Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group

EE4C03 STATISTICAL DIGITAL SIGNAL PROCESSING AND MODELING

18 January 2021, 9:00–12:20

Block 1 (9:00-10:30)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.

Upload answers during 10:25–10:35

This block consists of three questions (24 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (8 points)

(a) Let $x(n) = A\sin(\omega_0 n + \phi)$, where A and ω_0 are fixed constants, and ϕ is a random phase, uniformly distributed over $[-\pi, \pi]$. The signal is filtered by an FIR filter with impulse response

$$h(n) = \begin{cases} 1, & n = 0, \\ -\frac{1}{2}, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The output of the filter is $y(n) = h(n) * x(n)$.

Compute the autocovariance sequence $c_y(k)$.

(b) For a zero mean WSS process, let the autocorrelation sequence be given by

$$r(k) = b\, a^{|k|}\,, \qquad (|a| < 1)$$

and construct the $N \times N$ correlation matrix $\mathbf{R}$. Take $N > 2$.

- Derive the rank of $\mathbf{R}$.
- How can a random signal with this autocorrelation sequence be generated from white noise?

(c) *True* or *False:* is this a valid autocorrelation matrix?

$$\mathbf{R} = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

(Motivate your answer.)

(d) *True* or *False:* if $y(n) = x_1(n) + x_2(n)$, and $x_1(n)$ is a WSS process and $x_2(n)$ is a WSS process and not independent of $x_1(n)$, then $y(n)$ is a WSS process. (Motivate your answer: give a proof or give a counterexample.)

Question 2 (9 points)

A random sequence $x(n)$ has the autocorrelation sequence

$$r_x(0) = 1,\; r_x(1) = 0.8,\; r_x(2) = 0.5,\; r_x(3) = 0.1$$

(a) Use the Schur recursion to compute, step by step, the reflection coefficients $\Gamma_1, \Gamma_2, \Gamma_3$ and the modeling errors $\epsilon_1, \epsilon_2, \epsilon_3$.

(b) Draw the corresponding lattice filter schematic that filters the signal $x(n)$ into white noise. (Clearly indicate the inputs and outputs.)

(c) From the reflection coefficients, compute the model parameters $a_3(k)$ of a 3rd order model.

(d) Write down the 3×3 Yule-Walker equations and verify your result under (c).

(e) If we would have $\Gamma_3 = 0$, what conclusions can you draw from this? (What can you say about the random process, what can you say about $\Gamma_4, \Gamma_5, \cdots$ and $\epsilon_4, \epsilon_5, \cdots$.)

Question 3 (7 points)

We receive 1 second of a signal $x_a(t)$, sampled at a rate $f_s = 1000$ Hz to obtain a data sequence $x(n)$, $n = 0, \cdots, N-1$. The signal is assumed to be white Gaussian noise with variance σ^2 per sample, but it may be contaminated by a narrowband interferer. To detect it, Bartlett's method is used to estimate the power spectrum of $x(n)$. At each frequency we wish to obtain a variance of the estimate that is smaller than $\frac{1}{5}\sigma^2$.

(a) How should the number of segments K and the length of each segment L be chosen?

(b) What is best resolution (in Hz) that can be obtained using this technique?

(c) How can the resolution be improved? Give 3 possibilities. For each possibility, discuss the effect on the variance of the spectrum estimate.

(d) Suppose the bandwidth of the interferer is B Hz. To detect the interferer, does it help to improve the resolution? To answer this, consider the case where the resolution is smaller than B as well as the case where it is larger.

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18 January 2021, 9:00–12:20
Block 2 (10:50-12:20)
Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.
Upload answers during 12:15–12:25

This block consists of three questions (26 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 4 (9 points)

Consider a process $x(n)$ consisting of a sinusoid plus noise. Specifically, $x(n)$ is given by

$$x(n) = A(n)\cos(\omega n) + e(n),$$

where ω is a deterministic frequency, $A(n)$ is Gaussian with mean zero and variance σ_A^2, and $e(n)$ is white Gaussian with mean zero and variance σ_e^2. We assume $A(n)$ and $e(n)$ are independent. We want to estimate the amplitude $A(n)$ of this sinusoid by means of a simple zeroth order filter w applied to $x(n)$.

(a) What is the distribution of $x(n)$? What is its mean and its variance? Is $x(n)$ stationary?

(b) Whether $x(n)$ is stationary or not, can you think of a good filter w to estimate $A(n)$? In your answer, you may assume w is time-varying.

(c) Derive the normalized least mean squares (NLMS) algorithm to adaptively track the filter w that allows us to estimate $A(n)$ from $x(n)$.

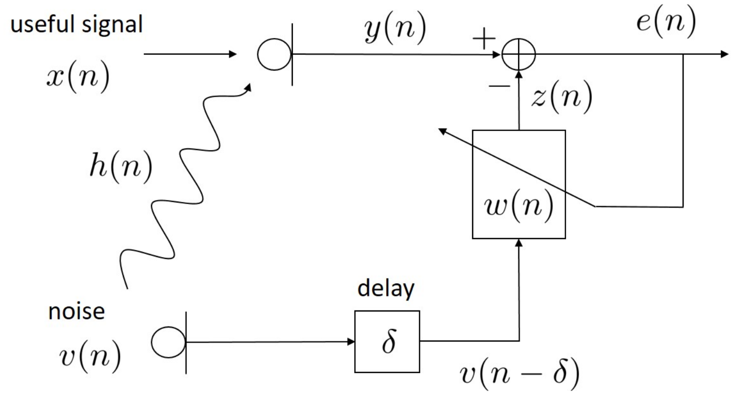
(d) Suppose we now want to estimate $A(n)$ from $x(n)$ using a Kalman filter. In that case, we first need to make some assumption on the dynamics of $A(n)$. Since we have no specific information about the dynamics, we simply assume

$$A(n+1) = A(n).$$

Using the above equation as state equation, derive the Kalman filter to estimate $A(n)$ from $x(n)$. Name at least two advantages of this Kalman filter over the NLMS filter described in (c)?

Question 5 (8 points)

Consider the noise canceling scenario depicted below.



The useful signal $x(n)$ that is picked up by the microphone is corrupted by noise $v(n)$ after it passes through the noise path $h(n)$. So the signal $y(n)$ measured by the microphone is $y(n) = x(n) + h(n) * v(n)$, where $*$ represents the convolution operator. To cancel the noise, we apply a separate microphone at the noise source, which only picks up the noise $v(n)$. We let the noise $v(n)$ pass through the filter $w(n)$ and subtract its output $z(n)$ from $y(n)$ to obtain $e(n) = y(n) - z(n)$. Minimizing the error $e(n)$ we hope that the filter $w(n)$ will be close to the noise path $h(n)$ such that $e(n)$ will be close to the desired signal $x(n)$. We assume that the noise $v(n)$ is white Gaussian with zero mean and variance σ^2 and that it is independent from the useful signal $x(n)$.

Suppose we can only afford estimating a filter $w(n)$ of a short order p which is much smaller than the order of $h(n)$. Hence, the filter taps of $w(n)$ can be grouped in a short vector $\mathbf{w} = [w(0), w(1), \ldots, w(p)]^T$. To compenstate for this limited order, we include a delay δ before the filter $w(n)$ and instead of filtering $v(n)$ we filter $v(n-\delta)$.

(a) First, express the optimal Wiener filter $\mathbf{w}$ as a function of $r_v(k) = E\{v(n)v(n-k)\}$ and $r_{yv}(k) = E\{y(n)v(n-k)\}$.

(b) Using the fact that $v(n)$ is white Gaussian with zero mean and variance σ^2, express $r_v(k)$ and $r_{yv}(k)$ as a function of $h(n)$.

(c) From the expressions in (b), write the optimal solution of $\mathbf{w}$ as a function of $h(n)$ and give the expression for the optimal error $e(n)$.

(d) Assume that we achieve our goal and $w(n) = h(n+\delta)$, for $n = 0, 1, \ldots, p$. In that case, what would be a good way to select δ?

(e) Is this still a good approach when $v(n)$ is not white? Also explain why. Describe in words what will change.

Question 6 (9 points)

Consider a stochastic process $x(n)$ with zero mean and correlation function $r_x(k)$. Now assume we sample this process with two low-rate samplers, one at rate $1/3$ and the other at rate $1/5$. So the sequence obtained from the first sampler is $x_1(n) = x(3n)$ whereas the one related to the second sampler is $x_2(n) = x(5n)$.

(a) Express the correlation matrix of the first 5 samples of $x_1(n)$ as a function of $r_x(k)$, i.e., use $r_x(k)$ to compute $\mathbf{R}_{x_1} = E\{\mathbf{x}_1\mathbf{x}_1^H\}$ with $\mathbf{x}_1 = [x(0), x(3), x(6), x(9), x(12)]^T$. Similarly express the correlation matrix of the first 3 samples of $x_2(n)$ as a function of $r_x(k)$, i.e., use $r_x(k)$ to compute $\mathbf{R}_{x_2} = E\{\mathbf{x}_2\mathbf{x}_2^H\}$ with $\mathbf{x}_2 = [x(0), x(5), x(10)]^T$.

(b) Express the cross-correlation matrix between the first 5 samples of $x_1(n)$ and the 3 first samples of $x_2(n)$ as a function of $r_x(k)$, i.e., use $r_x(k)$ to compute $\mathbf{R}_{x_1,x_2} = E\{\mathbf{x}_1\mathbf{x}_2^H\}$.

(c) How can these three correlation matrices be computed in practice if you have given $N_1 = 50$ samples of $x_1(n)$ and $N_2 = 30$ samples of $x_2(n)$?

(d) After computing the three correlation matrices $\mathbf{R}_{x_1}$, $\mathbf{R}_{x_2}$, and $\mathbf{R}_{x_1,x_2}$, which correlation values $r_x(k)$ can we estimate from $\mathbf{R}_{x_1}$, $\mathbf{R}_{x_2}$, and $\mathbf{R}_{x_1,x_2}$?

This result shows that even though we did not explicitly sample the process $x(n)$ at rate 1, we can still derive (part of) its correlation function from two low-rate samplers.

Now that we have estimated (part of) the correlation function $r_x(k)$, we can use this to interpolate the signals $x_1(n) = x(3n)$ and $x_2(n) = x(5n)$ in order to find the full rate 1 sequence $x(n)$. Assume for instance we want to estimate $x(n)$, for $n = 0, 1, 2, \ldots, 14$, from $\mathbf{x}_1$ and $\mathbf{x}_2$ jointly (or equivalently from $\mathbf{x}_0 = [\mathbf{x}_1^T, \mathbf{x}_2^T]^T$).

(e) Can you give the optimal Wiener filter expression to estimate $x(n)$, for $n = 0, 1, 2, \ldots, 14$, from $\mathbf{x}_1$ and $\mathbf{x}_2$? Try to express this filter as a function of $r_x(k)$.

(f) Do the correlation values from (d) provide enough information to estimate the full sequence $x(n)$, for $n = 0, 1, 2, \ldots, 14$, from $\mathbf{x}_1$ and $\mathbf{x}_2$? If so, explain why. If not, can you think of a way to circumvent this problem and still obtain a good estimate of $x(n)$, for $n = 0, 1, 2, \ldots, 14$?