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ET 4386 Estimation and Detection

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Example 1 - Lecture 6

This example is a continuation of an example given in lecture 5.
Let $x_1, ..., x_N$ be iid measurements from a Poisson (λ) distribution with marginal pmf

$$p(x_n; \lambda) = e^{-\lambda} \frac{\lambda^{x_n}}{x_n!},$$

and with expected value $E[x_n] = \lambda$.

(a) Calculate $\frac{\partial \ln p(x_n;\lambda)}{\partial \lambda}$ and show that the regularity condition is satisfied.

(b) Determine the CRLB for $\text{Var}[\hat{\lambda}]$ under the pmf $p(x; \lambda)$

(c) Give the MVU estimator for λ.

In the following we will consider λ to be a random variable instead of a deterministic parameter. Let the distribution of λ be uniform in the interval $[0, c]$.

(d) Calculate the MMSE estimator $\hat{\lambda} = E[\lambda | x_1, ..., x_N]$.
Make use of the the following relation:

$$\int_0^u x^{\nu-1} e^{-\mu x} dx = \mu^{-\nu} \gamma(\nu, \mu u), \tag{1}$$

where $\gamma(s, x)$ is known as the incomplete Gamma function, defined as $\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$.

(e) Calculate the MAP estimator λ_{MAP}.

Now let the distribution of λ be exponential with pdf $p(\lambda; a) = ae^{-a\lambda}$, with $\lambda \geq 0$, $E[\lambda] = \frac{1}{a}$ and $\text{var}[\lambda] = \frac{1}{a^2}$.

(f) Calculate the MMSE estimator $\hat{\lambda} = E[\lambda | x_1, ..., x_N]$.
Make use of the the following relation:

$$\int_0^\infty x^{\nu-1} e^{-\mu x} dx = \mu^{-\nu} \Gamma(\nu), \tag{2}$$

(g) Calculate the MAP estimator λ_{MAP}.

Answer Example 1 - Lecture 6

(a) $p(\mathbf{x};\lambda) = \prod_{i=1}^{N} e^{-\lambda}\frac{\lambda^{x_n}}{x_n!} = e^{-N\lambda}\frac{\lambda^{\left(\sum_{n=1}^{N} x_n\right)}}{\prod_{n=1}^{N} x_n!}$

$\frac{\partial \ln p(x;\lambda)}{\partial\lambda} = -N + \frac{\sum_{n=1}^{N} x_n}{\lambda}$

The regularity condition holds as, $E\left[\frac{\partial \ln p(x;\lambda)}{\partial\lambda}\right] = -N + \frac{E\left[\sum_{n=1^N} x_n\right]}{\lambda} = -N + \frac{N\lambda}{\lambda} = 0.$

(b) $\frac{\partial^2 \ln p(\mathbf{x};\lambda)}{\partial\lambda^2} = \frac{-\sum_{n=1}^{N} x_n}{\lambda^2}.$

$E\left[\frac{\partial^2 \ln p(\mathbf{x};\lambda)}{\partial\lambda^2}\right] = -\frac{N}{\lambda}$. The CRLB is then given by $\mathrm{Var}[\hat{\lambda}] \geq \frac{1}{-E\left[\frac{\partial^2 \ln p(\mathbf{x};\lambda)}{\partial\lambda^2}\right]} = \frac{\lambda}{N}$

(c) From question (a) we know that

$$\frac{\partial \ln p(x;\lambda)}{\partial\lambda} = -N + \frac{\sum_{n=1}^{N} x_n}{\lambda}.$$

This can be rewritten as

$$\frac{\partial \ln p(x;\lambda)}{\partial\lambda} = \underbrace{\frac{N}{\lambda}}_{I(\lambda)}\left(\underbrace{\frac{\sum_{n=1}^{N} x_n}{N}}_{\hat{\lambda}} - \lambda\right).$$

This is exactly the form

$$\frac{\partial \ln p(\mathbf{x};\lambda)}{\partial\lambda} = I(\lambda)(\hat{\lambda} - \lambda).$$

The MVU estimator is thus given by $\hat{\lambda} = \frac{\sum_{n=1}^{N} x_n}{N}$.

(d)

$$\hat{\lambda} = E[\lambda|x_1, ..., x_N] = \frac{\int_0^c \lambda e^{-N\lambda}\frac{\lambda^{\left(\sum_{n=1}^{N} x_n\right)}}{\prod_{n=1}^{N} x_n!}\frac{1}{c}d\lambda}{\int_0^c e^{-N\lambda}\frac{\lambda^{\left(\sum_{n=1}^{N} x_n\right)}}{\prod_{n=1}^{N} x_n!}\frac{1}{c}d\lambda} = \frac{\int_0^c e^{-N\lambda}\lambda^{1+\left(\sum_{n=1}^{N} x_n\right)}d\lambda}{\int_0^c e^{-N\lambda}\lambda^{\left(\sum_{n=1}^{N} x_n\right)}d\lambda}$$

Using Eq. (1) we then obtain

$$E[\lambda|x_1, ..., x_N] = \frac{N^{-(2+(\sum_{n=1}^{N} x_n))}\gamma(2 + \sum_{n=1}^{N} x_n, Nc)}{N^{-(1+(\sum_{n=1}^{N} x_n))}\gamma(1 + \sum_{n=1}^{N} x_n, Nc)} = \frac{1}{N}\frac{\gamma(2 + \sum_{n=1}^{N} x_n, Nc)}{\gamma(1 + \sum_{n=1}^{N} x_n, Nc)}.$$

Notice that

$$\lim_{x\to\infty}\gamma(s, x) = \Gamma(s)$$

and that $\Gamma(x + 1)/\Gamma(x) = x$. For $c \to \infty$ we thus obtain

$$\lim_{c\to\infty} E[\lambda|x_1, ..., x_N] = \frac{1}{N}\frac{\Gamma(2 + (\sum_{n=1}^{N} x_n))}{\Gamma(1 + (\sum_{n=1}^{N} x_n))} = \frac{1 + \sum_{n=1}^{N} x_n}{N}$$

(e) $\lambda_{MAP} = \arg\max_\lambda \log p(\lambda|x_1,...,x_N)$. Function $\log p(\lambda|x_1,...,x_N)$ is concave for $\lambda \in [0, c]$.

$$\frac{d}{d\lambda}\log p(\lambda|x_1,...,x_N) = \frac{d}{d\lambda}\log p(\lambda) + \log p(x_1,...,x_N|\lambda) - \log p(x_1,...,x_N) =$$

$$\frac{d}{\lambda} - N\lambda + (\sum_{n=1}^{N} x_n)\log(\lambda) = -N + \frac{(\sum_{n=1}^{N} x_n)}{\lambda}.$$

Thus, $\lambda_{MAP} = \frac{(\sum_{n=1}^{N} x_n)}{N}$ for $\frac{(\sum_{n=1}^{N} x_n)}{N} < c$. Otherwise, $\lambda_{MAP} = c$.

(f)

$$\hat{\lambda} = E[\lambda|x_1,...,x_N] = \frac{\int_0^\infty \lambda e^{-N\lambda}\frac{\lambda^{(\sum_{n=1}^{N} x_n)}}{\prod_{n=1}^{N} x_n!} a e^{-a\lambda} d\lambda}{\int_0^\infty e^{-N\lambda}\frac{\lambda^{(\sum_{n=1}^{N} x_n)}}{\prod_{n=1}^{N} x_n!} a e^{-a\lambda} d\lambda} = \frac{\int_0^\infty e^{-(a+N)\lambda}\lambda^{(1+\sum_{n=1}^{N} x_n)} d\lambda}{\int_0^\infty e^{-(a+N)\lambda}\lambda^{(\sum_{n=1}^{N} x_n)} d\lambda} =$$

$$\frac{\frac{1}{(N+a)^{2+\sum_{n=1}^{N} x_n}}\Gamma\left(2 + \sum_{n=1}^{N} x_n\right)}{\frac{1}{(N+a)^{1+\sum_{n=1}^{N} x_n}}\Gamma\left(1 + \sum_{n=1}^{N} x_n\right)} = \frac{1 + \sum_{n=1}^{N} x_n}{N + a}$$

If we define $\alpha = \frac{N}{N+a}$, we can write

$$E[\lambda|x_1,...,x_N] = \alpha\frac{\sum_{n=1}^{N} x_n}{N} + (1-\alpha)E[\lambda],$$

which clearly shows the trade-off between the data $(x_1,...,x_N)$ and the information of the prior by means of $E[\lambda]$.

(g) $\lambda_{MAP} = \arg\max_\lambda \log p(\lambda|x_1,...,x_N)$. Function $\log p(\lambda|x_1,...,x_N)$ is concave.

$$\frac{d}{d\lambda}\log p(\lambda|x_1,...,x_N) = \frac{d}{d\lambda}\log p(\lambda) + \log p(x_1,...,x_N|\lambda) - \log p(x_1,...,x_N) =$$

$$\frac{d}{\lambda} - (a+N)\lambda + (\sum_{n=1}^{N} x_n)\log(\lambda) = -(a+N) + \frac{(\sum_{n=1}^{N} x_n)}{\lambda}.$$

Thus, $\lambda_{MAP} = \frac{(\sum_{n=1}^{N} x_n)}{N+a}$