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ET 4386 Estimation and Detection

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Example 1 - Lecture 2

Let $x_1, ..., x_N$ be iid measurements from a Poisson (λ) distribution with marginal pmf

$$p(x_n; \lambda) = e^{-\lambda}\frac{\lambda^{x_n}}{x_n!},$$

and with expected value $E[x_n] = \lambda$.

(a) Calculate $\frac{\partial \ln p(x_n;\lambda)}{\partial \lambda}$ and show that the regularity condition is satisfied.

(b) Determine the CRLB for Var$[\hat{\lambda}]$ under the pmf $p(x; \lambda)$

(c) Give the MVU estimator for λ.

Example 2 - Lecture 2

Let $x[0]$ and $x[1]$ be two measurements of a constant in correlated Gaussian noise:

$$\mathbf{x} = A\mathbf{1} + \mathbf{w},$$

with $\mathbf{x} = [x[0]\ x[1]]^T$, $\mathbf{w} = [w[0]\ w[1]]^T$ and $\mathbf{1} = [1\ 1]^T$. The noise $\mathbf{w}$ is zero mean Gaussian with correlation matrix:

$$\mathbf{C} = \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Parameter ρ is the correlation coefficient between $w[0]$ and $w[1]$.

(a) Compute the CRLB for A.

(b) Explain what happens when $\rho = 1$, $\rho = -1$ and $\rho = 0$.

(b) Give the MVU estimator for A.

Answer Example 1 - Lecture 2

(a) $p(\mathbf{x}; \lambda) = \prod_{i=1}^{N} e^{-\lambda}\frac{\lambda^{x_n}}{x_n!} = e^{-N\lambda}\frac{\lambda^{\left(\sum_{n=1}^{N} x_n\right)}}{\prod_{n=1}^{N} x_n!}$

$\frac{\partial \ln p(x;\lambda)}{\partial \lambda} = -N + \frac{\sum_{n=1}^{N} x_n}{\lambda}$

The regularity condition holds as, $E\left[\frac{\partial \ln p(x;\lambda)}{\partial \lambda}\right] = -N + \frac{E\left[\sum_{n=1^N} x_n\right]}{\lambda} = -N + \frac{N\lambda}{\lambda} = 0.$

(b) $\frac{\partial^2 \ln p(\mathbf{x};\lambda)}{\partial \lambda^2} = \frac{-\sum_{n=1}^{N} x_n}{\lambda^2}$.
$E\left[\frac{\partial^2 \ln p(\mathbf{x};\lambda)}{\partial \lambda^2}\right] = -\frac{N}{\lambda}$. The CRLB is then given by $\text{Var}[\hat{\lambda}] \geq \frac{1}{-E\left[\frac{\partial^2 \ln p(\mathbf{x};\lambda)}{\partial \lambda^2}\right]} = \frac{\lambda}{N}$

(c) From question (a) we know that

$$\frac{\partial \ln p(x;\lambda)}{\partial \lambda} = -N + \frac{\sum_{n=1}^{N} x_n}{\lambda}.$$

This can be rewritten as

$$\frac{\partial \ln p(x;\lambda)}{\partial \lambda} = \underbrace{\frac{N}{\lambda}}_{I(\lambda)} \left(\underbrace{\frac{\sum_{n=1}^{N} x_n}{N}}_{\hat{\lambda}} - \lambda \right).$$

This is exactly the form

$$\frac{\partial \ln p(\mathbf{x};\lambda)}{\partial \lambda} = I(\lambda)(\hat{\lambda} - \lambda).$$

The MVU estimator is thus given by $\hat{\lambda} = \frac{\sum_{n=1}^{N} x_n}{N}$.

Answer Example 2 - Lecture 2

(a) $p(\mathbf{x};A) = (2\pi|\det(\mathbf{C})|)^{-1/2} \exp\left[-\frac{1}{2}(\mathbf{x} - A\mathbf{1})^T \mathbf{C}^{-1} (\mathbf{x} - A\mathbf{1})\right]$
$\frac{\partial \ln\, p(\mathbf{x};A)}{\partial A} = \mathbf{x}^T\mathbf{C}^{-1}\mathbf{1} - A\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}$
$\frac{\partial^2 \ln\, p(\mathbf{x};A)}{\partial A^2} = -\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}$
$\text{Var}[\hat{A}] \geq \frac{1}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}} = \frac{\sigma^2(1-\rho^2)}{2-2\rho} = \frac{\sigma^2}{2}(1+\rho)$

(b) When $\rho = 1$, the two noise samples $w[0]$ and $w[1]$ are fully correlated and $x[0]$ and $x[1]$ will be equal. This means we have only one independent sample and $\text{Var}[\hat{A}] = \sigma^2$. When $\rho = 0$ the two samples are completely independent and we get the result $\text{Var}[\hat{A}] = \sigma^2/2$. For $\rho = -1$, adding $x[0]$ and $x[1]$ will completely cancel the noise ($x[0] + x[1] = A$) and thus $\text{Var}[\hat{A}] \geq 0$.

(c) From $\frac{\partial \ln\, p(\mathbf{x};\mathbf{A})}{\partial A} = \mathbf{x}^T\mathbf{C}^{-1}\mathbf{1} - A\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}$ we can write

$$\frac{\partial \ln\, p(\mathbf{x};A)}{\partial A} = \mathbf{x}^T\mathbf{C}^{-1}\mathbf{1} - A\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1} = \underbrace{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}}_{I(A)} \left(\underbrace{\left(\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}\right)^{-1} \mathbf{x}^T\mathbf{C}^{-1}\mathbf{1}}_{\hat{A}} - A \right)$$

The MVU estimator is thus given by

$$\hat{A} = \left(\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}\right)^{-1} \mathbf{x}^T\mathbf{C}^{-1}\mathbf{1}.$$