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ET 4386 Estimation and Detection

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Exercises -Detection

Problem 1:

Problem 1a: Let $\mathbf{H} = \left[1, r, ..., r^{N-1}\right]^T$.

$$\begin{aligned} p(\mathbf{x}; \mathcal{H}_1) &= \frac{1}{(2\pi)^{\frac{N}{2}} \det^{\frac{1}{2}}(\mathbf{C})} \exp\left[-\tfrac{1}{2}(\mathbf{x} - A\mathbf{H})^T \mathbf{C}^{-1}(\mathbf{x} - A\mathbf{H})\right] \\ p(\mathbf{x}; \mathcal{H}_0) &= \frac{1}{(2\pi)^{\frac{N}{2}} \det^{\frac{1}{2}}(\mathbf{C})} \exp\left[-\tfrac{1}{2}\mathbf{x}^T \mathbf{C}^{-1}\mathbf{x}\right]. \end{aligned}$$

$$L(\mathbf{x}) = \frac{p(\mathbf{x}; \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} > \lambda.$$

$$\ln L(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} A\mathbf{H} - \frac{1}{2}\mathbf{H}^T \mathbf{C}^{-1}\mathbf{H}A^2 > \ln\lambda$$

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1}\mathbf{H}A > \ln\lambda + \frac{1}{2}\mathbf{H}^T \mathbf{C}^{-1}\mathbf{H}A^2 = \lambda'$$

Problem 1b: $T(\mathbf{x})$ is Gaussian distributed under both $\mathcal{H}_1$ and $\mathcal{H}_0$.

$$\mathrm{E}[T; \mathcal{H}_0] = \mathrm{E}[\mathbf{w}^T \mathbf{C}^{-1}\mathbf{H}A] = 0$$

$$\mathrm{E}[T; \mathcal{H}_1] = \mathrm{E}[(A\mathbf{H} + \mathbf{w})^T \mathbf{C}^{-1}\mathbf{H}A] = A^2\mathbf{H}^T \mathbf{C}^{-1}\mathbf{H} = \frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}$$

$$\mathrm{var}[T; \mathcal{H}_0] = \mathrm{E}[\left(\mathbf{w}^T \mathbf{C}^{-1}\mathbf{H}A\right)^2] = A^2\mathbf{H}^T \mathbf{C}^{-1}\mathbf{H} = \frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}$$

$$\mathrm{var}[T; \mathcal{H}_1] = \mathrm{E}[\left((A\mathbf{H} + \mathbf{w})^T \mathbf{C}^{-1}\mathbf{H}A - \mathrm{E}[(A\mathbf{H} + \mathbf{w})^T \mathbf{C}^{-1}\mathbf{H}A]\right)^2$$

$$= \mathrm{E}\left[\left(((A\mathbf{H} + \mathbf{w}) - \mathrm{E}\left[(A\mathbf{H} + \mathbf{w})\right])^T \mathbf{C}^{-1}\mathbf{H}A\right)^2\right] = \mathrm{E}\left[\left(\mathbf{w}^T \mathbf{C}^{-1}\mathbf{H}A\right)^2\right] = var[T; \mathcal{H}_0] = \frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}$$

$$P_{fa} = Q\left(\frac{\lambda'}{\sqrt{\frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}}}\right) \rightarrow \lambda' = Q^{-1}\left(P_{fa}\right)\sqrt{\frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}}$$

$$P_D = Q\left(\frac{\lambda' - \frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}}{\sqrt{\frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}}}\right) = Q\left(Q^{-1}\left(P_{fa}\right) - \sqrt{\frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}}\right),$$

Problem 1c: For $0 \leq r \leq 1$, $\sum_{n=0}^{N-1} r^{2n} = \frac{1-r^{2N}}{1-r^2}$ and $P_D = Q\left(Q^{-1}\left(P_{fa}\right) - \sqrt{\frac{A^2}{\sigma^2}\frac{1-r^{2N}}{1-r^2}}\right)$

When $N \to \infty$ for $0 \leq r \leq 1$, P_D will become $P_D = Q\left(Q^{-1}\left(P_{fa}\right) - \sqrt{\frac{A^2}{\sigma^2}\frac{1}{1-r^2}}\right)$ which will be smaller than 1 for $P_{fa} < 1$.

For $r = 1$ P_D will become $P_D = Q\left(Q^{-1}\left(P_{fa}\right) - \sqrt{\frac{NA^2}{\sigma^2}}\right)$ and for $N \to \infty$ P_D will approach 1. For $r \geq 1$ P_D will also approach 1 as $\lim_{N\to\infty} \sum_{n=0}^{N-1} r^{2n}$ will then $\to \infty$.

Problem 2:

Problem 2a: As the noise is white and Gaussian, the shape of the signal does not influence the detection performancce, but the power does. As both signals have an equal power, the detection performance will be equal.

Problem 2b: Using the matrix inversion lemma we can calculate $\mathbf{C}^{-1}$, that is, $\mathbf{C}^{-1} = \frac{1}{\sigma^2} - \frac{\frac{1}{\sigma^4}\mathbf{1}\mathbf{1}^T}{1+\frac{N}{\sigma^2}}$. We can use this result to calcualte the P_D:

$$P_D = Q\left(Q^{-1}(P_{fa}) - \sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}\right).$$

For $s_1[n]$ we then get $P_D = Q\left(Q^{-1}(P_{fa}) - \sqrt{\frac{A^2}{\frac{\sigma^2}{N}+1}}\right) = Q\left(Q^{-1}(P_{fa}) - \sqrt{\frac{A^2\frac{N}{\sigma^2}}{\frac{N}{\sigma^2}+1}}\right)$.

For $s_2[n]$ and (even) N we get $P_D = Q\left(Q^{-1}(P_{fa}) - \sqrt{A^2\frac{N}{\sigma^2}}\right)$. The P_D for even N and $s_2[n]$ will thus always be larger.

One can also argue that $s[n]$ should ideally equal the eigenvector of $\mathbf{C}$ that corresponds to the minimum eigenvalue. The largest eigenvalue is $\mathbf{1}$. This corresponds with $s_1[n]$. Signal $s_2[n]$ is at least orthogonal to this eigenvector and corresponds to the minimum eigenvalue. $s_2[n]$ will thus have the best detection performance.

Problem 3:

Problem 3a: LRT: $\frac{(1-p_1)^k p_1}{(1-p_0)^k p_0} \geq \lambda$

$\frac{(1-p_1)^k}{(1-p_0)^k} \geq \lambda\frac{p_0}{p_1}$

$k \geq \frac{\log\lambda\frac{p_0}{p_1}}{\log\left(\frac{1-p_1}{1-p_0}\right)} = \lambda'$

Problem 3b:

Problem 4: $\mathbf{s} \sim N(\mathbf{0}, \mathbf{C}_s)$ and $\mathbf{w} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$

$$T(\mathbf{x}) = \mathbf{x}^T\mathbf{C}_s\left(\mathbf{C}_s + \sigma^2\mathbf{I}\right)^{-1}\mathbf{x} = \mathbf{x}^T\boldsymbol{\Lambda}\mathbf{x}$$

with $\boldsymbol{\Lambda} = \text{diag}\left(\frac{\sigma_{s_0}^2}{\sigma_{s_0}^2+\sigma^2}, \frac{\sigma_{s_1}^2}{\sigma_{s_1}^2+\sigma^2}, ..., \frac{\sigma_{s_{N-1}}^2}{\sigma_{s_{N-1}}^2+\sigma^2}\right)$

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} x^2[n]\frac{\sigma_{s_n}^2}{\sigma_{s_n}^2 + \sigma^2}$$

Problem 5:

Problem 5a: We need to calculate $\hat{\mathbf{s}} = E[\mathbf{s}|\mathbf{x}]$. However, A and $\mathbf{w}$ are Gaussian (and thus also jointly Gaussian) distributed. In addition, the model is linear:

$$\mathbf{x} = \mathbf{1}A + \mathbf{w} = \mathbf{s} + \mathbf{w}.$$

In this case the MMSE estimator is given by $\hat{\mathbf{s}} = E[A|\mathbf{x}]\mathbf{1} = \left(\mathbf{C}_A^{-1} + \mathbf{H}^T\mathbf{C}_{\mathbf{w}}^{-1}\mathbf{H}^{-1}\right)^{-1}\mathbf{H}^T\mathbf{C}_{\mathbf{w}}^{-1}\mathbf{x} = \left(\frac{1}{N\sigma_A^2} + \frac{1}{\sigma^2}\right)^{-1}\frac{\bar{x}}{\sigma^2} = \frac{\sigma_A^2\bar{x}}{\sigma_A^2+\frac{\sigma^2}{N}}\mathbf{1}$

Problem 5b: NP: $T(\mathbf{x}) = \mathbf{x}^T\hat{\mathbf{s}}$

Problem 6:

Problem 6a: $\mathbf{s} = A\mathbf{H}$, $\mathbf{H} = \left[1, r^1, ..., r^{N-1}\right]^T$ with $\mathbf{s} \sim N(0, \sigma_A^2\mathbf{H}\mathbf{H}^T)$.

$$\hat{\mathbf{s}} = \mathbf{C}_s\left(\mathbf{C}_s + \sigma^2\mathbf{I}\right)^{-1}\mathbf{x}$$

Using the matrix inversion lemma it follows that

$$\hat{\mathbf{s}} = \mathbf{C}_s\left(\mathbf{C}_s + \sigma^2\mathbf{I}\right)^{-1}\mathbf{x} = \frac{\sigma_A^2}{\sigma^2}\mathbf{H}\mathbf{H}^T(1 - \frac{\mathbf{H}^T\mathbf{H}\sigma_A^2}{\sigma^2 + \mathbf{H}^T\mathbf{H}\sigma_A^2})\mathbf{x} = \frac{\sigma_A^2\mathbf{H}\mathbf{H}^T}{\sigma^2 + \mathbf{H}^T\mathbf{H}\sigma_A^2}\mathbf{x}$$

We then get

$$T(\mathbf{x}) = \mathbf{x}^T\hat{\mathbf{s}} = \frac{\sigma_A^2\mathbf{x}^T\mathbf{H}\mathbf{H}^T\mathbf{x}}{\sigma^2 + \mathbf{H}^T\mathbf{H}\sigma_A^2} = \frac{\left(\sum_{n=0}^{N-1} r^n x[n]\right)^2}{\left(\frac{\sigma^2}{\sigma_A^2} + \sum_{n=0}^{N-1} r^{2n}\right)}$$

or (using 14.7 vol - I):

$$\hat{A} = \left(\mathbf{C}_A^{-1} + \mathbf{H}^T\mathbf{C}_w^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{C}_w^{-1}\mathbf{x} = \left(\frac{1}{\sigma_A^2} + \frac{\sum_{n=0}^{N-1} r^{2n}}{\sigma^2}\right)^{-1}\frac{\sum_{n=0}^{N-1} r^n x[n]}{\sigma^2}$$

$$= \left(\frac{\sigma^2}{\sigma_A^2} + \sum_{n=0}^{N-1} r^{2n}\right)^{-1}\sum_{n=0}^{N-1} r^n x[n]$$

$$\hat{\mathbf{s}} = \hat{A}\mathbf{H}$$

$$T(\mathbf{x}) = \mathbf{x}^T\hat{\mathbf{s}} = \mathbf{x}^T\mathbf{H}\hat{A} = \frac{\left(\sum_{n=0}^{N-1} r^n x[n]\right)^2}{\left(\frac{\sigma^2}{\sigma_A^2} + \sum_{n=0}^{N-1} r^{2n}\right)}$$

Problem 7: $T(\mathbf{x}) = \mathbf{x}^T\hat{\mathbf{s}} = \mathbf{x}^T\mathbf{C}_s(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\mathbf{x} = \sum_{n=0}^{N-1}\frac{\sigma_{s_n}^2 x^2[n]}{\sigma_{s_n}^2 + \sigma^2}$

Problem 8:

Problem 8a: We have $w \sim N(\mathbf{0}, \mathbf{C}_w)$ and $s \sim N(\mathbf{0}, \mathbf{C}_s) = N(\mathbf{0}, \mathbf{C}_w\eta)$. So, $T(\mathbf{x}) = \mathbf{x}^T\mathbf{C}_w^{-1}\mathbf{C}_s(\mathbf{C}_s + \mathbf{C}_w)^{-1}\mathbf{x} = \frac{\mathbf{x}^T\mathbf{C}_w^{-1}\mathbf{x}\eta}{1+\eta} \geq \gamma$ and $T'(\mathbf{x}) = \mathbf{x}^T\mathbf{C}_w^{-1}\mathbf{x} \geq \gamma'$. We know that $\mathbf{x}^T\mathbf{C}^{-1}\mathbf{x} \sim \chi_N^2$ (whitening of $\mathbf{x}$)

Problem 8b:

$$\begin{array}{ll} \mathcal{H}_0 & \mathbf{x} \sim N(0, \mathbf{C}_w) \\ \mathcal{H}_1 & \mathbf{x} \sim N(0, (1+\eta)\mathbf{C}_w) \end{array}$$

so,

$$\begin{array}{ll} \mathcal{H}_0 & T(\mathbf{x}) \sim \chi^2_N \\ \mathcal{H}_1 & \frac{T(\mathbf{x})}{1+\eta} \sim \chi^2_N \end{array}$$

$P_{fa} = P(T(\mathbf{x}) \geq \gamma'; H_0) = Q_{\chi^2_N}(\gamma') \Rightarrow \gamma' = Q^{-1}_{\chi^2_N}(P_{fa})$
$P_D = P(T(\mathbf{x}) \geq \gamma'; H_1) = P(\frac{T(\mathbf{x})}{1+\eta} \geq \frac{\gamma'}{1+\eta}; H_1) = Q_{\chi^2_N}(\frac{\gamma'}{1+\eta})$.
Notice that for $N = 2$, χ^2_2-distributed RVs becomes exponentially distributed.
$Q_{\chi^2_N}(\gamma') = \int_{\gamma'}^{\infty} \frac{1}{2} e^{-\frac{x}{2}} dx = e^{-\frac{\gamma'}{2}} = P_{fa} \Rightarrow \gamma' = -2\log P_{fa}$. $P_D = Q_{\chi^2_N}(\frac{-2\log P_{fa}}{1+\eta}) = \int_{\frac{-2\log P_{fa}}{1+\eta}}^{\infty} \frac{1}{2} e^{-\frac{x}{2}} dx = e^{\frac{\log P_{fa}}{1+\eta}} = P_{fa}^{\frac{1}{1+\eta}}$

Problem 9: We can use the expression for general Gaussian detection. That is

$$\begin{aligned} T'(\mathbf{x}) &= \frac{1}{2}\mathbf{x}^T \left[\mathbf{C}_w^{-1}\mathbf{C}_s \left(\mathbf{C}_s + \mathbf{C}_w\right)^{-1}\right] \mathbf{x} + \mathbf{x}^T \left(\mathbf{C}_s + \mathbf{C}_w\right)^{-1} \boldsymbol{\mu}_s \\ &= \frac{1}{2}\mathbf{x}^T \frac{\sigma_s^2}{\sigma^2} \left(\sigma_s^2 + \sigma^2\right)^{-1} \mathbf{x} + \mathbf{x}^T \left(\sigma_s^2 + \sigma^2\right)^{-1} A\mathbf{1} \\ &= \frac{1}{2}\frac{\sigma_s^2}{\sigma^2} \left(\sigma_s^2 + \sigma^2\right)^{-1} \mathbf{x}^T\mathbf{x} + \frac{A}{\sigma_s^2 + \sigma^2}\mathbf{x}^T\mathbf{1} \\ &= \frac{N}{2}\frac{\sigma_s^2}{\sigma^2}\frac{1}{\sigma_s^2 + \sigma^2} \underbrace{\frac{1}{N}\sum_{n=0}^{N-1} x^2[n]}_{\text{estimate of variance}} + \frac{NA}{\sigma_s^2 + \sigma^2} \underbrace{\frac{1}{N}\sum_{n}^{N-1} x[n]}_{\text{estimate of mean}} \end{aligned}$$

From this we can clearly see the contribution in the detector based on the deterministic component (mean) of the data and the random component (variance) of the data.

Problem 10:

Problem 10a: Let $\mathbf{H} = \left[1, r, ..., r^{N-1}\right]^T$.

$$\begin{aligned} p(\mathbf{x}; A, \mathcal{H}_1) &= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\left(\mathbf{x} - A\mathbf{H}\right)^T\left(\mathbf{x} - A\mathbf{H}\right)\right] \\ p(\mathbf{x}; \mathcal{H}_0) &= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\mathbf{x}^T\mathbf{x}\right]. \end{aligned}$$

Determine the MLE of A:

$$p(\mathbf{x}; \hat{A}, \mathcal{H}_1) = \max_A p(\mathbf{x}; A, \mathcal{H}_1)$$

$$\frac{dp(\mathbf{x}; A, \mathcal{H}_1)}{dA} = 0$$

This leads to $\hat{A}_{MLE} = \frac{\mathbf{x}^T\mathbf{H} + \mathbf{H}^T\mathbf{x}}{\mathbf{H}^T\mathbf{H}} = \frac{\sum_{n=0}^{N-1} r^n x[n]}{\sum_{n=0}^{N-1} r^{2n}}$.

Problem 10b:

$$L_G(\mathbf{x}) = \frac{p(\mathbf{x}; \hat{A}_{MLE}, \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} = \frac{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - \hat{A}_{MLE} r^n)^2 \right]}{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} x^2[n] \right]} > \gamma$$

This can be written as

$$-\sum_{n=0}^{N-1} (x[n] - \hat{A}_{MLE} r^n)^2 + \sum_{n=0}^{N-1} x^2[n] = -\hat{A}_{MLE}^2 \sum_{n=0}^{N-1} r^{2n} + 2\hat{A}_{MLE} \sum_{n=0}^{N-1} x[n] r^n =$$

$$-\left(\frac{\sum_{n=0}^{N-1} r^n x[n]}{\sum_{n=0}^{N-1} r^{2n}} \right)^2 \sum_{n=0}^{N-1} r^{2n} + 2 \frac{\sum_{n=0}^{N-1} r^n x[n]}{\sum_{n=0}^{N-1} r^{2n}} \sum_{n=0}^{N-1} x[n] r^n = \frac{\left(\sum_{n=0}^{N-1} r^n x[n] \right)^2}{\sum_{n=0}^{N-1} r^{2n}} > \gamma'$$

This can be written as $\hat{A}_{MLE}^2 > \gamma''$