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ET 4386 Estimation and Detection

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Exercises -Detection

Problem 1: Binary detection problem with $\mathbf{w} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$ and deterministic signal $s[n] = Ar^n$:

$$\begin{array}{ll} \mathcal{H}_0 & x[n] = w[n] \\ \mathcal{H}_1 & x[n] = Ar^n + w[n] \end{array}$$

Problem 1a: Find the NP detector.

Problem 1b: Determine the detection performance.

Problem 1c: What happens as $N \to \infty$ for $0 \leq r \leq 1$, $r = 1$ and $r \geq 1$?

Problem 2: To optimize the detection probability of a signal in WGN, different signals are investigated. These are

$$s_1[n] = A$$

and

$$s_2[n] = A(-1^n),$$

both for $n = 0, ..., N-1$.

Problem 2a: Which signal will have the best detection performance?

Now consider the case where the noise has correlation matrix $\mathbf{C} = \sigma^2\mathbf{I} + P\mathbf{1}\mathbf{1}^T$

Problem 2b: Which signal will have the best detection performance?

Problem 3: Given is a geometrically distribute random variable k, which is the number of failures before the first succes in a series of Bernoulli trials. The pmf is given by

$$f(k;p) = (1-p)^k p.$$

We want to make a binary decision on the distribution of k, which is given by the following two hypotheses:

$$\begin{array}{ll} \mathcal{H}_0 & k \sim f(k;p_0) \\ \mathcal{H}_1 & k \sim f(k;p_1) \end{array}$$

Problem 3a: Find the NP detector $T(k)$.

Problem 3b: Determine the detection performance as a function of P_{fa}.

Problem 4: Find the NP detector for the problem of a random Gaussian signal $s[n]$ for $n = 0, ..., N-1$ in white Gaussian noise. The covariance matrix $\mathbf{C}_s$ is given by $\mathbf{C}_s = \text{diag}(\sigma^2_{s_0}, \sigma^2_{s_1}, ..., \sigma^2_{s_{N-1}})$ and $\mathbf{s} \sim N(\mathbf{0}, \mathbf{C}_s)$.

Problem 5: We want to detect a random DC level A embedded in WGN with variance σ^2. The two hypotheses are given by

$$\begin{array}{ll} \mathcal{H}_0 & x[n] = w[n] \\ \mathcal{H}_1 & x[n] = A + w[n] \end{array}$$

for $n = 0, ..., N-1$ and $A \sim N(0, \sigma^2_A)$.

Problem 5a: Find the MMSE estimator of the signal $\mathbf{s}$.

Problem 5b: Find the NP detector $T(\mathbf{x})$.

Problem 6: We have the following binary detection problem

$$\begin{array}{ll} \mathcal{H}_0 & x[n] = w[n] \\ \mathcal{H}_1 & x[n] = Ar^n + w[n] \end{array}$$

with $0 < r < 1$ and $A \sim N(0, \sigma^2_A)$ and $w[n]$ white Gaussian noise with variance σ^2. A and $w[n]$ are independent.

Problem 6a: Find the test statistic $T(\mathbf{x})$.

Problem 7: Find the NP detector for the problem of a random zero-mean signal $s[n]$ with covariance $\mathbf{C}_s = diag(\sigma^2_{s_0}, \sigma^2_{s_1}, ..., \sigma^2_{s_{N-1}})$ embedded in WGN with variance σ^2. Assume tht the data samples observed are $x[n]$ for $n = 0, 1, ..., N-1$.

Problem 8:

Problem 8a: Find the NP detector for the case where $\mathbf{x}$ has dimension $N \times 1$ and where

$$\begin{array}{ll} \mathcal{H}_0 & x[n] = w[n] \\ \mathcal{H}_1 & x[n] = s[n] + w[n] \end{array}$$

with $\mathbf{w} \sim N(\mathbf{0}, \mathbf{C}_w)$ and $\mathbf{s} \sim N(\mathbf{0}, \mathbf{C}_s) = N(\mathbf{0}, \mathbf{C}_w \eta)$ with $\eta > 0$.

Problem 8b: Determine P_{fa} and P_D for general N as well as for $N = 2$. Hint: if $\mathbf{x} \sim N(\mathbf{0}, \mathbf{C})$, then $\mathbf{x}^T \mathbf{C}^{-1} \mathbf{x} \sim \chi^2_N$.

Problem 9: Let

$$\begin{array}{ll} \mathcal{H}_0 & x[n] = w[n] \\ \mathcal{H}_1 & x[n] = s[n] + w[n] \end{array}$$

with $w[n] \sim N(0, \sigma^2)$ and $s[n] \sim N(A, \sigma^2_s)$. Give the NP detector for the case that the IID samples $n = 0, ..., N-1$ are observed.

Problem 10: We wish to detect a damped exponential $s[n] = Ar^n$, where A is unknown and r is known $(0 < r < 1)$, in WGN with known variance σ^2, based on N samples for $x[n]$.

Problem 10a: Determine the MLE for A.

Problem 10b: Determine the GLRT that decides $\mathcal{H}_1$.