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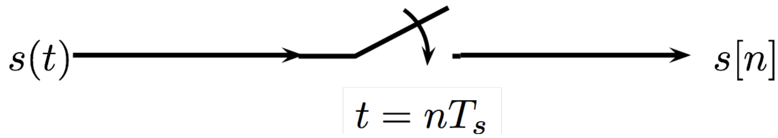
ET 4386 Estimation and Detection

April 13th 2017, 13.30–16:30

This is a closed book exam. One double sided self-handwritten A4 formula sheet is allowed.

This exam has four questions (40 points in total).

Question 1 (10 points)



Continuous time sinusoids of the form $a_i \cos(2\pi f_i t) + b_i \sin(2\pi f_i t)$ are summed to produce a signal $s(t)$:

$$s(t) = \sum_{i=1}^{M} a_i \cos(2\pi f_i t) + \sum_{i=1}^{M} b_i \sin(2\pi f_i t).$$

This signal is sampled, as shown in the above picture, at the sampling instants $t = nT_s$ to get the discrete time signal

$$s[n] = \sum_{i=1}^{M} a_i \cos(2\pi f_i T_s n) + \sum_{i=1}^{M} b_i \sin(2\pi f_i T_s n).$$

Typically, the samples are taken over an interval $0 \leq t \leq NT_s$ with some inaccuracy, leading to the following discrete-time model with additive noise term $w[n]$:

$$x[n] = \sum_{i=1}^{M} a_i \cos(2\pi f_i T_s n) + \sum_{i=1}^{M} b_i \sin(2\pi f_i T_s n) + w[n] \quad n = 0, 1, \ldots, N-1.$$

Here, $w[n]$ is zero-mean white Gaussian noise with variance σ^2. Further, we assume the following

- The sampling period $T_s = 1$ second.
- The "known" frequencies are harmonically related as $f_i = i/N$.
- The number of sinusoids $2M < N$.

We wish to estimate the amplitudes $\boldsymbol{\theta} = [a_1, a_2, \ldots, a_M, b_1, b_2, \ldots, b_M]^T$ of these sinusoids from the noisy observations $\mathbf{x} = [x[0], x[1], \ldots, x[N-1]]^T$. This estimation problem can be written as a linear model

$$\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}.$$

You might need the following hint to solve this question.

Hint: Use the fact that $\mathbf{H}$ matrix that you will construct is orthogonal, that is, $\mathbf{H}^T\mathbf{H} = \frac{N}{2}\mathbf{I}$.

(2.5 p) (a) Set up a linear model of the form

$$\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$$

by giving the entries of $\mathbf{H}$. Derive the least squares estimator (LSE) of $\boldsymbol{\theta}$. Also, explicitly give the expressions of the least squares estimates $\hat{a}_i$ and $\hat{b}_i$.

(2.5 p) (b) Is the LSE unbiased? Also, compute the error covariance matrix $\mathbf{C}_{\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}} = E\left[(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^H\right]$.

(2.5 p) (c) Now, let us assume that the amplitudes a_i and b_i are random variables with a prior probability density function. That is, we will now assume that $\boldsymbol{\theta}$ is Gaussian distributed as

$$\boldsymbol{\theta} \sim \mathcal{N}(\mathbf{0}, \sigma_\theta^2 \mathbf{I}).$$

Derive the minimum mean squared error (MMSE) estimator of $\boldsymbol{\theta}$. explicitly give the expressions of the MMSE estimates of $\hat{a}_i$ and $\hat{b}_i$.

(2.5 p) (d) Compute the posterior covariance matrix $\mathbf{C}_{\boldsymbol{\theta}|\mathbf{x}}$. Compare this posterior covariance matrix with the error covariance matrix derived in *part b* of this question. What happens to the MMSE estimates as $\sigma_\theta^2 \to \infty$?

Question 2 - 10 points

We observe the data $y[n]$, for $n = 0, .., N-1$, where $y[n]$ has the pdf

$$p(y[n];\lambda) = \frac{\lambda}{2} e^{-\lambda |y[n]|} \text{ with } \lambda > 0.$$

(2 p) (a) Determine the Cramér-Rao lower bound (CRLB) for λ as a function of N.

(2 p) (b) Does an unbiased estimator for λ exist that achieves the CRLB for a finite number N? If not, explain why. If so, determine this MVU estimator.

(2 p) (c) Determine the MLE for λ.

Now we consider a Bayesian approach to estimate λ. Let us therefore consider the conditional pdf

$$p(y[n]|\lambda) = \frac{\lambda}{2} e^{-\lambda |y[n]|} \text{ with } \lambda > 0.$$

and the prior

$$p(\lambda) = \begin{cases} \frac{1}{\beta} & c \leq \lambda \leq \beta + c \\ 0 & \text{otherwise.} \end{cases}$$

(2 p) (d) Determine the MAP estimator for λ using N samples for the cases: 1) $c > 0$ and 2) $c = 0$.

(2 p) (e) Give the expression for the MMSE estimator $E[\lambda|\mathbf{y}]$ in terms of integrals, with $\mathbf{y} = [y[0], y[1], ..., y[N-1]]^T$. (Do not solve the integrals). How does the prior $p(\lambda)$ and its parameters c and β influence $E[\lambda|\mathbf{y}]$?

Question 3 - 10 points

We would like to transmit binary information over a noisy channel. The information will be transmitted in sequences of N samples. If the signal is detected in this sequence of N samples (that is $\mathcal{H}_1$), a '1' is decided, and if no signal is detected (that is $\mathcal{H}_0$) a '0' is decided.

First, we test the use of the signal $s_1[n] = A[n]$ with $A[n] \sim \mathcal{N}(0, \sigma_A^2)$ and $A[n]$ uncorrelated across time. That is, every time index n that a signal sample is transmitted, this consists of a new (uncorrelated) realisation of the random variable $A[n]$.

In vector form the hypotheses can be written as

$$\begin{aligned} \mathcal{H}_0 : \quad & \mathbf{x} = \mathbf{w} \\ \mathcal{H}_1 : \quad & \mathbf{x} = \mathbf{s}_1 + \mathbf{w}, \end{aligned}$$

with $\mathbf{w} = [w[0], w[1], ..., w[N-1]]^T$, $\mathbf{s}_1 = [A[0], A[1], ..., A[N-1]]^T$, $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$ and $\mathbf{s}_1 \sim \mathcal{N}(\mathbf{0}, \sigma_A^2\mathbf{I})$.

(1 p) (a) Give the MMSE estimator $\hat{\mathbf{s}}_1 = E[\mathbf{s}_1|\mathbf{x}]$.

(2 p) (b) Give the Neyman-Pearson detector $T(\mathbf{x})$ and explain how we can interpret this detector.

(2 p) (c) Calculate the optimal threshold for the Neyman-Pearson detector $T(\mathbf{x})$ and give the detection probability P_D in terms of P_{FA}, σ^2 and σ_A^2.
Hint: a variable y is χ^2_ν (chi-squared) distributed with ν degrees of freedom if $y = \sum_{i=1}^{\nu} y_i^2$ with $y_i \sim \mathcal{N}(0,1)$.

Now we test the use of the signal $\mathbf{s}_2 = A\mathbf{1}$ with $A \sim \mathcal{N}(0, \sigma_A^2)$ and $\mathbf{1}$ the all ones vector $\mathbf{1} = [1, 1, ..., 1]^T$. That is, the sequence $\mathbf{s}_2 = A\mathbf{1}$ consists of a randomly drawn constant value. In vector form these hypotheses can be written as

$$\begin{aligned} \mathcal{H}_0 : \quad & \mathbf{x} = \mathbf{w} \\ \mathcal{H}_1 : \quad & \mathbf{x} = \mathbf{s}_2 + \mathbf{w}. \end{aligned}$$

(1 p) (d) Give the distribution of $\mathbf{x}$ under both $\mathcal{H}_0$ and $\mathcal{H}_1$.

(1 p) (e) Calculate the MMSE estimator $\hat{\mathbf{s}}_2 = E[\mathbf{s}_2|\mathbf{x}]$.

(1 p) (f) Give the Neyman-Pearson detector $T(\mathbf{x})$. Do not calculate the threshold.

The covariance matrices $\mathbf{C}_{\mathbf{s_1}|\mathbf{x}}$ and $\mathbf{C}_{\mathbf{s_2}|\mathbf{x}}$ are given by

$$\mathbf{C}_{\mathbf{s_1}|\mathbf{x}} = \sigma_A^2 \left(1 - \frac{\sigma_A^2}{\sigma_A^2 + \sigma^2}\right) \mathbf{I}$$

and

$$\mathbf{C}_{\mathbf{s_2}|\mathbf{x}} = \sigma_A^2 \left(1 - \frac{\sigma_A^2}{\frac{\sigma^2}{N} + \sigma_A^2}\right) \mathbf{1}\mathbf{1}^T,$$

respectively. Lets further denote with $[\hat{\mathbf{s}}_\mathbf{1}]_i$ and $[\hat{\mathbf{s}}_\mathbf{2}]_i$ the ith element of the vectors $\hat{\mathbf{s}}_\mathbf{1}$ and $\hat{\mathbf{s}}_\mathbf{2}$, respectively, with $i = 0, ..., N-1$.

(2 p) (g) Based on these covariance matrices, determine the Bayesian mean-squared errors $Bmse([\hat{\mathbf{s}}_\mathbf{1}]_i)$ and $Bmse([\hat{\mathbf{s}}_\mathbf{2}]_i)$. Also argue which of the two detectors will be more effective without explicitly calculating the P_D.

Question 4 (10 points)

In an optical communication system, bits are signaled by turning on and off a laser source. Suppose during a bit period, a laser transmits n photons according a Poisson process with "probability mass function"

$$P(N = n) = e^{-\lambda}\lambda^n/n!.$$

Here, n can only be "integers" and $n!$ is read as "n factorial". We want to count the number of photons n that arrive and use this observed data to make a decision. We do this by solving the binary hypothesis testing problem

$$\mathcal{H}_0 : \lambda = \lambda_0; \quad \text{photon arrival rate when laser is off (0 bit)}$$

and

$$\mathcal{H}_1 : \lambda = \lambda_1; \quad \text{photon arrival rate when laser is on (1 bit)}.$$

Assume that $\lambda_1 > \lambda_0$.

(3.5 p) (a) Find the Neyman-Pearson test to choose hypothesis $\mathcal{H}_0$ or $\mathcal{H}_1$. Show that the test statistic equals the number of photons in a bit period.

(3.5 p) (b) Give the expressions for the probability of detection P_d and probability of false alarm P_f for this hypothesis testing problem. It is sufficient if you give the expressions as a function of the threshold.

(3 p) (c) Assuming that both the hypotheses occur with equal probability, that is, $P(\mathcal{H}_0) = P(\mathcal{H}_1) = 1/2$, derive the "test statistic". Also, give the expression for the probability of error P_e.