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ET 4386 Estimation and Detection

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Exercises - Estimation

Notice that below are only the short answers without explanation and workout. In the exam you must also give the calculations/workout that you need to get to the answer. The road to the answer is more important than the answer itself!

Problem 1:

Problem 1a:

$$var[\hat{A}] \geq \frac{\sigma^2}{\sum_{n=0}^{N-1} r^{2n}} = \begin{cases} \frac{\sigma^2}{N}, & \text{for} r = 1 \\ \frac{\sigma^2(1-r^2)}{1-r^{2N}}, & \text{otherwise.} \end{cases}$$

Problem 1b:

$$\hat{A}_{MVU} = \begin{cases} \frac{\sum_{n=0}^{N-1} x[n]}{N}, & \text{for} r = 1 \\ \frac{\sum_{n=0}^{N-1} x[n] r^n (1-r^2)}{1-r^{2N}}, & \text{otherwise.} \end{cases}$$

$$var[\hat{A}_{MVU}] = CRLB$$

Problem 1c:

$$\hat{A}_{BLUE} = \begin{cases} \frac{\sum_{n=0}^{N-1} x[n]}{N}, & \text{for} r = 1 \\ \frac{\sum_{n=0}^{N-1} x[n] r^n (1-r^2)}{1-r^{2N}}, & \text{otherwise.} \end{cases}$$

Problem 1d: If $r = 1$, $var[\hat{A}_{MVU}] \to 0$ if $r < 1$, $var[\hat{A}_{MVU}] = \sigma^2(1-r^2)$

Problem 1e:

$$E[A|\mathbf{x}] = \begin{cases} \frac{\sum_{n=0}^{N-1} x[n]}{\frac{\sigma^2}{\sigma_A^2} + N}, & \text{for} r = 1 \\ \frac{\sum_{n=0}^{N-1} x[n] r^n}{\frac{\sigma^2}{\sigma_A^2} + \frac{1-r^{2N}}{(1-r^2)}}, & \text{otherwise.} \end{cases}$$

Problem 1f: Use 10.33. $Bmse = \frac{\sigma^2}{\frac{\sigma^2}{\sigma_A^2} + \sum_{n=0}^{N-1} r^{2n}}$

Problem 1g: The data model is linear and the data is Gaussian. The LMMSE will be equal to the MMSE estimatr calculated by question 1e.

Problem 2:

Problem 2a:

$$E[\theta|\mathbf{x}] = \frac{1-N}{2-N} \frac{\beta^{-N+2} - \max[x[n]]^{-N+2}}{\beta^{-N+1} - \max[x[n]]^{-N+1}}$$

Problem 2b: . For $\beta \rightarrow$ the prior becomes non-informative.

Problem 2c:

$$E[\theta|\mathbf{x}] \approx \frac{1-N}{2-N} \max[x[n]] \approx \max[x[n]]$$

little prior knowledge.

Problem 2d: $\hat{\theta} = \max[x[n]]$

Problem 2e: $\hat{\theta} = \max[x[n]]$

Problem 3:

Problem 3a:

$$\hat{\mathbf{s}} = \frac{\sigma_S^2}{\sigma_S^2 + \sigma^2}\mathbf{x}$$

Problem 3b: $Bmse(\hat{\mathbf{s}}_i) = \sigma_S^2(\mathbf{I} - \frac{\sigma_S^2}{\sigma_S^2+\sigma^2}\mathbf{I})$

Problem 4:

Problem 4a: $\hat{A}_{MAP} = \max\left(\frac{\sum_{n=0}^{N-1} x[n] - \sigma^2\lambda}{N}, 0\right)$.

Problem 4b: $\hat{A}_{LMMSE} = \frac{\lambda\frac{\sigma^2}{N}}{\lambda^2\frac{\sigma^2}{N}+2} + \frac{\frac{2\sum_{n=0}^{N-1} x[n]}{\sigma^2}}{\lambda^2+2\frac{N}{\sigma^2}}$

Problem 5:

Problem 5a: $\hat{\theta}_{MAP} = X + 1$

Problem 6:

Problem 6a: $\frac{\partial p(\mathbf{x}|\theta)}{\partial\theta} = \frac{N}{\theta} - \sum_{n=0}^{N-1} x[n]$

$\frac{\partial^2 p(\mathbf{x}|\theta)}{\partial\theta^2} = -\frac{N}{\theta^2} \rightarrow \ I(\theta) = \frac{N}{\theta^2}$

$\text{Var}[\hat{\theta}] \geq \frac{\theta^2}{N}$

Problem 6b: $\frac{\partial p(\mathbf{x}|\theta)}{\partial\theta} = \frac{N}{\theta} - \sum_{n=0}^{N-1} x[n]$ cannot be written in the form $\frac{\partial p(\mathbf{x}|\theta)}{\partial\theta} = I(\theta)(\hat{\theta} - \theta)$. An unbiased estimator that reaches the bound does not exist.

Problem 6c: $\frac{\partial p(\mathbf{x}|\theta)}{\partial\theta} = \frac{N}{\theta} - \sum_{n=0}^{N-1} x[n] = 0 \Rightarrow \hat{\theta}_{MLE} = \frac{N}{\sum_{n=0}^{N-1} x[n]}$

Problem 6d: $\theta_{MAP} = \frac{N}{\sum_{n=0}^{N-1} x[n] + \lambda} = \frac{1}{\frac{\sum_{n=0}^{N-1} x[n}{N}] + \frac{\lambda}{N}}$

Problem 6e: $\hat{\theta}_{MMSE} = \frac{\int_0^\infty \lambda\theta^{N+1} e^{-\theta\left(\lambda + \sum_{n=0}^{N-1} x[n]\right)} d\theta}{\int_0^\infty \lambda\theta^{N} e^{-\theta\left(\lambda + \sum_{n=0}^{N-1} x[n]\right)} d\theta} = \frac{\Gamma(N+2)}{(\sum_{n=0}^{N-1} x[n])^{N+2}} \frac{(\sum_{n=0}^{N-1} x[n])^{N+1}}{\Gamma(N+1)} = \frac{(N+1)!}{N!} \frac{1}{\sum_{n=0}^{N-1} x[n]} = \frac{N+1}{\sum_{n=0}^{N-1} x[n]}$

These two integrals can be calcualted using partial integration, or, using a table of integrals.