Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group

ET 4386 Estimation and Detection

April 11th 2019,

This is a closed book exam. One double sided self-handwritten A4 formula sheet is allowed.

This exam has four questions (38 points in total).

You might need the expression for the Gaussian probability density function in some questions. Below, we therefore give the general definition:

Let $\mathbf{x} \sim \mathcal{N}(\mathbf{m}, \mathbf{C})$ be an $N \times 1$ random Gaussian distributed vector with $N \times 1$ mean vector $\mathbf{m}$ and $N \times N$ covariance matrix $\mathbf{C}$. The probability density function of $\mathbf{x}$ is then given by

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{N}{2}} (\det \mathbf{C})^{\frac{1}{2}}} \exp\left[-\frac{1}{2} (\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x} - \mathbf{m})\right]$$

Question 1 (10 points)

In a factory producing LED light bulbs, each fabricated LED light bulb is checked to function correctly. This can be modeled as a Bernoulli random trial, where the probability of a defect is $\frac{1}{\theta}$, with $\theta \geq 1$. Let X_n denote the number of such Bernoulli trials, until the first defect light bulb is detected. In that case, X_n is known to be a Geometric random variable with probability mass function (PMF)

$$P_{X_n}(x_n;\theta) = \begin{cases} \frac{1}{\theta}\left(1-\frac{1}{\theta}\right)^{x_n} & x_n = 0, 1, 2, ... \\ 0 & \text{otherwise,} \end{cases}$$

with expected value $E[X_n] = \theta - 1$ and variance $var[X_n] = \theta(\theta - 1)$.

Now assume that we have N such independent Geometric random processes, each thus with X_n trials until the first defect light bulb is detected. The joint pmf $P_{X_1,...,X_N}(x_1, ..., x_N; p)$ is then given by

$$P_{X_1,...,X_N}(x_1, ..., x_N; \theta) = \prod_{n=1}^{N} P_{X_n}(x_n; \theta).$$

In this exercise we are interested in estimating the probability $\frac{1}{\theta}$ by using these N independent Geometric random processes.

(2 pts) (a) Show that the Fisher information $I_n(\theta)$ based on a single observation of the random variable X_n, i.e., using $P_{X_n}(x_n; \theta)$, is given by $I_n(\theta) = \frac{1}{\theta(\theta-1)}$.

(2 pts) (b) Determine the Fisher information $I_N(\theta)$ for the sequence of N observations of X_n, i.e., for $X_1, ..., X_N$.

(2 pts) (c) Calculate the maximum likelihood estimator $\hat{\theta}_{MLE}$ based on the N independent observations of X_n, i.e., using $P_{X_1,...,X_N}(x_1, ..., x_N; \theta)$.

(2 pts) (d) Determine whether the maximum likelihood estimator based on N observations X_n reaches the Cramér-Rao lower bound.

(2 pts) (e) Based on prior information it is known that $1 \leq \theta \leq \theta_{max}$ and that all values of θ in this interval are equally likely. Explain how you could take this prior information into account.

Question 2 (14 points)

In this question we revisit aspects of the Fisher information, and the Cramèr-Rao lower bound (CRLB) on a parameter θ. Recollect that the score function is defined as:

$$\frac{\partial}{\partial\theta}\ln p_X(x;\theta).$$

If its first moment is 0, i.e. $E_X\{\frac{\partial}{\partial\theta}\ln p_X(x;\theta)\} = 0$, then the pdf $p_X(x,\theta)$ satisfies the "regularity condition". The Fisher information $I(\theta)$, is then defined as the variance (second moment) of the score function

$$I(\theta) = E\left\{\left(\frac{\partial}{\partial\theta}\ln p_X(x;\theta)\right)^2\right\} = -E\left\{\frac{\partial^2}{\partial\theta^2}p_X(x;\theta)\right\}.$$

In questions **(a), (b) and (c)** below, check the regularity condition of the distributions parameterized by θ and calculate, if possible, the Fisher information. If it is impossible to calcualte $I(\theta)$, explain why.

(3 pts) (a) Gaussian with zero mean and variance θ:

$$p_X(x;\theta) = (2\pi\theta)^{-1/2}\exp\{-x^2/(2\theta)\}.$$

Hint: The fourth order moment for this normal distribution is given by $E[X^4] = 3\theta^2$.

(3 pts) (b) Exponential distribution parameterized by θ:

$$p_X(x;\theta) = \theta\exp(-\theta x), \qquad x \geq 0.$$

Hint: The first and second moments for an exponential distribution are given by $E[X] = 1/\theta$ *and* $E[X^2] = 2/\theta^2$*, respectively.*

(3 pts) (c) Uniform distribution parameterized by θ:

$$p_X(x;\theta) = \begin{cases} \frac{1}{\theta} & \text{for} \quad x \in [0,\theta]; \\ 0 & \quad\text{otherwise.} \end{cases}$$

(2 pts) (d) Calculate, if possible, the Cramèr-Rao lower bound for the variance of the unknown parameter θ for the distribution given in question (a), (b) and (c). If it is impossible to calculate the Cramèr-Rao low bound, then explain with reasons.

(3 pts) (e) Argue for the distributions in questions (a), (b) and (c), whether or not an unbiased estimator for θ exists that achieves the CRLB bound. If it does exist, give the estimator. If you cannot find such an estimator, explain with reason.

Question 3 (9 points)

Given is a known signal $s[n] = r^n$ with $0 < r < 1$. We would like to detect whether $s[n]$ is present in a white Gaussian noise signal $w[n]$ with variance σ^2. To do so, we distinguish between two hypotheses. That are,

$$\begin{aligned} \mathcal{H}_0 &: \quad x[n] = w[n] \qquad\qquad n = 0, 1, ..., N-1 \\ \mathcal{H}_1 &: \quad x[n] = s[n] + w[n] \quad n = 0, 1, ..., N-1. \end{aligned}$$

(2 p) (a) Give the log-likelihood ratio (LLR) and determine the Neyman-Pearson detector $T(\mathbf{x})$ and threshold γ'.

(2 p) (b) Give the distribution of the test statistic under both hypotheses including the expected values and variances.

(2 p) (c) Determine an expression for the threshold γ' that maximizes the detection performance under a given false alarm probability P_{FA}.

(2 p) (d) Calculate the detection probablity P_D as a function of the false alarm probability P_{FA} and the variance σ^2.

(1 p) (e) Explain two ways how the detection probablity P_D can be increased.

Question 4 (5 points)

A sensor needs to be characterized under lab conditions. Let the lab ideal measured values be denoted by $s[n], n = 0, 1, \ldots, N$, then the sensor response is described by the following equation

$$x[n] = \alpha s[n] + \alpha + w[n] \quad n = 0, 1, \ldots, N, \tag{1}$$

where $x[n]$ is the sensor measurement, α is the unknown sensor parameter, and w is the noise on the system that is uncorrelated (uncorrelated over time and uncorrelated with $s[n]$) with $w[n] \sim \mathcal{N}(0, \sigma^2)\ \forall\ n < N$.

(2 pts) (a) Derive the maximum likehood estimator (MLE) for α.

(1 pts) (b) A sensor failure is typically indicated by $\alpha = 0$, and the sensor is functional when $\alpha > 0.5$. State the binary hypothesis for detecting sensor failure versus functionality.

(2 pts) (c) Derive the likelihood ratio test (LRT) for detecting the sensor functionality and determine the Neyman-Pearson detector $T(\mathbf{x})$ and threshold γ'.