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ET 4386 Estimation and Detection - Answers

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This is a closed book exam. One double sided self-handwritten A4 formula sheet is allowed.

This exam has four questions (38 points in total).

You might need the expression for the Gaussian probability density function in some questions. Below, we therefore give the general definition:

Let $\mathbf{x} \sim \mathcal{N}(\mathbf{m}, \mathbf{C})$ be an $N \times 1$ random Gaussian distributed vector with $N \times 1$ mean vector $\mathbf{m}$ and $N \times N$ covariance matrix $\mathbf{C}$. The probability density function of $\mathbf{x}$ is then given by

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{N}{2}} (\det \mathbf{C})^{\frac{1}{2}}} \exp\left[-\frac{1}{2} (\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x} - \mathbf{m})\right]$$

Question 1 (10 points)

$$P_{X_1,...,X_N}(x_1,...,x_N;\theta) = \frac{1}{\theta^N}\prod_{n=1}^{N}\left(1-\frac{1}{\theta}\right)^{x_n-1}.$$

In this exercise we are interested in estimating the probability $\frac{1}{\theta}$ by using these N independent Geometric random processes.

(2 pts) (a) $\log\{P_{X_n}(x_n;\theta)\} = -\log(\theta)+x_n\log(1-\frac{1}{\theta}) = -\log(\theta)+x_n\log(\theta-1)-x_n\log(\theta) = x_n\log(\theta-1)-(1+x_n)\log(\theta)$
$\frac{\partial\log\{P_{X_n}(x_n;\theta)\}}{\partial\theta} = \frac{x_n}{\theta-1}-\frac{1+x_n}{\theta}$
$\frac{\partial^2\log\{P_{X_n}(x_n;\theta)\}}{\partial\theta^2} = -\frac{x_n}{(\theta-1)^2}+\frac{1+x_n}{\theta^2}$ $I(\theta) = -E[-\frac{x_n}{(\theta-1)^2}+\frac{1+x_n}{\theta^2}] = \frac{1}{\theta(\theta-1)}$

(2 pts) (b) The Fisher information matrix is additive for independent observations: $I_N(\theta) = \sum_{n=1}^{N} I_n(\theta) = \frac{N}{\theta(\theta-1)}$

(2 pts) (c) $\frac{\partial\log\left\{P_{X_1,...,x_N}(x_1,...,x_N;\theta)\right\}}{\partial\theta} = \frac{\sum_n x_n}{\theta-1}-\frac{N+\sum_n x_n}{\theta} = 0.$
$\hat{\theta}_{MLE} = \frac{\sum x_n+N}{N} = \frac{\sum x_n}{N}+1$

(2 pts) (d) The Cramér-Rao lower bound is given by $\frac{1}{I_N(\theta)} = \frac{\theta(\theta-1)}{N}$, thus for any unbiased estimator it holds that $var(\hat{\theta}) \geq \frac{\theta(\theta-1)}{N}$ and $var\hat{\theta}_{MLE} = \frac{Nvar(x_n)}{N^2} = \frac{\theta(\theta-1)}{N}$. So, the MLE estimator reaches the CRLB.

Another way would be to make use of the fact that an unbiased etimator that attains the CRLB bound can be found if and only if

$$\frac{\partial\log\{P_{X_1,...,x_N}(x_1,...,x_N;\theta)\}}{\partial\theta} = I(\theta)(g(\mathbf{x})-\theta) = \frac{\sum_n x_n}{\theta-1}-\frac{N+\sum_n x_n}{\theta}$$

$$= \frac{N}{\theta(\theta-1)}\left(\theta-N+\sum_n x_n\right).$$

Thus, the MVU is equal to the MLE and thus reaches the MLE the CRLB.

(2 pts) (e) We could calculate the MMSE or MAP estimator by turning this information into a prior pdf that is uniformly distributed in the interval $1<\theta<\theta_{max}$ with constant value $1/(\theta_{max}-1)$.

Question 2 (14 points)

(3 pts) (a) Using the chain rule for differentiation and the fact that:

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) = \frac{1}{x}, \qquad x > 0, \tag{1}$$

then Fisher information is given by:

$$E\left\{\left(\frac{\frac{\mathrm{d}}{\mathrm{d}\theta}p(x;\theta)}{p(x;\theta)}\right)^2\right\}.$$

First we compute

$$\frac{\mathrm{d}}{\mathrm{d}\theta}p(x;\theta) = \left(-\frac{1}{2}\frac{1}{\sqrt{2\pi\theta^3}} + \frac{x^2}{2\theta^2}\frac{1}{\sqrt{2\pi\theta}}\right)\exp\{-x^2/(2\theta)\}.$$

Then

$$\frac{\frac{\mathrm{d}}{\mathrm{d}\theta}p(x;\theta)}{p(x;\theta)} = \left(-\frac{1}{2\theta} + \frac{x^2}{2\theta^2}\right).$$

Therefore Fisher information,

$$\begin{aligned} I(\theta) &= E\left\{\left(\frac{\frac{\mathrm{d}}{\mathrm{d}\theta}p(x;\theta)}{p(x;\theta)}\right)^2\right\}, \\ &= E\left\{\frac{1}{4\theta^2} - 2\frac{1}{2\theta}\frac{x^2}{2\theta^2} + \frac{x^4}{4\theta^4}\right\}, \\ &= \frac{1}{4\theta^2} - 2\frac{1}{\theta}\frac{\theta}{2\theta^2} + \frac{3\theta^2}{4\theta^4}, \\ &= \frac{1}{2\theta^2}. \end{aligned}$$

Where we have made use of the hint, i.e. $E\{x^4\} = 3\theta^2$.
$\log(p_X(x;\theta)) = \frac{-1}{2}\frac{1}{\theta} + \frac{x^2}{2\theta^2}$.
$E[\log(p_X(x;\theta))] = \frac{-1}{2}\frac{1}{\theta} + \frac{\theta}{2\theta^2} = 0$. The regularity condition holds.

(3 pts) (b) $\log(p_X(x;\theta)) = \log(\theta) - \theta x$. $\frac{\partial \log(p_X(x;\theta))}{\partial\theta} = \frac{1}{\theta} - x$
$E\left[\frac{\partial \log(p_X(x;\theta))}{\partial\theta}\right] = \int_0^\infty(\frac{1}{\theta} - x)\theta e^{-\theta x}dx = \left[-\frac{1}{\theta}e^{-\theta x}\right]_0^\infty - E[X] = 0$
The distribution thus satisfies the regularity condition.
$I(\theta) = E[(\frac{1}{\theta} - X)^2] = \frac{1}{\theta^2}$

(3 pts) (c) The uniform pdf does not satisfy the regularity condition. Hence: $\frac{\partial \log(p_X(x;\theta))}{\partial\theta} = \frac{-1}{\theta}$.
$E[\frac{\partial \log(p_X(x;\theta))}{\partial\theta}] = E[\frac{-1}{\theta}] = \int_0^\theta \frac{-1}{\theta}\frac{1}{\theta}d\theta = [\frac{1}{\theta}]_0^\theta \neq 0$. The order of integration and differentiation cannot be changed here, as the boundaries do depend on the integration variable. Therefore, it holds that this pdf is not regular.

(2 pts) (d) The CRLB is given by $var[\hat{\theta}] \geq \frac{1}{I(\theta)}$. For (a): $var[\hat{\theta}] \geq 2\theta^2$, for (b): $var[\hat{\theta}] \geq \theta^2$, while for (c) it is not defined as the pdf is not regular.

(3 pts) (e) For (a): We can write $\frac{\partial \log(p_X(x;\theta))}{\partial\theta} = I(\theta)(\hat{\theta}-\theta) = \frac{1}{2\theta^2}(x^2-\theta)$, $\hat{\theta}_{MVU} = x^2$
For (b) We can write $\frac{\partial \log(p_X(x;\theta))}{\partial\theta} = I(\theta)(\hat{\theta}-\theta) = \frac{1}{\theta^2}(\theta - \theta^2 x)$ An unbiased etimator that attains the CRLB bound can be found if and only if

$$\frac{\partial \log p(\mathbf{x};\lambda)}{\partial\lambda} = I(\lambda)(g(\mathbf{x}) - \lambda).$$

In this case, it is impossible to write $\frac{\partial \log p(\mathbf{x};\lambda)}{\partial\lambda}$ in this form and thus there is no unbiased etimator that attains the CRLB bound.
For (c): The MVU cannot be determined as the fisher information cannot be calculated.

Question 3 (9 points)

(2 p) (a) $T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s[n]$ and $\gamma' = \sigma^2 \log(\gamma) + \frac{1}{2}\sum_{n=0}^{N-1} s^2[n]$.

(2 p) (b)

$$T(\mathbf{x}) = \begin{cases} \mathcal{N}(0, \mathcal{E}\sigma^2) \text{ under } \mathcal{H}_0 \\ \mathcal{N}(\mathcal{E}, \mathcal{E}\sigma^2) \text{ under } \mathcal{H}_1 \end{cases},$$

with $\mathcal{E} = \sum_{n=0}^{N-1} r^{2n} = \frac{1-r^{2N}}{1-r^2}$.

(2 p) (c) $P_{fa} = Q(\gamma'/\sqrt{\sigma^2\mathcal{E}})$. $\gamma' = \sqrt{\sigma^2\mathcal{E}}Q^{-1}(P_{fa})$

(2 p) (d) $P_D = Q\left(\frac{\sqrt{\sigma^2\mathcal{E}}Q^{-1}(P_{fa})}{\sqrt{\sigma^2\mathcal{E}}} - \sqrt{\frac{\mathcal{E}}{\sigma^2}}\right) = Q\left(Q^{-1}(P_{fa}) - \sqrt{\frac{\mathcal{E}}{\sigma^2}}\right)$

(1 p e) Increase the false alarm probability and increase $\mathcal{E}$, which implies to increase N.

Question 4 (5 points)

Solution 4a The problem can be rewritten in vector notation as follows

$$\mathbf{x} = \alpha(\mathbf{s} + \mathbf{1}_N) + \mathbf{w} = \mathbf{y} + \mathbf{w} \tag{2}$$

where $\mathbf{x}, \mathbf{s}$, $\mathbf{w}$ are the N dimensional vectors containing the sensor measurements, lab measurements and noise, and $\mathbf{1}_N$ is a column vector of ones. The PDF is then given by

$$p(\mathbf{x}; \alpha) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp[-0.5\sigma^{-2}(\mathbf{x} - \alpha(\mathbf{s} + \mathbf{1}_N))^T(\mathbf{x} - \alpha(\mathbf{s} + \mathbf{1}_N))] \tag{3}$$

Taking the derivative of the log likelihood, we have

$$\frac{\partial \; \ln p(\mathbf{x}; \alpha)}{\partial \alpha} = 2\alpha(\mathbf{s} + \mathbf{1}_N)^T(\mathbf{s} + \mathbf{1}_N) - 2\mathbf{x}^T(\mathbf{s} + \mathbf{1}_N), \tag{4}$$

which when set to zero, yields

$$\hat{\alpha}_{MLE} = \frac{\mathbf{x}^T(\mathbf{s} + \mathbf{1}_N)}{(\mathbf{s} + \mathbf{1}_N)^T(\mathbf{s} + \mathbf{1}_N)}.$$

Solution 4b The binary hypothesis for this problem is

$$\begin{array}{lll} \mathcal{H}_0: & \mathbf{x} = \mathbf{w} & \text{(sensor failure)} \\ \mathcal{H}_1: & \mathbf{x} = \alpha(\mathbf{s} + \mathbf{1}_N) + \mathbf{w} & \text{(sensor functioning)} \end{array}$$

Solution 4c The likelihood ratio test (LRT) decides $\mathcal{H}_1$ if

$$\frac{p(\mathbf{x}; \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} > \gamma, \tag{5}$$

Now, let $\mathbf{y} = \alpha(\mathbf{s} + \mathbf{1}_N)$. Subsequently, the LRT takes the following form

$$\frac{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp[-0.5\sigma^{-2}(\mathbf{x} - \mathbf{y})^T(\mathbf{x} - \mathbf{y})]}{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp[-0.5\sigma^{-2}\mathbf{x}^T\mathbf{x}]} > \gamma \tag{6}$$

Taking log on both sides, we have

$$\begin{aligned} -0.5\sigma^{-2}[-2\mathbf{x}^T\mathbf{y} + \mathbf{y}^T\mathbf{y}] &> \ln \gamma \\ \mathbf{x}^T\mathbf{y} &> \sigma^2 \ln \gamma + 0.5\mathbf{y}^T\mathbf{y} = \gamma' \quad \text{(rearranging terms)} \\ \alpha\mathbf{x}^T(\mathbf{s} + \mathbf{1}_N) &> \sigma^2 \ln \gamma + 0.5\alpha^2(\mathbf{s}^T\mathbf{s} + 2\mathbf{s}^T\mathbf{1}_N + 1) \quad \text{(substituting } \mathbf{y}\text{)} \\ \alpha\mathbf{x}^T(\mathbf{s} + \mathbf{1}_N) &> \gamma' \end{aligned} \tag{7}$$

For $\alpha > 0$, the NP test is to decide $\mathcal{H}_1$ if

$$\mathbf{x}^T(\mathbf{s} + \mathbf{1}_N) > \frac{\gamma'}{\alpha} = \gamma'', \tag{8}$$

while for $\alpha < 0$, the test decides $\mathcal{H}_1$ if

$$\mathbf{x}^T(\mathbf{s} + \mathbf{1}_N) < \frac{\gamma'}{\alpha} = \gamma''. \tag{9}$$