Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group

ET 4386 Estimation and Detection - Answers

January 26th 2018, 9:00–12:00

This is a closed book exam. One double sided self-handwritten A4 formula sheet is allowed.

This exam has four questions (40 points in total).

Answer Question 1 (10 points)

(a) $\frac{\partial \ln f(x;p)}{\partial p} = \frac{x}{p} - \frac{m-x}{1-p} = \frac{x-mp}{p(1-p)}$

The regularity condition holds as $E\left[\frac{x-mp}{p(1-p)}\right] = \frac{E[x]-mp}{p(1-p)} = 0$.

(b) $\frac{\partial^2 \ln f(x;p)}{\partial p^2} = \frac{-p(1-p)m-(x-mp)(1-2p)}{p^2(1-p)^2}$ $E\left[\frac{\partial^2 f(x;p)}{\partial p^2}\right] = \frac{-m}{p(1-p)}$.

The CRLB is then given by $\mathrm{var}(\hat{p}) \geq \frac{p(1-p)}{m}$

(c) From Question (a) we know that

$$\frac{\partial \ln f(x;p)}{\partial p} = \frac{x}{p} - \frac{m-x}{1-p} = \frac{x-mp}{p(1-p)}$$

This can be rewritten as $\frac{\partial \ln f(x;p)}{\partial p} = \frac{m}{p(1-p)}\left(\frac{x}{m} - p\right)$, which is exactly the form $\frac{\partial \ln f(x;p)}{\partial p} = I(p)(\hat{p}-p)$. The MVU estimator is the $\hat{p}_{MVU} = \frac{x}{m}$

(d) $\frac{\partial \ln f(x_1,\ldots,X_N;p)}{\partial p} = \sum_{n=1}^{N} \frac{x_n - m_n p}{p(1-p)}$ and $\frac{\partial^2 \ln f(x_1,\ldots,X_N;p)}{\partial p^2} = -\frac{1}{p(1-p)} \sum_{n=1}^{N} m_n$

The CRLB is now $\mathrm{Var}[\hat{p}] \geq \frac{p(1-p)}{\sum_{n=1}^{N} m_n}$

(e) It follows that $\frac{\partial \ln f(x_1,\ldots,X_N;p)}{\partial p}$ can be written in the form $\frac{\partial \ln f(x_1,\ldots,X_N;p)}{\partial p} = I(p)(\hat{p}-p)$. $\frac{\partial \ln f(x_1,\ldots,X_N;p)}{\partial p} = \frac{\sum_{n=1}^{N} m_n}{p(1-p)}\left(\frac{\sum_{n=1}^{N} x_n}{\sum_{n=1}^{N} m_n} - p\right)$

Answer Question 2

(a)

$$\frac{\binom{m}{x} p_1^x (1-p_1)^{m-x}}{\binom{m}{x} p_0^x (1-p_0)^{m-x}} \geq \lambda$$

$$T(x) = x \geq \frac{\log \lambda - m \log\left(\frac{1-p_1}{1-p_0}\right)}{\log\left(\frac{p_1}{p_0}\right) - \log(\frac{1-p_1}{1-P-0})} = \lambda'$$

(b) $P_{fa} = P(T(x) > \lambda' | \mathcal{H}_0) = \sum_{x=\lambda'}^{m} f(x;p_0) = \sum_{x=\lambda'}^{m} \binom{m}{x} p_0^x (1-p_0)^{m-x}$

(c) $P_d = P(T(x) > \lambda' | \mathcal{H}_1) = \sum_{x=\lambda'}^{m} f(x;p_1) = \sum_{x=\lambda'}^{m} \binom{m}{x} p_1^x (1-p_1)^{m-x}$

(d) $P_{fa} = Q\left(\frac{\lambda' - mp_0}{\sqrt{mp_0(1-p_0)}}\right)$. $\lambda' = Q^{-1}(p_{fa})\sqrt{mp_0(1-p_0)} + mp_0$.

(e) $P_d = Q\left(\frac{Q^{-1}(p_{fa})\sqrt{mp_0(1-p_0)} + mp_0 - mp_1}{\sqrt{mp_1(1-p_1)}}\right)$

Answer Question 3

(a) The conditional probability mass function of x given y:

$$\begin{aligned} p(x|y) = \frac{p(x,y)}{p(y;\lambda_1+\lambda_2)} &= \frac{p(x, y = x+n)}{p(y;\lambda_1+\lambda_2)} \\ &= \frac{p(x;\lambda_1)p(y-x;\lambda_2)}{p(y;\lambda_1+\lambda_2)} \\ &= \frac{y!}{x!(y-x)!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{y-x}. \end{aligned}$$

This is a binomial distribution.

(b) The MMSE estimator is given by the conditional mean of x, given y:

$$\hat{x} = E(x|y) = \frac{\lambda_1 y}{\lambda_1+\lambda_2}.$$

Notice that the measured counts y are corrected by a factor $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

(c) The mean of the estimator $E(\hat{x})$ is given by

$$E(\hat{x}) = \frac{\lambda_1 E(y)}{\lambda_1+\lambda_2} = \lambda_1.$$

This means the MMSE estimator is conditionally unbiased: $E(\hat{x}) = E(x)$.

(d) The Bayesian mean-squared error is

$$E(x-\hat{x})^2 = E_y(\mathrm{var}(x|y)) = \lambda_1\lambda_2.$$

($E_y(\cdot)$ denotes the expectation operator over the distribution of y.)

Answer Question 4

(a) We can write the measurement equations as the following linear model

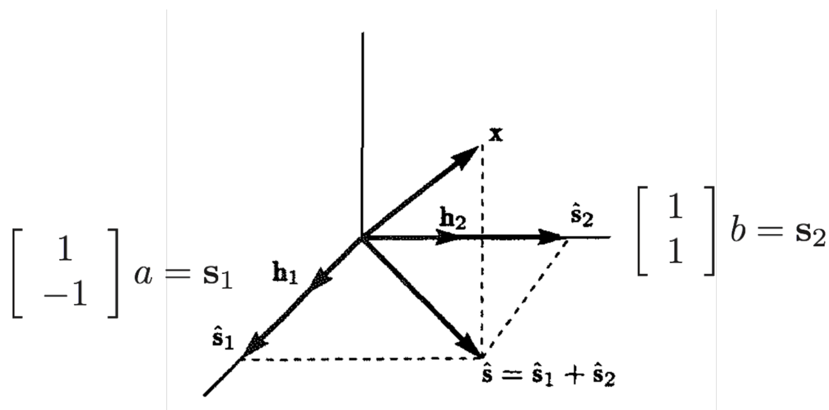
$$\mathbf{x} = \begin{bmatrix} x[1] \\ x[2] \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} a + \begin{bmatrix} 1 \\ 1 \end{bmatrix} b + \begin{bmatrix} w[1] \\ w[2] \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} w[1] \\ w[2] \end{bmatrix}$$
$$= \mathbf{H}\boldsymbol{\theta} + \mathbf{w}.$$

Note that $\mathbf{H}$ is a orthogonal matrix (up to a scale.)

The least squares estimator for $\boldsymbol{\theta}$ is given by $\hat{\boldsymbol{\theta}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x}$, i.e.,

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x[2] - x[1] \\ x[1] + x[2] \end{bmatrix}.$$

(b) Since $\frac{1}{\sqrt{(2)}}\mathbf{H}$ is orthogonal, the geometric illustration is a bit more simpler.



(c) To derive the GLRT, we use the least squares estimator (which is also the maximum likelihood estimator) that we obtained before in the likelihood ratio to decide on $\mathcal{H}_1$ if

$$L_G(\mathbf{x}) = \frac{p(\mathbf{x}; \hat{\boldsymbol{\theta}}, \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} > \gamma$$

with the test statistic

$$L_G(\mathbf{x}) = T(\mathbf{x}) = \frac{\mathbf{x}^T\mathbf{x}}{\sigma^2} = \frac{\hat{a}^2 + \hat{b}^2}{\sigma^2}.$$

(d) For $T(\mathbf{x}) = \mathbf{x}^T\mathbf{x}/\sigma^2$ with length-2 vector $\mathbf{x}$, we have

$$T(\mathbf{x}) = \begin{cases} \chi_2^2 & \text{under } \mathcal{H}_0 \\ \chi_2'^2(\lambda) & \text{under } \mathcal{H}_1 \end{cases}$$

with $\lambda = (a^2 + b^2)/\sigma^2$. Note that $\chi_2'^2(\lambda)$ denotes the non-central chi-squared distribution with a non-centrality parameter λ. So P_{fa} and P_d are given by

$$P_{fa} = Q_{\chi_2^2}(\gamma)$$

and

$$P_d = Q_{\chi_2'^2(\lambda)}(\gamma).$$