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ET 4386 Estimation and Detection

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This exam has three questions (32 points in total) and is a closed book exam. One double sided self-handwritten A4 formula sheet is allowed.

The following formulas might be useful for some of the exercises:

- The general expression for the Gaussian probability density function:

  Let $\mathbf{x} \sim \mathcal{N}(\mathbf{m}, \mathbf{C})$ be an $N \times 1$ random Gaussian distributed vector with $N \times 1$ mean vector $\mathbf{m}$ and $N \times N$ covariance matrix $\mathbf{C}$. The probability density function of $\mathbf{x}$ is then given by

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{N}{2}} (\det \mathbf{C})^{\frac{1}{2}}} \exp\left[-\frac{1}{2} (\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x} - \mathbf{m})\right].$$

- The Woodbury identity may also be useful, which is as follows

$$(\mathbf{A} + \mathbf{C})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}(\mathbf{A}^{-1} + \mathbf{C}^{-1})^{-1}\mathbf{A}^{-1},$$

  for matrices $\mathbf{A}, \mathbf{C}$ of appropriate sizes.

Question 1 (12 points)

In functional magnetic resonance imaging (fMRI), local brain activity is determined by detecting changes in the blood flow. The technique assumes that local blood flow in the brain and neural activity are coupled: When neurons in a zone of the brain become active, local blood flow to those regions increases. Deoxigenated blood gets displaced by oxigenated blood. This creates a difference in potential, **giving rise to a negative signal peak** on that region for a brief moment, before going back to the original level. After proper signal normalization, a sliding time-window is applied to obtain a set of N consecutive signal samples. The detection model per quantized area in the fMRI image –call it "pixel"– is then the following (depicted in Fig. 1 for one time instant.):

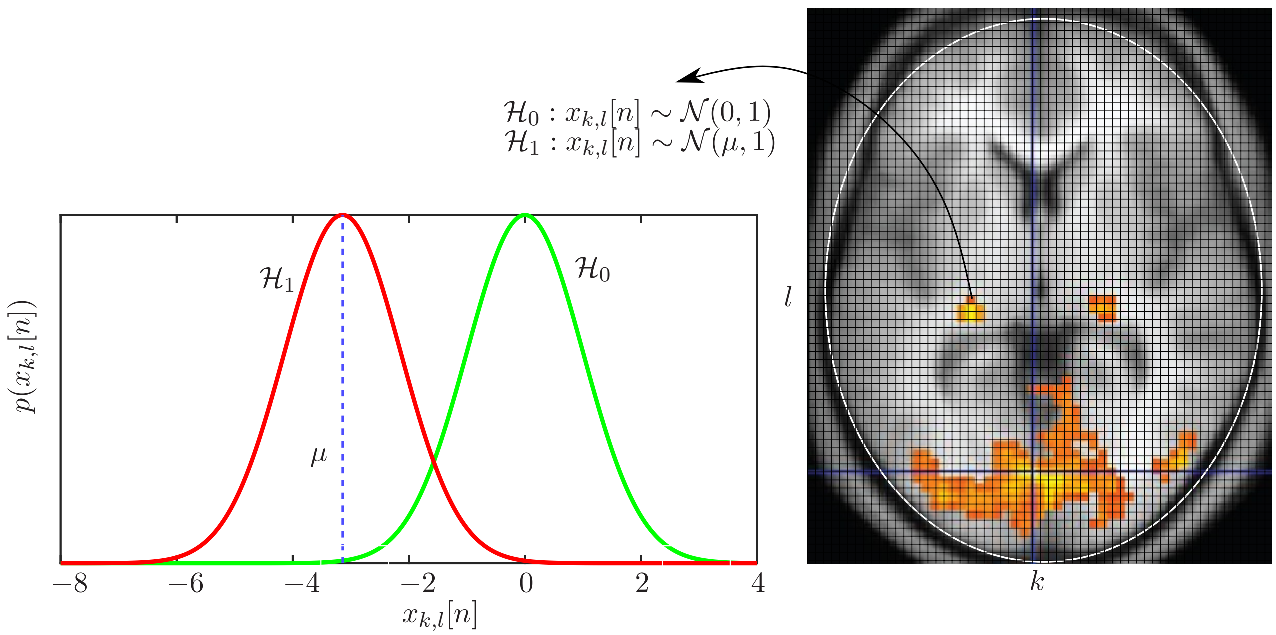

Figure 1: Functional MRI brain scan. Detection of neural activity in a given brain zone for one time instant.

$$\mathcal{H}_0 : x_{k,l}[n] \sim \mathcal{N}(0,1), \text{ for } n = 0, \ldots, N-1$$
$$\mathcal{H}_1 : x_{k,l}[n] \sim \mathcal{N}(\mu,1), \text{ for } n = 0, \ldots, N-1,$$

where $x_{k,l}[n]$ is the signal pixel at position k,l with k,l integers, **and** μ **is unknown but negative**. Hypothesis $\mathcal{H}_0$ means no neural activity in that region, while hypothesis $\mathcal{H}_1$ means the aforementioned blood flow phenomenon is observed and neural activity in the pixel zone is inferred.

Recall the parameter μ is unknown, but known to be negative. Given this restriction, please answer the questions below. In all questions **consider the detection of neural activity at only one single pixel** and for $n = 0, ..., N-1$.

(2 pts) (a) Calculate the maximum likelihood estimate of the unknown parameter μ.

(2 pts) (b) Derive the test statistic function $T_G(\mathbf{x})$, corresponding to the generalized likelihood ratio test (GLRT) detector. Do not explicitly derive the optimal threshold.

(2 pts) (c) Now assume μ is known. Derive the test statistic function $T_N(\mathbf{x})$, corresponding to the Neyman-Pearson (NP) detector. Do not explicitly derive the optimal threshold.

(2 pts) (d) Determine the optimal threshold as a function of the false alarm probability P_{FA} for the NP detector in Question 1(c) AND the corresponding detection probability P_D.

(2 pts) (e) Argue whether a uniformly most powerful (UMP) test exists for this specific problem. If it exists, derive it.

(2 pts) (f) Explain in words (no calculation), how the detection performance of the GLRT detector will be, compared to the detection probability of the NP detector from Question 1(c). Give your argumentation.

Question 2 (10 points)

Consider a signal of interest $\mathbf{s} = A\mathbf{h}$, where $\mathbf{h}$ is an N-dimensional deterministic vector and $A \sim \mathcal{N}(0, \sigma_A^2)$.

(1 pts) (a) Find the first 2 moments of $\mathbf{s}$. What is the rank of the second moment?

(2 pts) (b) Let $\mathbf{s}$ be embedded in Gaussian noise, that is,

$$\mathbf{x} = \mathbf{s} + \mathbf{w},$$

where $\mathbf{w} \sim \mathcal{N}(0, \mathbf{C}_w)$ represents the measurement noise. Propose a Bayesian estimator for $\mathbf{s}$, given the measurements $\mathbf{x}$. Give an explicit expression in terms of the moments derived in Question 2(a).

(1 pts) (c) A measurement system failure is indicated by $\mathbf{h} = \mathbf{0}$. State the binary hypothesis test for system failure versus system functionality.

(2 pts) (d) Design an NP detector for detecting the signal of interest, assuming the second-order moments are invertible. Do not explicitly evaluate the threshold.
Hint: Use the Woodbury identity, which is also given on the frontpage of the exam.

(2 pts) (e) Describe the functioning of the detector using a block diagram. What is the relationship between the Bayesian estimator in Question 2(b) and the designed NP detector in Question 2(d)? Justify the conventional name used for such an NP detector.

(2 pts) (f) Now let $\mathbf{C}_w = \sigma^2\mathbf{I}$ and $\mathbf{h} = [1, \mathbf{0}_{N-1}^T]^T$, then give an explicit expression of the test statistic for the NP detector in Question 2(d). Explain the implication of $\sigma_A^2 >> \sigma_w^2$ on the detector functionality.

Question 3 (10 points)

Given is the following discrete-time signal,

$$x[n] = A\sin(2\pi f n + \phi) + w[n] \quad n = 0, 1, \ldots, N-1, \tag{1}$$

where A is the amplitude, f the frequency with $0 < f < 0.5$, ϕ the phase and $w[n]$ the noise. Let $\mathbf{x} = [x[0], x[1], ..., x[N-1]]^T$ and $\mathbf{w} = [w[0], w[1], ..., w[N-1]]^T$ with $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$.

In this question we will first investigate estimation of the amplitude A under the condition that f and ϕ are known.

(2 pts) (a) Let $f(\mathbf{x}; A, f, \phi)$ be probability density function of $\mathbf{x}$, parameterised by A, ϕ and f. Calculate $\frac{\partial \ln f(\mathbf{x}; A, f, \phi)}{\partial A}$ and show that the regularity condition is satisfied.

(2 pts) (b) Calculate the Fisher information $I(A)$ and the Cramer-Rao lower bound (CRLB) for A.

(2 pts) (c) Derive the minimum variance unbiased estimator $\hat{A}_{MVU}$ for A.

(2 pts) (d) Specify the distribution of $\hat{A}_{MVU}$, including all distributional parameters like expected value and variance.

(2 pts) (e) Give the maximum likelihood estimator (MLE) $\hat{A}_{MLE}$ for A.