Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group
Course: ET4386 Estimation and Detection theory (2020-2021)

# Excercises : Estimation theory

1. Consider the data $\{x[0], x[1], \ldots, x[N-1]\}$, where each sample is distributed as $\mathcal{U}[0, \theta]$, and the samples are IID.
   - Is there an unbiased estimator for $\theta$?
   - If yes, what is the estimator ? If no, why not ?
   - Show that the regularity condition does not hold for $x[n] \sim \mathcal{U}[0, \theta]$, which are IID.
2. Consider the data $x[n] = Ar^n + w[n]$ for $n = 0, 1, \ldots, N-1$ where $w[n]$ is WGN with variance $\sigma^2$. Derive the CRLB for $A$, and show that an efficient estimator exists and find its variance.
3. Consider the measurement model $\mathbf{x} \sim \mathcal{N}(A, 0.5A)$. Show that the regularity conditions hold for the estimation of A (a) Find the CRLB bound for $A$ (if it exists). Furthermore, find the following estimators for $A$ (if they exist) (b) MLE, (c) BLUE, and (d) LS.
4. For $N$ IID observation from the PDF $\mathcal{N}(A, \sigma^2)$, where $A$ and $\sigma^2$ are both unknown, find the MLE of the SNR $\alpha = A^2/\sigma^2$
5. The observed samples $x[0], x[1], \ldots, x[N-1]$ are IID according to the following PDFs:
   - Laplacian: $p(x[n]; \mu) = 0.5 \; exp(-x[n] - \mu)$
   - Gaussian: $p(x[n]; \mu) = (2\pi)^{-0.5} exp(-0.5(x[n] - \mu)^2)$

   Find the BLUE and LS of the mean $\mu$. Discuss the properties of the respective estimators.
6. To minimize the LS error, the data $\mathbf{x}$ is orthogonally projected to determine the signal estimate $\hat{\mathbf{s}}$. Prove $\|\hat{\mathbf{s}}\|_2^2 + \|\mathbf{x} - \hat{\mathbf{s}}\|_2^2 = \|\mathbf{x}\|_2^2$.
7. The data x[n] for $n = 0, 1, \ldots, N-1$ are observed, each sample having the conditional PDF

   $$p(x[n]|\theta) = \begin{cases} exp[(-(x[n] - \theta)] & x[n] > \theta \\ 0 & x[n] < \theta \end{cases},$$

   and conditioned on $\theta$ the observations are independent. The prior PDF is

   $$p(\theta) = \begin{cases} exp[-\theta] & \theta > 0 \\ 0 & \theta < 0 \end{cases}.$$

   Find the MMSE estimator of $\theta$.

8. If $[x, y]^T \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$, where

$$\mathbf{C} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

let $g(y) = p(x_0, y)$ for some $x = x_0$. Prove that $g(y)$ is maximized for $y = \rho x_0$. Also, show that $E(y|x_0) = \rho x_0$. If $\rho = 0$, what is the MMSE estimator of $y$ based on $x$ ?

9. Consider the data model

$$x[n] = Acos(2\pi f_0 n + \phi) + w[n]$$

where $A = \sqrt{(a^2 + b^2)}$ and $\phi = arctan(\frac{-b}{a})$. if $\theta = [a\ b]^T \sim \mathcal{N}(\mathbf{0}, \sigma_\theta^2 \mathbf{I})$, show that the PDF of A is Rayleigh, the PDF of $\phi$ is $\mathcal{U}[0, 2\pi]$ and that $A$ and $\phi$ are independent.

10. Consider the data

$$x[n] = Ar^n + w[n] \qquad n = 0, 1, \ldots, N-1$$

where $A$ is a parameter to be estimated, $r$ is a known constant, and $w[n]$ is zero mean white noise with variance $\sigma^2$. The parameter $A$ is modeled as a random variable with mean $\mu_A$ and variance $\sigma_A^2$ and is independent of $w[n]$. Find the LMMSE estimator of $A$ and the minimum Bayesian MSE.

11. A Gaussian random vector $\mathbf{x} = [x_1\ x_2]^T$ has zero mean and covariance matrix $\mathbf{C}_{xx}$. If $x_2$ is to be linearly estimated based on $x_1$. Find the estimator that minimizes the Bayesian MSE, and find the minimum MSE