Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group

ET 4386 Estimation and Detection

April 12th 2018, 13:30–16:30

This is a closed book exam. One double sided self-handwritten A4 formula sheet is allowed.

This exam has four questions (40 points in total).

You might need the expression for the Gaussian probability density function in some questions. Below, we therefore give the general definition:

Let $\mathbf{x} \sim \mathcal{N}(\mathbf{m}, \mathbf{C})$ be an $N \times 1$ random Gaussian distributed vector with $N \times 1$ mean vector $\mathbf{m}$ and $N \times N$ covariance matrix $\mathbf{C}$. The probability density function of $\mathbf{x}$ is then given by

$$p\left(\mathbf{x}\right) = \frac{1}{\left(2\pi\right)^{\frac{N}{2}} \left(\det \mathbf{C}\right)^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\left(\mathbf{x}-\mathbf{m}\right)^T \mathbf{C}^{-1}\left(\mathbf{x}-\mathbf{m}\right)\right]$$

Question 1 (10 points)

Let x be the number of failures before the first success in a series of Bernoulli trials with probability $\frac{1}{\theta}$, with $\theta \geq 1$. The distribution of x is then Geometric with probability mass function (pmf)

$$f(x;\theta) = \frac{1}{\theta}\left(1 - \frac{1}{\theta}\right)^x,$$

with $E[x] = \theta - 1$ and $\text{Var}[x] = \theta(\theta - 1)$. In this question we wish to estimate θ.

(2 pts) (a) Calculate $\frac{\partial \ln f(x;\theta)}{\partial \theta}$ and show that the regularity condition is satisfied.

(1 pts) (b) Show that the maximum likelihood estimator (MLE) for θ is given by $\hat{\theta}_{MLE} = x + 1$.

(3 pts) (c) Calculate the Fisher information $I(\theta)$ under the pmf $f(x;\theta)$ and determine the Cramér-Rao lower bound (CRLB) for $\text{var}(\hat{\theta})$.

(2 pts) (d) Calculate the variance of the MLE estimator $\hat{\theta}_{MLE}$, that is, $\text{Var}[\hat{\theta}_{MLE}]$

(2 pts) (e) Give the minimum variance unbiased estimator (MVU) estimator for θ.

Question 2 (10 points)

Let $x[0]$ and $x[1]$ be two measurements of a constant A in correlated Gaussian noise. Using vector notation we can model this as

$$\mathbf{x} = A\mathbf{1} + \mathbf{w}$$

with $\mathbf{x} = [x[0]\ x[1]]^T$, $\mathbf{w} = [w[0]\ w[1]]^T$ and $\mathbf{1} = [1\ 1]^T$. The noise $\mathbf{w}$ is zero mean Gaussian with correlation matrix $\mathbf{C}$. The inverse of $\mathbf{C}$ is given by

$$\mathbf{C}^{-1} = \frac{1}{\sigma^2(1-\rho^2)} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}.$$

Parameter ρ, with $-1 \leq \rho \leq 1$ is the normalized correlation coefficient, capturing the correlation between $w[0]$ and $w[1]$.

(2,5 pts) (a) Determine the Cramér-Rao lower bound (CRLB) for $\text{var}(\hat{A})$.

(2,5 pts) (b) Derive the MVU estimator $\hat{A}_{MVU}$ for A.

(2,5 pts) (c) Determine the value of ρ that minimizes the CRLB and give the interpretation of the effect that this ρ-value has on the remaining noise in the estimator $\hat{A}$.

Now we assume that A is a continuous random variable with probability density function

$$f(A) = \begin{cases} \lambda e^{-\lambda A} & \text{for } A \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

and $E[A] = \frac{1}{\lambda}$ and $\text{Var}[A] = \frac{1}{\lambda^2}$.

(2,5 pts) (d) Derive the maximum a posteriori estimator (MAP) $\hat{A}_{MAP}$.

Question 3 (10 points)

Let x be the number of failures before the first success in a series of Bernoulli trials with probability $\frac{1}{\theta}$, with $\theta \geq 1$. The distribution of x is then Geometric with probability mass function (pmf)

$$f(x;\theta) = \frac{1}{\theta}\left(1 - \frac{1}{\theta}\right)^x,$$

with $E[x] = \theta - 1$ and $\text{Var}[x] = \theta(\theta - 1)$.

We have to make a binary decision on the distribution of x. In both cases x is Geometrical distributed, but with a different θ-value. Formally we can write this as

$$\begin{aligned} \mathcal{H}_0 &: X \sim f_X(x;\theta_0) = \frac{1}{\theta_0}\left(1 - \frac{1}{\theta_0}\right)^x \\ \mathcal{H}_1 &: X \sim f_X(x;\theta_1) = \frac{1}{\theta_1}\left(1 - \frac{1}{\theta_1}\right)^x, \end{aligned}$$

with $\theta_0 \geq \theta_1$. The binary decision can be made by comparing the Neyman-Pearson detector $T(x)$ with a threshold λ'.

(2.5 pts) (a) Determine the Neyman-Pearson detector $T(x)$.

(2.5 pts) (b) Determine the false alarm probability P_{fa} and show that the optimal Neyman-Pearson threshold λ' can be written as $\lambda' = \frac{\log(P_{fa})}{\log\left(1-\frac{1}{\theta_0}\right)}$. *Hint:* $\sum_{n=N}^{\infty} r^n = \frac{r^N}{1-r}$ *for* $|r| < 1$.

(2.5 pts) (c) Calculate the detection probability P_D as a function of the false alarm probability P_{fa}.

Now suppose θ_1 is not known, but θ_0 is known.

(2.5 pts) (d) Derive the generalised likelihood ratio test (GLRT) for this problem. That is, give the test statistic $T(x)$ and the threshold λ'.

Question 4 (10 points)

In this question we analyze the transmission of a bit T ($T = 1$ or $T = 0$) of information over a noisy channel. We do this by transmission of a known (deterministic) signal $s[n]$ of length N, i.e., $n = 0, ..., N-1$. If signal $s[n]$ is not detected, the receiver decides that $T = 0$ and if signal $s[n]$ is detected, the receiver decides that $T = 1$.

Let us first consider the case where we model the noise in the channel by a stationary zero-mean Gaussian process $w[n]$, which is uncorrelated over time with variance σ^2 i.e., $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$. We can formulate this detection problem as the following binary hypothesis test

$$\begin{aligned} \mathcal{H}_0: &\quad x[n] = w[n] &\quad n = 0, ..., N-1 \\ \mathcal{H}_1: &\quad x[n] = s[n] + w[n] &\quad n = 0, ..., N-1. \end{aligned}$$

In all questions below, we assume that the energy of $s[n]$ is constant, i.e., $\mathbf{s}^T\mathbf{s} = \alpha$.

(2 pts) (a) Determine the Neyman-Pearson detector $T(\mathbf{x})$ and give the statistics (expected value and variance) of $T(\mathbf{x})$ under both hypotheses.

(2 pts) (b) Give an expression for the detection probability P_D as a function of the false alarm probability P_{FA} and the statistics of $T(\mathbf{x})$.

(2 pts) (c) Argue which signal shape for $s[n]$ will lead to the best detection performance P_D (taking into account that $\mathbf{s}^T\mathbf{s} = \alpha$).

Now we consider a situation where the noise is not anymore stationary, but where the variance changes over time. The distribution of $\mathbf{w}$ is given by $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$, with $\mathbf{C} = \text{diag}(\sigma_0^2, \sigma_1^2, ..., \sigma_{N-1}^2)$.

(1 pts) (d) Determine the Neyman-Pearson detector $T(x)$ under these new noise statistics and give the statistics (Expected value and variance) of $T(x)$ under both hypotheses.

(3 pts) (e) Explain which signal $s[n]$ will lead to the best detection performance P_D under these new noise statistics (again taking into account that $\mathbf{s}^T\mathbf{s} = \alpha$).