Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group

ET 4386 Estimation and Detection

January 26th 2018, 9:00–12:00

This is a closed book exam. One double sided self-handwritten A4 formula sheet is allowed.

This exam has four questions (40 points in total).

Question 1 (10 points)

Let x be the number of successes in m independent Bernoulli distributed trials with probability p. The distribution of x is then Binomial with probability mass function (pmf)

$$f(x;p) = \binom{m}{x} p^x (1-p)^{m-x},$$

with $E[x] = pm$. In this question we wish to estimate p.

(2 pts) (a) Calculate $\frac{\partial \ln f(x;p)}{\partial p}$ and show that the regularity condition is satisfied.

(2 pts) (b) Determine the Cramér-Rao lower bound (CRLB) for $\text{var}(\hat{p})$ under the pmf $f(x;p)$

(2 pts) (c) Give the minimum variance unbiased estimator (MVU) estimator for p.

Now, we have N independent Binomial processes, each with m_n independent trials and x_n successes with probability p for $n = 1, ..., N$. The joint pmf $f(x_1, ..., x_N; p)$ is therefore given by

$$f(x_1, ..., x_N; p) = \prod_{n=1}^{N} \binom{m_n}{x_n} p^{x_n} (1-p)^{m_n - x_n}.$$

(2 pts) (d) Determine the CRLB for $\text{var}(\hat{p})$ under the joint pmf $f(x_1, ..., x_N; p)$.

(2 pts) (e) Give the MVU estimator for p based on the N independent Binomial processes.

Question 2

Let x be the number of successes in m independent Bernoulli distributed trials with probability p. The distribution of x is then Binomial with pmf

$$f_X(x;p) = \binom{m}{x} p^x (1-p)^{m-x},$$

with $E[x] = pm$. We have to make a binary decision on the distribution of x. In both cases X is Binomial distributed, but with a different p-value. Formally we can write this as

$$\begin{aligned} \mathcal{H}_0 &: X \sim f_X(x;p_0) = \binom{m}{x} p_0^x (1-p_0)^{m-x} \\ \mathcal{H}_1 &: X \sim f_X(x;p_1) = \binom{m}{x} p_1^x (1-p_1)^{m-x} \end{aligned}$$

The binary decision can be made by comparing the Neyman-Pearson detector $T(x)$ with a threshold λ'.

(2 pts) (a) Determine the Neyman-Pearson detector $T(x)$.

(2 pts) (b) Determine the false alarm probability P_{fa}. Give the P_{fa} as a function of the threshold λ'.

(2 pts) (c) Give the detection probability P_D as a function of the threshold λ'.

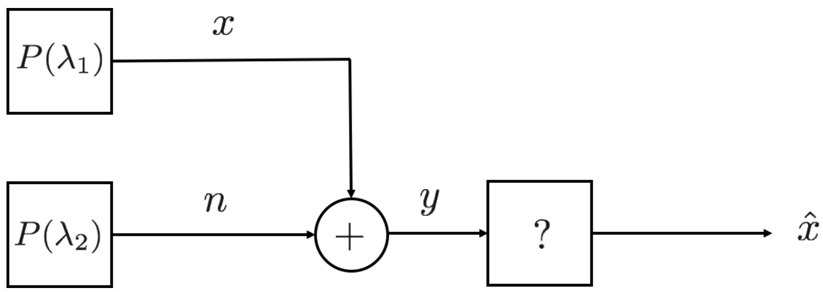
Under the given Binomial distribution, it is hard to find an analytic expression of λ'. However, for large m, $x \sim \mathcal{N}(mp, \sqrt{mp(1-p)})$.

(2 pts) (d) Given an expression for the threshold λ' as a function of P_{fa}. Use the Gaussian approximation of the Binomial distribution for large m given above.

(2 pts) (e) Give the Detection P_D as a function of the P_{fa}. Do this again using the Gaussian approximation of the Binomial distribution for large m given above.

Question 3 - 10 points

Consider the following photon counter, which suffers from shot noise.



A light source emits x photons, but due to the noisy photon counter we measure y photons. To proceed, we assume that x and n are independent Poisson random variables, and y is their sum. That is,

$$x \sim p(x; \lambda_1) = \frac{\lambda_1^x e^{-\lambda_1}}{x!}, \; x = 0, 1, 2, \ldots$$

$$n \sim p(n; \lambda_2) = \frac{\lambda_2^n e^{-\lambda_2}}{n!}, \; n = 0, 1, 2, \ldots$$

$$y \sim P(y; \lambda_1 + \lambda_2) = \frac{(\lambda_1 + \lambda_2)^y e^{-(\lambda_1 + \lambda_2)}}{y!}, \; y = 0, 1, 2, \ldots$$

Our problem is to estimate x from y. To do so, answer the below questions.

Hint: To answer Question 3, you might need the following facts.

The probability of getting exactly k successes in n trials with probability p is given by the binomial distribution with probability mass function:

$$p(k; n, p) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k},$$

with $E[k] = pn$ and $\text{var}(k) = np(1-p)$.

(2.5 pts) (a) Compute the conditional probability mass function of x, given y.

(2.5 pts) (b) Find the minimum mean-squared error (MMSE) estimator for x.

(2.5 pts) (c) Find the mean of your MMSE estimator for x.

(2.5pts) (d) Find the Bayesian mean-squared error of your MMSE estimator for x.

Question 4 (10 points)

This question has 4 subquestions divided into two parts on estimation and detection.

Estimation: Two sensors measure DC along with a DC offset as

$$x[n] = (-1)^n a + b + w[n], \quad n = 1, 2,$$

with n the index of the sensor. We want to estimate both the DC denoted by a and the DC offset denoted by b. The noise $\mathbf{w} = [w[1], w[2]]^T$ is Gaussian distributed as $\mathbf{w} \sim \mathcal{N}(0, \sigma^2\mathbf{I})$.

(2.5 pts) (a) Find the least squares estimator of $\boldsymbol{\theta} = [a, b]^T$. Give the explicit expressions for the estimators $\hat{a}$ and $\hat{b}$.

(2.5 pts) (b) Illustrate through a sketch the geometric viewpoint of your least squares estimator.

Detection: Now we proceed to the detection problem of detecting DC plus DC offset in noise. Consider the binary hypothesis testing problem

$$\begin{aligned} \mathcal{H}_0 &: \ x[n] = w[n] \quad n = 1, 2 \\ \mathcal{H}_1 &: \ x[n] = (-1)^n a + b + w[n] \quad n = 1, 2. \end{aligned}$$

Note that a and b are not known, and as before the noise $\mathbf{w} = [w[1], w[2]]^T$ is Gaussian distributed as $\mathbf{w} \sim \mathcal{N}(0, \sigma^2\mathbf{I})$.

(2.5 pts) (c) Derive the generalized likelihood detector (GLRT) for this problem.

(2.5 pts) (d) Determine the false alarm probability P_{fa} and detection probability P_d. Give the P_{fa} and P_d as a function of the threshold γ and the error function of the test statistic. To do so, you will have to first determine the probability distribution of the test statistic.