Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group

ET 4386 Estimation and Detection

Januray 25th 2019, 9:00–12:00

This is a closed book exam. One double sided self-handwritten A4 formula sheet is allowed.

This exam has four questions (36 points in total).

You might need the expression for the Gaussian probability density function in some questions. Below we give the general definition:

Let $\mathbf{x} \sim \mathcal{N}(\mathbf{m}, \mathbf{C})$ be an $N \times 1$ random Gaussian distributed vector with $N \times 1$ mean vector $\mathbf{m}$ and $N \times N$ covariance matrix $\mathbf{C}$. The probability density function of $\mathbf{x}$ is then given by

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{N}{2}} (\det \mathbf{C})^{\frac{1}{2}}} \exp\left[-\frac{1}{2} (\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x} - \mathbf{m})\right]$$

Question 1 (10 points)

Let

$$\begin{aligned} \mathcal{H}_0 &: \quad \mathbf{x} = \mathbf{w} \\ \mathcal{H}_1 &: \quad \mathbf{x} = \mathbf{s} + \mathbf{w} \end{aligned}$$

with $\mathbf{w} \sim N(\mathbf{0}, \mathbf{C})$ and $\mathbf{s} \sim N(\boldsymbol{\mu}_s, \eta\mathbf{C})$ with η a scalar with $\eta \geq 0$, $\mathbf{x} \in \mathbb{R}^N$, $\mathbf{s} \in \mathbb{R}^N$ and $\mathbf{w} \in \mathbb{R}^N$.

(2 pts) (a) Determine the test statistic $T(\mathbf{x})$ for the Neyman-Pearson (NP) detector and show that $T(\mathbf{x})$ can be written as the sum of a term that depends linearly on $\mathbf{x}$, and a term that depends quadratically on $\mathbf{x}$.

(0.5 pts) (b) Show that in general it holds that, if $\mathbf{x} \sim N(\mathbf{0}, \mathbf{C})$, then $\mathbf{x}^T\mathbf{C}^{-1}\mathbf{x} \sim \chi^2_N$.
Hint: Consider for your argumentation 1) the vector $\mathbf{y} = \mathbf{C}^{-1/2}\mathbf{x}$ *2)* $\mathbf{x}^T\mathbf{C}^{-1}\mathbf{x} = \mathbf{y}^T\mathbf{y}$ *and 3) calculate* $E[\mathbf{y}\mathbf{y}^T]$.

(0.5 pts) (c) Show that if $\mathbf{x} \sim N(\mathbf{0}, (\eta + 1)\mathbf{C})$, then $\frac{\mathbf{x}^T\mathbf{C}^{-1}\mathbf{x}}{\eta+1} \sim \chi^2_N$.

Consider the case $\boldsymbol{\mu}_s = \mathbf{0}$ while $\eta > 0$.

(2 pts) (d) Give the corresponding test statistic $T(\mathbf{x})$ and derive an expression for the the false alarm probability P_{FA} and the detection probability P_D as a function of P_{FA}. *Hint: Make use of the statements given in Questions (b) and (c).*

For the remaining questions, consider the case that $\boldsymbol{\mu}_s \neq \mathbf{0}$, while $\eta = 0$.

(2 pts) (e) Give the corresponding test statistic $T(\mathbf{x})$ and determine the statistics $E[T(\mathbf{x}); \mathcal{H}_0]$, $E[T(\mathbf{x}); \mathcal{H}_1]$, $var[T(\mathbf{x}); \mathcal{H}_0]$ and $var[T(\mathbf{x}); \mathcal{H}_1]$.

(1 pts) (f) Derive an expression for the false alarm probability P_{FA} and the detection probability P_D as a function of P_{FA}.

Let us now consider the two dimensional case, i.e., $N = 2$. Let $0 < \rho < 1$ be the correlation coefficient between the two elements in vector $\mathbf{w}$, and let

$$\mathbf{C} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

with its two eigenvectors given by $\mathbf{v_1} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(2 pts) (g) Give the vector $\boldsymbol{\mu}_s$ that leads to the best detection performance.

Question 2 (10 points)

Let X be a Rayleigh distributed random variable with probability density function (pdf)

$$f_X(x; a) = axe^{-ax^2/2},$$

with $x > 0$ and $a > 0$ and variance $\text{VAR}[X] = \frac{2-\pi/2}{a^2}$. We have a single realisation of the random variable X (that is x) and want to make a binary decision on the pdf that generated this realisation x. In both cases x is Rayleigh distributed, but with a different a-value. Formally we can write this as

$$\begin{aligned} \mathcal{H}_0 &: X \sim f_X(x; a_0) = a_0 x e^{-a_0 x^2/2} \\ \mathcal{H}_1 &: X \sim f_X(x; a_1) = a_1 x e^{-a_1 x^2/2}, \end{aligned}$$

with $a_1 < a_0$. The binary decision can be made by comparing the Neyman-Pearson detector $T(x)$ with a threshold.

(2,5 pts) (a) Determine the Neyman-Pearson detector $T(X)$.

(2,5 pts) (b) Determine the threshold as a function of the false alarm probability P_{fa}.

(2,5 pts) (c) Give the detection probability P_D as a function of the false alarm probability P_{fa}.

(2,5 pts) (d) Make a sketch of the two pdfs $f_X(x; a_0)$ and $f_X(x; a_1)$ and indicate the threshold, probability of false alarm, and the probability detection.

Question 3 (10 points)

In this question we revisit the problem of finding the DC component from a signal corrupted by white Gaussian noise when the noise variance is *unknown.*

Given N data samples drawn i.i.d. from a $\mathcal{N}(\mu, \sigma^2)$ distribution, where both μ and σ^2 are unknown, determine the following:

(1 pt) (a) Write down the data model for $\mathbf{x} = \big[x[0], \ldots, x[N-1]\big]^T$.

(1 pt) (b) Show that the sample mean

$$\bar{x} = \left(\frac{1}{N}\right) \mathbf{x}^T \mathbf{1} = \left(\frac{1}{N}\right) \sum_{n=0}^{N-1} x[n],$$

where $\mathbf{1} = [1, \ldots, 1]^T$, is an unbiased estimator for μ.

In the following call

$$V = (\mathbf{x} - \bar{x}\underline{1})^T (\mathbf{x} - \bar{x}\underline{1}) = \sum_{n=0}^{N-1} (x[n] - \bar{x})^2 .$$

(3 pts) (c) Show that the estimator

$$\hat{\sigma}^2_{(N)} = \left(\frac{1}{N}\right) V,$$

is a *biased* estimator for σ^2.

(3 pts) (d) Show that the estimator

$$\hat{\sigma}^2_{(N-1)} = \left(\frac{1}{N-1}\right) V,$$

is an *unbiased* estimator for σ^2.

(2 pts) (e) Show that the estimator $\hat{\sigma}^2_{(N)}$ attains a lower mean squared error (MSE), as an estimator of the variance σ^2, than the estimator $\hat{\sigma}^2_{(N-1)}$.
Hint, consider that $E[V^2] = (N^2 - 1)\sigma^4$.

Question 4 (6 points)

A sensor needs to be calibrated under lab conditions using a 2-point measurement setup. Let the measured lab values be denoted by $\mathbf{s} = [s[0], s[1]]^T$. The sensor response is then described by the following equation

$$\mathbf{x} = \alpha \mathbf{s} + \mathbf{w}, \tag{1}$$

where $\mathbf{x} = [x[0], x[1]]^T$ are the sensor measurements, α is the unknown sensor gain and $\mathbf{w} = [w[0], w[1]]^T \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$ is the noise, where

$$\mathbf{C} = \begin{bmatrix} \sigma^2 & \rho \\ \rho & \sigma^2 \end{bmatrix}.$$

(2 pts) (a) Derive the Cramér Rao lower bound (CRLB) for α based on (1).

(2 pts) (b) Derive the minimum-variance unbiased estimate (MVUE) for α and show that the MVUE attains the derived CRLB for α

(1 pts) (c) What simplified form does the MVUE take when $\rho = 0$? Support your answer with a derivation.

(1 pts) (d) Is the MVUE feasible when $\rho = \sigma^2$? Support your choice with an argument.

Hint: If required, use the following identity

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$