatrix} G_1 \\ -G_2 \end{bmatrix}.$$

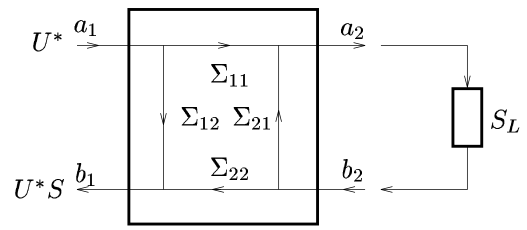

Figure 9. Θ (or Σ) generates Hankel norm approximants via S and parameterized by S_L.

Because Θ is J-inner: $\Theta^* J \Theta = J$, this is equivalent to $[G_1^* \;\; G_2^*] := [S^* \;\; I]\Theta$, and using $S = -\Delta\Gamma^{-1} + UT'^*\Gamma^{-1}$ we obtain

$$\Gamma[G_1^* \;\; G_2^*] = T'[U^* \;\; 0]\Theta \; + \; [-\Delta^* \;\; \Gamma]\Theta \tag{8.1}$$

However, according to lemma 8.6,

$$\begin{array}{lcl} [U^* \;\; 0]\,\Theta & \in & [\mathcal{L} \;\; \mathcal{L}] \\ [-\Delta^* \;\; \Gamma]\,\Theta & \in & [\mathcal{L} \;\; \mathcal{L}]\,. \end{array}$$

This implies $\mathcal{H}(G_2^*) \subset \mathcal{H}(T')$, for let U_1 be a minimal inner factor of a (left) external factorization of T', such that $U_1^* T' \in \mathcal{L}$ (*i.e.*, $\mathcal{H}(U_1) = \mathcal{H}(T')$), then (8.1) shows that also $U_1^* \Gamma G_2^* \in \mathcal{L}$. Hence s-dim $\mathcal{H}(T') \geq$ s-dim $\mathcal{H}(G_2^*) = N$. $\square$

So all S of the form $S = (\Theta_{11} S_L - \Theta_{12})(\Theta_{22} - \Theta_{21} S_L)^{-1}$ with $S_L \in \mathcal{U}$, $\| S_L \| \leq 1$ give rise to Hankel norm approximants of T. It is well known that this type of expression for S is a chain scattering transformation of S_L by Θ; consequently, S is the transfer of port a_1 to b_1 if $b_2 = a_2 S_L$, as in figure 9. This is readily verified using the standard relations between Θ and Σ:

$$\begin{aligned} S &= \Sigma_{12} + \Sigma_{11} S_L \,(I - \Sigma_{21} S_L)^{-1} \Sigma_{22} \\ &= -\Theta_{12}\Theta_{22}^{-1} + (\Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{21}) S_L \,(I - \Theta_{22}^{-1}\Theta_{21} S_L)^{-1}\Theta_{22}^{-1} \\ &= \left[-\Theta_{12}(I - \Theta_{22}^{-1}\Theta_{21} S_L) + (\Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{21}) S_L\right] (I - \Theta_{22}^{-1}\Theta_{21} S_L)^{-1}\Theta_{22}^{-1} \\ &= (\Theta_{11} S_L - \Theta_{12})\,(\Theta_{22} - \Theta_{21} S_L)^{-1} \end{aligned}$$

The reverse question is: are all Hankel norm approximants obtained this way? That is, given some T' whose strictly upper part is a Hankel norm approximant of T, is there a corresponding upper and contractive S_L such that T' is given by $T' = T + \Gamma S^* U$, with S as above. This problem is addressed in the next theorem. The main issue is to prove that S_L as defined by the equations is upper.

8.3. Generating all approximants

THEOREM 8.8. *Let T, Γ, U and Θ be as in theorem 8.7, and let N be the number of Hankel singular values of $\Gamma^{-1}T$ that are larger than 1. Let be given a bounded operator $T' \in \mathcal{X}$ such that*

(1) $\| \Gamma^{-1}(T - T') \| \leq 1$,

(2) *the state dimension sequence of T_a = (upper part of T') is at most equal to N.*

Define $S = U(T'^* - T^*)\Gamma^{-1}$. Then there is an operator S_L with $(S_L \in \mathcal{U}, \| S_L \| \le 1)$ such that

$$S = (\Theta_{11} S_L - \Theta_{12})(\Theta_{22} - \Theta_{21} S_L)^{-1}$$

(i.e., Θ generates all Hankel norm approximants). The state dimension of T_a is precisely equal to N.

PROOF The main line of the proof runs in parallel with [17], but differs in detail. In particular, the 'winding number' argument to determine state dimensions is replaced by theorem 8.4 and its corollary 8.5. The proof consists of five steps.

1. From the definition of S, and using the factorization $T = \Delta^* U$, we know that

   $$\| S \| = \| U(T'^* - T^*)\Gamma^{-1} \| = \| \Gamma^{-1}(T' - T) \| \ \le \ 1$$

   so S is contractive. Since $S = -\Delta\Gamma^{-1} + UT'^*\Gamma^{-1}$, where Δ and U are upper, the anti-causal state dimension sequence of S is at most equal to N, since it depends exclusively on T'^*, for which this is given.

2. Define

   $$[G_1^* \ \ G_2^*] := [S^* \ \ I]\Theta\,. \tag{8.2}$$

   Then $\mathcal{H}(G_1^*) \subset \mathcal{H}(T')$ and $\mathcal{H}(G_2^*) \subset \mathcal{H}(T')$.

   PROOF Using $S = -\Delta\Gamma^{-1} + UT'^*\Gamma^{-1}$, equation (8.2) can be rewritten as

   $$\Gamma[G_1^* \ \ G_2^*] = T'[U^* \ \ 0]\Theta \ + \ [-\Delta^* \ \ \Gamma]\Theta$$

   According to lemma 8.6,

   $$\begin{array}{lcl} [U^* \ \ 0]\,\Theta & \in & [\mathcal{L} \ \ \mathcal{L}] \\ [-\Delta^* \ \ \Gamma]\,\Theta & \in & [\mathcal{L} \ \ \mathcal{L}]\,. \end{array}$$

   As in the proof of theorem 8.7, this implies $\mathcal{H}(G_1^*) \subset \mathcal{H}(T')$ and $\mathcal{H}(G_2^*) \subset \mathcal{H}(T')$.

3. Equation (8.2) can be rewritten using $\Theta^{-1} = J\Theta^* J$ as

   $$\begin{bmatrix} S \\ -I \end{bmatrix} = \Theta \begin{bmatrix} G_1 \\ -G_2 \end{bmatrix} \tag{8.3}$$

   G_2 is boundedly invertible, and S_L defined by $S_L = G_1 G_2^{-1}$ is well defined and contractive: $\| S_L \| \le 1$. In addition, S satisfies $S = (\Theta_{11} S_L - \Theta_{12})(\Theta_{22} - \Theta_{21} S_L)^{-1}$ as required.

   PROOF Θ is boundedly invertible: $\Theta^{-1} = J\Theta^* J$ with $\|\Theta^{-1}\| = \|\Theta\|$. Hence $\Theta\Theta^* \ge \epsilon I$ (for some $\epsilon > 0$) and

   $$\begin{array}{rcl} G_1^* G_1 + G_2^* G_2 & = & [S^* \ \ I]\,\Theta\Theta^* \begin{bmatrix} S \\ I \end{bmatrix} \\ & \ge & \epsilon\,(S^* S + I) \\ & \ge & \epsilon\, I\,. \end{array}$$

We also have from the J-unitarity of Θ and the contractivity of S that

$$G_1^* G_1 \le G_2^* G_2 \tag{8.4}$$

Together, this shows that $G_2 \ge 1/2\,\epsilon\, I$, and hence G_2 is boundedly invertible (but not necessarily in $\mathcal{U}$). With $S_L = G_1 G_2^{-1}$, equation (8.4) shows that $S_L^* S_L \le 1$, and hence $\| S_L \| \le 1$. Evaluating equation (8.3) gives

$$\begin{aligned} G_2^{-1} &= \Theta_{22} - \Theta_{21} S_L \\ S\, G_2^{-1} &= \Theta_{11} S_L - \Theta_{12} \end{aligned} \tag{8.5}$$

and hence $S = (\Theta_{11} S_L - \Theta_{12})(\Theta_{22} - \Theta_{21} S_L)^{-1}$.

4. $G_2^{-1} \in \mathcal{U}$, the space $\mathcal{H}(T')$ has the same dimension as $\mathcal{H}(\Theta_{22}^{-*})$, and $\mathcal{H}(G_1^*) \subset \mathcal{H}(G_2^*)$.

   PROOF According to equation (8.5), G_2^{-1} satisfies

$$\begin{aligned} G_2^{-1} &= \Theta_{22}\,(I - \Theta_{22}^{-1}\Theta_{21} S_L) \\ G_2 &= (I - \Theta_{22}^{-1}\Theta_{21} S_L)^{-1}\Theta_{22}^{-1}\,. \end{aligned}$$

   Let p be the dimension sequence of anti-causal states of G_2^{-1}, and $N_2 \le N$ be the number of anti-causal states of G_2, with N the number of anti-causal states of Θ_{22}^{-1}. Application of corollary 8.5 with $A = \Theta_{22}$ and $B = (I - \Theta_{22}^{-1}\Theta_{21} S_L)$ shows that $N_2 = N + p$, and hence $N_2 = N$ and $p = 0$: $G_2^{-1} \in \mathcal{U}$, and $\mathcal{H}(G_2^*)$ has dimension N. Step 2 claimed $\mathcal{H}(G_2^*) \subset \mathcal{H}(T')$, and because T' has at most N anti-causal states, we must have that in fact $\mathcal{H}(G_2^*) = \mathcal{H}(T')$, and hence $\mathcal{H}(G_1^*) \subset \mathcal{H}(G_2^*)$.

5. $S_L \in \mathcal{U}$.

   PROOF This can be inferred from $G_2^{-1} \in \mathcal{U}$, and $\mathcal{H}(G_1^*) \subset \mathcal{H}(G_2^*)$, as follows. $S_L \in \mathcal{U}$ is equivalent to $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 S_L) = 0$, and

$$\begin{aligned} \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 S_L) &= \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 G_1 G_2^{-1}) \\ &= \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\,\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 G_1) G_2^{-1}\,) \end{aligned}$$

   since $G_2^{-1} \in \mathcal{U}$. Using $\mathcal{H}(G_1^*) \subset \mathcal{H}(G_2^*)$, or $\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 G_1) \subset \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 G_2)$ we obtain that

$$\begin{aligned} \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 S_L) &\subset \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\,\mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 G_2) G_2^{-1}\,) \\ &= \mathbf{P}_{\mathcal{L}_2 Z^{-1}}(\mathcal{U}_2 G_2 G_2^{-1}) \qquad (\text{since } G_2^{-1} \in \mathcal{U}) \\ &= 0\,. \end{aligned}$$

$\square$

9. CONCLUDING DISCUSSION

The theorems of sections 4–6 contain an implicit proof of an equivalent of Nehari's theorem to the present setting, for operators T which have a strictly stable, uniformly observable realization. If Γ in (4.2) is chosen such that all local Hankel singular values are uniformly smaller than one, then $T' = (B'\Theta_{22}^{-1}\Gamma)^*$ obtained through theorem 6.1 will be a lower ($\in \mathcal{L}$) operator and the state sequence x_- of section 7 will be of dimension zero: $\#(\mathcal{B}_-) = 0$. Such a T' is known as the Nehari extension of T.

THEOREM 9.1. *If T is a bounded upper operator which has a locally finite strictly stable and uniformly observable realization, and if Γ is a boundedly invertible hermitian diagonal operator, then*

$$\| \Gamma^{-1} T \|_H \;=\; \inf_{T' \in \mathcal{L}} \| \Gamma^{-1}(T - T') \| \,. \tag{9.1}$$

PROOF Let $d = \| \Gamma^{-1} T \|_H$ and consider the operator $(d+\epsilon)^{-1}\Gamma^{-1}T$ for some $\epsilon > 0$. Then $r := \| (d+\epsilon)^{-1}\Gamma^{-1}T \|_H < 1$ and theorem 6.1 applies. Since the largest singular value of any local Hankel operator of $(d+\epsilon)^{-1}\Gamma^{-1}T$ is majorized by r, we have that the sequence of singular values larger than one is zero, and $T' = (B'\Theta_{22}^{-1}\Gamma(d+\epsilon))^*$ is a lower operator. We have from theorem 6.1 that

$$\| (d+\epsilon)^{-1}\Gamma^{-1}(T - T') \| \;\leq\; 1$$

by construction, and hence

$$\| \Gamma^{-1}(T - T') \| \;\leq\; d + \epsilon \,.$$

Letting $\epsilon \downarrow 0$ achieves (9.1). The reverse inequality is obvious from proposition 2.1. $\square$

A comparable result (but with less conditions on T) has been obtained by Gohberg, Kaashoek and Woerdeman [35, 31, 32] in the context of block matrix and operator matrix extensions. A state space realization of the 'maximum entropy' Nehari extension T' can be obtained as a special instance of the method in section 7, and does not need the upward recursions because the dimension of x_- is zero. Omitting the details, we claim that for $\Gamma = I$ and $\| H_T \| < 1$ the realization of T'^* can be given in terms of a realization $\{A, B, C, 0\}$ of T that is in normal output form as

$$\begin{aligned}
D' &= -C^*MA(I - A^*MA)^{-1}B^* \\
C' &= A(I - A^*MA)^{-1}B^* \\
B' &= -C^*MA(I - (I - A^*MA)^{-1}B^*B) \\
A' &= A(I - (I - A^*MA)^{-1}B^*B)
\end{aligned}$$

where M satisfies

$$M^{(-1)} \;=\; A^*MA + B^*B \,.$$

ACKNOWLEDGEMENTS

The authors wish to extend warm feelings of gratitude to Prof. Harry Dym of the Weizmann Institute in Israel, who undertook a careful reading of the original manuscript, made many suggestions for improvement, and contributed a nice proof of a key point in the proof of theorem 6.1. Also constructive comments and criticism of anonymous reviewers are hereby acknowledged. Furthermore, we acknowledge partial support by the commission of the EC under the ESPRIT BRA program 6632 (NANA2).

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MCS 1991: primary 47A57, secondary 47N70, 93B36.